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Representation of Canonical Commutation Relations in a Gauge Theory, the Aharonov-Bohm Effect, and the Dirac-Weyl Operator

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Abstract

We consider a representation of canonical commutation relations (CCR) appearing in a (non-Abelian) gauge theory on a non-simply connected region in the two-dimensional Euclidean space. A necessary and sufficient condition for the representation to be equivalent to the Schrödinger representation of CCR is given in terms of Wilson loops. A representation inequivalent to the Schrödinger representation gives a mathematical expression for the (non-Abelian) Aharonov-Bohm effect. Some aspects of the Dirac-Weyl operator associated with the representation of CCR are discussed.

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1 Introduction

Let $\mathcal{H}$ be a Hilbert space. For a linear operator T on $\mathcal{H}$, we denote its domain by $D(T)$. We say that a set $\{Q_j, P_j\}_{j=1}^d$ of self-adjoint operators on $\mathcal{H}$ is a representation of the canonical commutation relations (CCR) with d degrees of freedom if there exists a dense subspace $\mathcal{D}$ of $\mathcal{H}$ such that (i) $\mathcal{D} \subset \cap_{j,k=1}^d [D(Q_jP_k) \cap D(P_kQ_j) \cap D(Q_jQ_k) \cap D(P_jP_k)]$ and (ii) Q_j and P_j satisfy the CCR

$$\begin{aligned} [Q_j, P_k] &= i\hbar\delta_{jk}, \\ [Q_j, Q_k] &= 0, \quad [P_j, P_k] = 0, \quad j, k = 1, \cdots, d, \end{aligned}$$

on $\mathcal{D}$, where $\hbar$ is the Planck constant divided by 2π.

As is well known, a standard representation of the CCR is the *Schrödinger representation* $\{Q_j^S, P_j^S\}_{j=1}^d$ which is given as follows: $\mathcal{H} = L^2(\boldsymbol{R}^d), Q_j^S = x_j$ (the multiplication operator by the jth coordinate x_j), $P_j^S = -i\hbar D_j$ (D_j is the generalized partial differential operator in x_j), J$\mathcal{D} = \mathcal{S}(\boldsymbol{R}^d)$ (the Schwartz space of rapidly decreasing C^∞ functions on $\boldsymbol{R}^d$) or $\mathcal{D} = C_0^\infty(\boldsymbol{R}^d)$ (the space of C^∞ functions on $\boldsymbol{R}^d$ with compact support).

In relation to the Schrödinger representation, it is convenient to introduce a technical term : A set $\{Q_j, P_j\}_{j=1}^d$ of self-adjoint operators on a Hilbert space $\mathcal{H}$ is called a *Schrödinger d-system* if there exist mutually orthogonal closed subspaces $\mathcal{H}_\alpha$ of $\mathcal{H}$ such that $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$ with the following properties: (i) each $\mathcal{H}_\alpha$ reduces all Q_j, P_j ; (ii) the set $\{Q_j, P_j\}_{j=1}^d$ is, in each $\mathcal{H}_\alpha$, unitarily equivalent to the Schrödinger representation $\{Q_j^S, P_j^S\}_{j=1}^d$ [21].

Since the pioneering work of von Neumann [19], many studies have been done in connection with representation theory of CCR (see, e.g., [21] and references therein). A set $\{Q_j, P_j\}_{j=1}^d$ of self-adjoint operators on a Hilbert space $\mathcal{H}$ is called a *Weyl representation* with d degrees of freedom if Q_j and P_j satisfy the *Weyl relations*

$$\begin{aligned} e^{itQ_j}e^{isP_k} &= e^{-ist\hbar\delta_{jk}}e^{isP_k}e^{itQ_j}, \\ e^{itQ_j}e^{isQ_k} &= e^{isQ_k}e^{itQ_j}, \quad e^{itP_j}e^{isP_k} = e^{isP_k}e^{itP_j}, \\ & j,k = 1,\cdots,d, \ \ s,t \in \boldsymbol{R}. \end{aligned}$$

The Schrödinger representation $\{Q_j^S, P_j^S\}_{j=1}^d$ is a Weyl representation of CCR. Von Neumann established a uniqueness theorem in the sense that, if $\mathcal{H}$ is separable, then every Weyl representation of CCR with d degrees of freedom is a Schrödinger d-system ([19], [21]).

It follows from the von Neumann theorem that a Weyl representation is a representation of CCR. But the converse is not true. Namely, there exist representations of CCR which are not Weyl ones and hence not Schrödinger systems. Such examples have been discussed by some authors (e.g., [13], [16], [26], [27] and references therein). These examples, however, do not seem to have something to do with physics (with possible exception [13]).

In what follows, we say that a representation of CCR is an *equivalent* (resp., *inequivalent*) representation if it is (resp., not) a Schrödinger system.

Recently H. Reeh found a physically interesting inequivalent representation of CCR [23] : He considered a quantum system of a charged particle moving in the plane $\boldsymbol{R}^2$ under the influence of a perpendicular magnetic field concentrated at the origin and showed that, if the value of the magnetic flux is not in a discrete set, then the representation of CCR satisfied by the position and the physical (kinetic) momentum operators of the particle is an inequivalent representation. This inequivalent representation is interesting in that it may be regarded as a mathematical expression of the Aharonov-Bohm effect [1], although the quantum system under consideration is an idealized one[1].

Motivated by the work of Reeh just mentioned, a systematic mathematical approach was undertaken to analyze a two-dimensional quantum system of a charged particle with a perpendicular magnetic field which may be strongly singular at arbitrarily fixed points $\boldsymbol{a}_1, \cdots \boldsymbol{a}_N$ in $\boldsymbol{R}^2$ [3]. If the magnetic field is concentrated on the set $\{\boldsymbol{a}_n\}_{n=1}^N$, then the position and the physical momentum operators of the particle give a representation of CCR with two degrees of freedom. Mathematical aspects concerning this representation were clarified, including a complete characterization of the representation in terms of "local

[1]A characterization of the Aharonov-Bohm effect in terms of representations of local currents is given in [17]. The author is grateful to Prof. G.A. Goldin for informative comments in this respect.

quantization" of magnetic flux [2]. Moreover, in connection with this work, mathematical analysis of the Dirac-Weyl operator defined in terms of the physical momentum operator has been made in some detail [4, 6].

From the view-point of gauge theory, these studies were concerned with an Abelian gauge theory on the non-simply connected region

$$M := \boldsymbol{R}^2 \setminus \{\boldsymbol{a}_n\}_{n=1}^N. \tag{1.1}$$

It is natural to ask about that the non-Abelian case is like. This question was pursued in [5] and some results in the Abelian case have been extended to the non-Abelian case. As in the Abelian case, if the gauge field strength is concentrated on $\{\boldsymbol{a}_n\}_{n=1}^N$, then the position and the physical momentum operators of a quantum mechanical particle interacting with the gauge field give a representation of the CCR with two degrees of freedom. This representation is characterized in terms of Wilson loops of the gauge potential. An inequivalent representation appearing in this case may be regarded as a "non-Abelian Aharonov-Bohm effect".

The purpose of the present paper is to give a survey of results obtained in [3] – [5] as well as some additional new results.

2 Representation of CCR in a gauge theory

We consider a gauge theory on the non-simply connected region M given by (1.1). As the gauge group, we take the unitary group $U(p)$ of order p ($p \geq 1$). Since the Lie algebra of $U(p)$ is the algebra of $p \times p$ anti-Hermitian matrices, which we denote by $M_p^{\mathrm{ah}}(\boldsymbol{C})$, a gauge potential in the present case is given by an $M_p^{\mathrm{ah}}(\boldsymbol{C})$-valued 1-form

$$A(\boldsymbol{r}) := A_1(\boldsymbol{r})dx + A_2(\boldsymbol{r})dy, \quad \boldsymbol{r} = (x, y) \in M,$$

on M, where $A_j(\boldsymbol{r}), j = 1, 2$, are $M_p^{\mathrm{ah}}(\boldsymbol{C})$-valued functions. We assume that *each A_j is continuously differentiable on M*, unless otherwise stated. The gauge filed strength is an $M_p^{\mathrm{ah}}(\boldsymbol{C})$-valued 2-form given by

$$F(A) := dA + A \wedge A = F_{12} dx \wedge dy$$

with

$$F_{12} = D_x A_2 - D_y A_1 + [A_1, A_2],$$

where D_x and D_y are partial differential operators in the distribution sense in x and y, respectively.

We say that A is *flat* if $F(A) = 0$ on M.

Remark : By a theorem in distribution theory, A is flat if and only if there exist a nonnegative integer L and $p \times p$ matrices $T_n^{\alpha,\beta} \in M_p^{\mathrm{ah}}(\boldsymbol{C}), n = 1, \cdots, N, \alpha, \beta = 0, 1, \cdots, L$, such that

[2]Recently, H.Kurose and H.Nakazato [18] have taken another approach to this subject ; they construct a $*$-representation of the Weyl algebra with two degrees of freedom induced by a one-dimensional representation of the fundamental group of the non-simply connected space M (see (1.1) below) and prove that the $*$-representation is uintarily equivalent to the $*$-algebra generated by the position and the physical momentum operators considered in [3]. Their approach can be generalized to the non-Abelian case discussed below in the present paper.

$$D_x A_2(\boldsymbol{r}) - y A_1(\boldsymbol{r}) + [A_1(\boldsymbol{r}), A_2(\boldsymbol{r})] = \sum_{n=1}^{N} \sum_{\alpha,\beta=0}^{L} T_n^{\alpha,\beta} D_x^\alpha D_y^\beta \delta(\boldsymbol{r} - \boldsymbol{a}_n), \qquad (2.1)$$

where $\delta(\boldsymbol{r})$ is the Dirac delta distribution on $\boldsymbol{R}^2$. It is an open problem to find *all solutions* (up to gauge transformations) to this nonlinear partial differential equations (or, equivalently, to give a complete characterization of $M_p^{\mathrm{ah}}(\boldsymbol{C})$-valued, flat 1-forms) (see §6 for a partial result).

Henceforth, we use a system of units where $\hbar = 1$. The physical (kinetic) momentum operator $\boldsymbol{P} = (P_1, P_2)$ of a quantum mechanical particle interacting with the gauge potential A is given by

$$P_1 = -iD_x - iA_1, \quad P_2 = -iD_y - iA_2,$$

acting in $L^2(\boldsymbol{R}^2; \boldsymbol{C}^p)$, the Hilbert space of $\boldsymbol{C}^p$-valued square integrable functions on $\boldsymbol{R}^2$.

For an open set D in $\boldsymbol{R}^2$, we denote by $C_0^m(D; \boldsymbol{C}^p)$ the set of $\boldsymbol{C}^p$-valued, m times continuously differentiable functions on D with compact support. We denote by q_1, q_2 the multiplication operators by x and y, respectively. The following proposition is easily shown:

Proposition 2.2. *Suppose that A is flat. Then $\{q_j, P_j\}_{j=1}^2$ satisfies the CCR with two degrees of freedom*

$$[q_j, P_k] = i\delta_{jk}, \ [q_j, q_k] = 0, \ [P_j, P_k] = 0, \quad j, k = 1, 2.$$

on $C_0^2(M; \boldsymbol{C}^p)$.

This proposition shows that, if A is flat and each P_j is essentially self-adjoint, then $\{q_j, \bar{P}_j\}_{j=1}^2$ gives a representation of the CCR with two degrees of freedom, where $\bar{P}_j$ denotes the closure of P_j. Then it is an interesting problem to find a necessary and sufficient condition for the representation to be a Schrödinger 2-system. To solve the problem, we first examine if q_j and $\bar{P}_j$ satisfy the Weyl relations with two degrees of freedom.

3 Commutation relations of the unitary groups generated by the position and the physical momentum operators

Let $M_p(\boldsymbol{C})$ be the set of $p \times p$ complex matrices and B be an $M_p(\boldsymbol{C})$-valued, continuous, piecewise differentiable function on the interval $[a, b]$. Then one can define the *product integral* for B by

$$\prod_a^b e^{B(\tau)d\tau} := \lim_{n\to\infty} e^{B(t_n)(t_n - t_{n-1})} e^{B(t_{n-1})(t_{n-1} - t_{n-2})} \cdots e^{B(t_1)(t_1 - t_0)},$$

where $a = t_0 < t_1 < \cdots < t_n = b, \max_j |t_j - t_{j-1}| \to 0 \ (n \to \infty)$ [14].

Let C be a continuous, piecewise differentiable path in M and $\gamma(\tau) = (\gamma_1(\tau), \gamma_2(\tau)), \tau \in [a, b]$ $(a < b, a, b \in \boldsymbol{R})$ be a parametrization of it. The *Wilson loop* of C with a gauge potential A is defined by

$$W_A(C) := Pe^{-\int_C A} := \prod_a^b e^{-\{A_1(\gamma(\tau))\dot{\gamma}_1(\tau)+A_2(\gamma(\tau))\dot{\gamma}_2(\tau)\}d\tau},$$

where $\dot{\gamma}_j(\tau) = d\gamma_j(\tau)/d\tau, j = 1, 2$. It follows that $W_A(C) \in U(p)$.

We write $\boldsymbol{a}_n = (a_{n1}, a_{n2})$. For $t, s \in \boldsymbol{R}$, let

$$\boldsymbol{R}_s = \boldsymbol{R} \setminus \{a_{n1}, a_{n1} - s | n = 1, \cdots, N\}, \quad \boldsymbol{R}_t = \boldsymbol{R} \setminus \{a_{n2}, a_{n2} - t | n = 1, \cdots, N\},$$

and

$$M_{s,t} = \boldsymbol{R}_s \times \boldsymbol{R}_t.$$

For each $(x, y) \in M_{s,t}$, we define a closed path $C_{x,y;s,t}$ in M by

$$\begin{aligned} C_{x,y;s,t} &= \{(x + \tau s, y) | 0 \le \tau \le 1\} \cup \{(x + s, y + \tau t) | 0 \le \tau \le 1\} \\ &\quad \cup \{(x + s - \tau s, y + t) | 0 \le \tau \le 1\} \cup \{(x, y + t - \tau t) | 0 \le \tau \le 1\}, \end{aligned}$$

which is the rectangle starting from and ending at (x, y) :$(x, y) \to (x + s, y) \to (x + s, y + t) \to (x, y + t) \to (x, y)$. With this path, we can define a $U(p)$-valued function on $M_{s,t}$ by

$$W_{s,t}^A(x, y) = W_A(C_{x,y;s,t}), \quad (x, y) \in M_{s,t}.$$

For each s, t, $W_{s,t}^A$ is continuous on $M_{s,t}$. Since the two-dimensional Lebesgue measure $\boldsymbol{R}^2 \setminus M_{s,t}$ is zero, $W_{s,t}^A$ can be regarded as an almost everywhere (a.e.) finite function on $\boldsymbol{R}^2$ (with respect to the Lebesgue measure). Hence the multiplication by the function $W_{s,t}^A$ defines a unique unitary operator on $L^2(\boldsymbol{R}^2;^p)$. We denote this unitary operator by the same symbol $W_{s,t}^A$.

In the rest of this section, we assume the following

Assumption (P) : *Each P_j is essentially self-adjoint.*[3]

We have the following result concerning the commutation relations on the unitary groups generated by $\bar{P}_1$ and $\bar{P}_2$:

Theorem 3.1. *For all $s, t \in \boldsymbol{R}$,*

$$e^{is\bar{P}_1}e^{it\bar{P}_2} = \left(W_{s,t}^A\right)^* e^{it\bar{P}_2}e^{is\bar{P}_1}.$$

The idea of proof of this theorem is to apply the Trotter-Kato product formula [22, p.297, Theorem VIII.31]:

$$e^{is\bar{P}_j} = \mathrm{s} - \lim_{n\to\infty} \left(e^{isp_j/n}e^{sA_j/n}\right)^n,$$

where $\mathrm{s} - \lim$ denotes strong limit and $p_1 = -iD_x, p_2 = -iD_y$. See [5] for the details.

Remarks:

(i) Theorem 3.1 may be regarded as a mathematical expression for the (non-Abelian) Aharonov-Bohm effect.

[3] A class of gauge potentials satisfying this assumption will be given in §5.

(ii) Theorem 3.1 is interesting also from an operator-theoretical point of view. L S and T be self-adjoint operators on a Hilbert space. We say that S and T *strongly commute* if their spectral measures commute. A necessary and sufficient condition for S and T to strongly commute is that for all $a, b \in \boldsymbol{R}$, $e^{iaT}e^{ibS} = e^{ibS}e^{iaT}$ [22, §VIII.5]. As already shown (Proposition 2.1), $\bar{P}_1$ and $\bar{P}_2$ commute on $C_0^2(M; \boldsymbol{C}^p)$, provided that A is flat. Theorem 3.1 shows, however, that, even in such a case, $\bar{P}_1$ and $\bar{P}_2$ do not necessarily strongly commute. The case where $\bar{P}_1$ and $\bar{P}_2$ do not strongly commute corresponds to the Aharonov-Bohm efect.

In the same way as in the proof of Theorem 3.1, we can obtain the following result:

Theorem 3.2. For all $s, t \in \boldsymbol{R}$,

$$e^{isq_j}e^{it\bar{P}_k} = e^{-ist\delta_{jk}}e^{it\bar{P}_k}e^{isq_j}, \quad j, k = 1, 2.$$

Theorems 3.1 and 3.2 imply the following theorem:

Theorem 3.3. The set $\{e^{itq_j}, e^{it\bar{P}_j} | t \in \boldsymbol{R}, j = 1, 2\}$ of unitary operators satisfies the Weyl relations with two degrees of freedom if and only if $W_{s,t}^A = I$ for all $s, t \in \boldsymbol{R}$.

As a corollary of Theorem 3.3, we obtain the following:

Corollary 3.4. Suppose that A is flat. Then the representation $\{q_j, \bar{P}_j\}_{j=1}^2$ of CCR is a Schrödinger 2-system if and only if $W_{s,t}^A = I$ for all $s, t \in \boldsymbol{R}$.

Thus a complete characterization of the representation $\{q_j, \bar{P}_j\}_{j=1}^2$ of CCR is given in terms of the Wilson loops of the rectangles $C_{x,y;s,t}$.

In the Abelian case $p = 1$, we have

$$W_A(C) = e^{i\Phi_A(C)},$$

where $\Phi_A(C) := i\int_C A$ is the magnetic flux passing through the interior of the loop C. The condition $W_{s,t}^A = I, \forall s, t \in \boldsymbol{R}$, is equivalent to that, for each (s, t), $\Phi_A(C_{x,y;s,t}) \in 2\pi\boldsymbol{Z}$ a.e.(x, y), where $\boldsymbol{Z}$ is the set of integers. In this case we say that the magnetic flux is *locally quantized* [3].

As a generalization of this notion to the non-Abelian case, we say that the "gauge flux" is *locally quantized* if $W_{s,t}^A = I$ for all $s, t \in \boldsymbol{R}$.

Remark : Suppose that A is flat. Let $\boldsymbol{r}_0$ be any point in M and $C_{\boldsymbol{r}_0}$ be a loop at $\boldsymbol{r}_0$. Let $[C_{\boldsymbol{r}_0}]$ be the homotopclass of loops at $\boldsymbol{r}_0$ to which $C_{\boldsymbol{r}_0}$ belongs. Then the mapping $[C_{\boldsymbol{r}_0}] \to W_A(C_{\boldsymbol{r}_0})$ gives a p-dimensional unitary representation of the fundamental group of M (cf.[17, 18]).

4 Condition for the local quantization of the gauge flux

Let $C_\varepsilon^{\boldsymbol{r}}(\boldsymbol{a}_n)$ be the circle with center $\boldsymbol{a}_n$, radius $\varepsilon > 0$ and initial point $\boldsymbol{r}$ ($|\boldsymbol{r} - \boldsymbol{a}_n| = \varepsilon$) (the direction is taken to be anticlockwise). We set

$$\delta_0 = \min_{n \neq m\,;\ n,m=1,\cdots,N} |\boldsymbol{a}_n - \boldsymbol{a}_m|$$

Theorem 4.1. *The equality $W^A_{s,t} = I$ holds for all $s, t \in \boldsymbol{R}$ if and only if A is flat and there exists a constant $\delta \in (0, \delta_0)$ such that, for all $\varepsilon < \delta$ and some $\boldsymbol{r}_n$ with $|\boldsymbol{r}_n - \boldsymbol{a}_n| = \varepsilon$, $W_A(C_\varepsilon^{\boldsymbol{r}_n}(\boldsymbol{a}_n)) = I, \quad n = 1, \cdots, N,.$*

This theorem can be proven by employing some results in the theory of product integrals [14]. See [5, §3].

By Theorem 4.1 and Corollary 3.4, we obtain the following result:

Theorem 4.2. *Suppose that A isf flat and Assumption (P) is satisfied. Then the representation $\{q_j, \bar{P}_j\}_{j=1}^2$ of CCR is a Schrödinger 2-system if and only if there exists a constant $\delta \in (0, \delta_0)$ such that, for all $\varepsilon < \delta$ and some $\boldsymbol{r}_n$ with $|\boldsymbol{r}_n - \boldsymbol{a}_n| = \varepsilon$, $W_A(C_\varepsilon^{\boldsymbol{r}_n}(\boldsymbol{a}_n)) = I, \ n = 1, \cdots, N.$*

5 Essential self-adjointness of the physical momentum operator

Let $S_j = \boldsymbol{R} \setminus \{a_{nj}\}_{n=1}^N$ $(j = 1, 2)$ and

$$\mathcal{D}_1^m = C_0^m(\boldsymbol{R} \times S_2; \boldsymbol{C}^p), \mathcal{D}_2^m = C_0^m(S_1 \times \boldsymbol{R}; \boldsymbol{C}^p), \quad m = 0, 1, 2 \cdots.$$

Definition 5.1. We say that an $M_p^{\mathrm{ah}}(\boldsymbol{C})$-valued 1-form A is in the class $\mathcal{A}_m$ if there exist $U(p)$-valued functions $g_1 \in C^{m+1}(\boldsymbol{R} \times S_2; U(p))$ and $g_2 \in C^{m+1}(S_1 \times \boldsymbol{R}; U(p))$ such that $A_1 = g_1^{-1} D_x g_1, \quad A_2 = g_2^{-1} D_y g_2.$

Theorem 5.2. *Suppose that $A \in \mathcal{A}_{m-1} (m \geq 1)$. Then each P_j is essentially self-adjoint on $\mathcal{D}_j^m$.*

Proof : We have $P_j \psi = g_j^{-1} p_j g_j \psi, \psi \in \mathcal{D}_j^m$ and g_j is a projection on the space $\mathcal{D}_j^m$. Since p_j is essentially self-adjoint on $\mathcal{D}_j^m$, the desired result follows. □

Theorem 5.3. *Suppose that $A_j \in C^m(M; M_p^{\mathrm{ah}}(\boldsymbol{C}))$ $(j = 1, 2)$ $(m \geq 1)$ and $A = A_1 dx + A_2 dy$ is flat on M. Then $A \in \mathcal{A}_m$. In particular, each P_j is essentially self-adjoint on $\mathcal{D}_j^{m+1}$.*

The idea of proof of this theorem is to decompose $\boldsymbol{R} \times S_2$ (resp. $S_1 \times \boldsymbol{R}$) as a union of simply-connected regions and to use a lemma of the Poincaré type [25] on each simply-connected region. See [5, Theorem 4.3].

Combining Theorem 5.3 with Theorem 4.2, we obtain the following result:

Theorem 5.4. *Suppose that $A_j \in C^m(M; M_p^{\mathrm{ah}}(\boldsymbol{C}))$ $(j = 1, 2)$ $(m \geq 1)$ and $A = A_1 dx + A_2 dy$ is flat on M. Then each P_j is essentially self-adjoint on $\mathcal{D}_j^{m+1}$. Moreover, the representation $\{q_j, \bar{P}_j\}_{j=1}^2$ of CCR is a Schrödinger system if and only if there exists a constant $\delta \in (0, \delta_0)$ such that, for all $\varepsilon < \delta$ and some $\boldsymbol{r}_n$ with $|\boldsymbol{r}_n - \boldsymbol{a}_n| = \varepsilon$, $W_A(C_\varepsilon^{\boldsymbol{r}_n}(\boldsymbol{a}_n)) = I, \quad n = 1, \cdots, N.$*

6 A characterization for a class of flat gauge potentials

Let A be a flat gauge potential on M. We fix a point $\boldsymbol{r}_0$ in M and denote by $C_{\boldsymbol{r}_0,n}$ a loop at $\boldsymbol{r}_0$ going around $\boldsymbol{a}_n$ in such a way that the intersection of the interior domain of $C_{\boldsymbol{r}_0,n}$

and the set $\{\boldsymbol{a}_n\}_{n=1}^N$ is $\{\boldsymbol{a}_n\}$. Then, for each $n = 1, \cdots, N$, the unitary operator

$$U_n := W_A(C_{\boldsymbol{r}_{0,n}})$$

depends only on the homotopy class of $C_{\boldsymbol{r}_{0,n}}$. We introduce a class of flat gauge potentials on M:

Definition 6.1. *Let A be a gauge potential on M. We say that A is in the set $\mathcal{L}_0$ if A is flat and $[U_n, U_m] = 0, m, n = 1, \cdots, N$.*

For $p \times p$ Hermitian matrices $T_n, n = 1, \cdots, N$, we define $M_p^{\mathrm{ah}}(\boldsymbol{C})$-valued functions $B_j(\cdot; T_1, \cdots, T_n), j = 1, 2$, by

$$B_1(\boldsymbol{r}; T_1, \cdots, T_n) = i \sum_{n=1}^N \frac{(y - a_{n2})}{|\boldsymbol{r} - \boldsymbol{a}_n|^2} T_n, \quad B_2(\boldsymbol{r}; T_1, \cdots, T_n) = -i \sum_{n=1}^N \frac{(x - a_{n1})}{|\boldsymbol{r} - \boldsymbol{a}_n|^2} T_n,$$

and set

$$B(\boldsymbol{r}; T_1, \cdots, T_n) = B_1(\boldsymbol{r}; T_1, \cdots, T_n) dx + B_2(\boldsymbol{r}; T_1, \cdots, T_n) dy. \tag{6.1}$$

Note that, if $[T_n, T_m] = 0, m, n = 1, \cdots, N$, then $B(\boldsymbol{r}; T_1, \cdots, T_n)$ is a flat gauge potential. The following theorem characterizes the class $\mathcal{L}_0$:

Theorem 6.2.[4] *A gauge potential A is in the class $\mathcal{L}_0$ if and only if there exist a family $\{T_n\}_{n=1}^N$ of commuting $p \times p$ Hermitian matrices and a $U(p)$-valued, twice continuously differentiable function g on M such that $g(\boldsymbol{r}_0) = I$ and*

$$A = gB(\cdot; T_1, \cdots, T_n) g^{-1} - (dg) g^{-1}. \tag{6.2}$$

Proof: We give only an outline of proof. Let $C_{\boldsymbol{r}_0}$ be a loop at $\boldsymbol{r}_0$. Then we can show that, if T_n's commute each other, then

$$W_{B(\cdot; T_1, \cdots, T_n)}(C_{\boldsymbol{r}_0}) = W_{-B}(C_{\boldsymbol{r}_0})^{-1} = e^{2\pi i \sum_{\boldsymbol{a}_n \in D_{\boldsymbol{r}_0}} k_n T_n}, \tag{6.3}$$

where $D_{\boldsymbol{r}_0}$ is the interior domain of the loop $C_{\boldsymbol{r}_0}$ and k_n is the rotation number of $C_{\boldsymbol{r}_0}$ with respect to $\boldsymbol{a}_n$.
Necessity : Let $A \in \mathcal{L}_0$. Then there exists a family $\{T_n\}_{n=1}^N$ of commuting $p \times p$ Hermitian matrices such that

$$U_n = e^{2\pi i k_n T_n}, n = 1, \cdots, N. \tag{6.4}$$

With these T_n's, we define a 1-form $B := B(\cdot; T_1, \cdots, T_n)$ by (6.1). Then, by (6.3), we have

$$W_A(C_{\boldsymbol{r}_0}) = W_B(C_{\boldsymbol{r}_0}).$$

We denote by $C_{\boldsymbol{r}_0}^{\boldsymbol{r}}$ a path from $\boldsymbol{r}_0$ to $\boldsymbol{r}$ in M. The commutativity of T_n's and U_n's ensures that

$$\widetilde{B}(\boldsymbol{r}) := W_A(C_{\boldsymbol{r}_0}^{\boldsymbol{r}}) B(\boldsymbol{r}) W_A(C_{\boldsymbol{r}_0}^{\boldsymbol{r}})^{-1}$$

depends only on $\boldsymbol{r}$. We then introduce

$$\widetilde{A}_j = A_j - \widetilde{B}_j, \quad j = 1, 2.$$

[4]This result has been obtained through joint work with H. Kurose.

By using a basic theorem in product integration [14, p.21, Theorem 3.2], we can show that

$$W_{\widetilde{A}}(C^{\boldsymbol{r}}_{\boldsymbol{r}_0}) = W_A(C^{\boldsymbol{r}}_{\boldsymbol{r}_0})W_{-B}(C^{\boldsymbol{r}}_{\boldsymbol{r}_0}),$$

which implies that the function

$$g(\boldsymbol{r}) := W_{\widetilde{A}}(C^{\boldsymbol{r}}_{\boldsymbol{r}_0})$$

depends only on $\boldsymbol{r}$. With this g we can see that $g(\boldsymbol{r}_0) = I$ and (6.2) holds.

Sufficiency: The flatness of A follows from a direct computation and the flatness of $B(\cdot; T_1, \cdots, T_n)$. By a theorem on product integration [14, p.21, Theorem 3.2], we have

$$W_A(C^{\boldsymbol{r}}_{\boldsymbol{r}_0}) = g(\boldsymbol{r})W_B(C^{\boldsymbol{r}}_{\boldsymbol{r}_0}),$$

which, together with condition $g(\boldsymbol{r}_0) = I$, implies (6.4). Hence $A \in \mathcal{L}_0$. □

For $A \in \mathcal{L}_0$, $\{q_j, \bar{P}_j\}_{j=1}^2$ gives a representation of CCR (Theorem 5.3). On this representation we have the following result:

Theorem 6.3. *Let $A \in \mathcal{L}_0$ and $T_n, n = 1, \cdots, N$, be as in Theorem 6.2. Then $\{q_j, \bar{P}_j\}_{j=1}^2$ is a Schrödinger 2-system if and only if, for each $n = 1, \cdots, N$, all the eigenvalues of T_n are integers.*

Proof: We have

$$W_A(\varepsilon^{\boldsymbol{r}}(\boldsymbol{a}_n)) = W_A(C^{\boldsymbol{r}}_{\boldsymbol{r}_0})e^{2\pi i T_n}W_A(C^{\boldsymbol{r}}_{\boldsymbol{r}_0})^{-1},$$

which, together with Theorem 5.4, gives the desired. □

Remark : (i) Let $\{T_n\}_{n=1}^N$ be a family of commuting $p \times p$ Hermitian matrices and define A by (6.2) with a function $g \in C^2(M; U(p))$. Then A is flat. If g has no singularity at $\boldsymbol{a}_n, n = 1, \cdots, N$, and can be extended to a function in $C^2(\boldsymbol{R}^2; U(p))$, then one can show that A satisfies the following distribution equation (cf. (2.1)):

$$D_xA_2 - D_yA_1 + [A_1, A_2] = -2\pi i \sum_{n=1}^{N} g(\boldsymbol{a}_n)T_n g(\boldsymbol{a}_n)^{-1}\delta(\boldsymbol{r} - \boldsymbol{a}_n). \tag{6.5}$$

If g has singularity at some $\boldsymbol{a}_n$'s, then derivatives of the delta functions $\delta(\boldsymbol{r} - \boldsymbol{a}_n)$ may appear on the right–hand side of (6.5), depending on the form of singularity of g. See [4, §II] for the Abelian case.

(ii) The class $\mathcal{L}_0$ is a special class of flat gauge potentials. It is an open problem to give a complete (explicit) characterization of general flat gauge potentials.

7 Dirac-Weyl operator

In what follows, the domain $D(S + T)$ of the sum $S + T$ of two linear operators S and T on a Hilbert space is always taken to be $D(S) \cap D(T)$, unless otherwise stated.

Let $\sigma_j, j = 1, 2, 3$, be the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let A be an $M_p^{\mathrm{ah}}(\boldsymbol{C})$-valued, continuously differentiable 1-form on M (not necessarily flat). Then the Dirac-Weyl operator with this gauge potential is given by

$$\not{D} = \sigma_1 \otimes \bar{P}_1 + \sigma_2 \otimes \bar{P}_2$$

acting in the Hilbert space

$$\boldsymbol{C}^2 \otimes L^2(\boldsymbol{R}^2; \boldsymbol{C}^p) = L^2(\boldsymbol{R}^2; \boldsymbol{C}^p) \bigoplus L^2(\boldsymbol{R}^2; \boldsymbol{C}^p) = \left\{ \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \Big| \, \psi_j \in L^2(\boldsymbol{R}^2; \boldsymbol{C}^p), j = 1, 2 \right\}.$$

It is easy to see that $\not{D}$ is a symmetric operator. One of the important problems concerning the operator $\not{D}$ is to prove its (essential) self-adjointness or to construct self-adjoint extensions of it. This problem is not so trivial, because the gauge potential $A(\boldsymbol{r})$ can be strongly singular at $\boldsymbol{r} = \boldsymbol{a}_n, n = 1, \cdots, N$. We first present some results on this aspect.

7.1 Self-adjoint extensions of the minimal Dirac operator

Let

$$\not{D}_{\min} = \not{D} | C_0^\infty(M; \boldsymbol{C}^p),$$

the restriction of $\not{D}$ to $C_0^\infty(M; \boldsymbol{C}^p)$. We call it the *minimal Dirac operator*. By the reason mentioned above, one can not expect that $\not{D}_{\min}$ is essentially self-adjoint. To construct self-adjoint extensions of $\not{D}_{\min}$, we take a method used in [7].

We can write

$$\not{D}_{\min} = \left(\begin{array}{cc} 0 & D_- \\ D_+ & 0 \end{array} \right),$$

where

$$D_\pm = P_1 \pm i P_2, \quad D(D_\pm) = C_0^\infty(M; \boldsymbol{C}^p).$$

It is easy to see that

$$D_+ \subset D_-^*, \quad D_- \subset D_+^*. \tag{7.1}$$

In particular, $D_\pm$ are closable.

Theorem 7.1. *The following operators $\not{D}_j, j = 1, 2$, are self-adjoint extensions of $\not{D}_{\min}$:*

$$\not{D}_1 = \left(\begin{array}{cc} 0 & D_+^* \\ \bar{D}_+ & 0 \end{array} \right), \quad \not{D}_2 = \left(\begin{array}{cc} 0 & \bar{D}_- \\ D_-^* & 0 \end{array} \right),$$

where $\bar{D}_\pm$ denote the closures of $D_\pm$, respectively.

In the case where the gauge flux is locally quantized, we can prove the following theorem:

Theorem 7.2. *Suppose that the gauge flux is locally quantized. Then $\not{D}$ is a self-adjoint extension of $\not{D}_{\min}$. Moreover,*

$$\not{D}^2 = \bar{P}_1^2 + \bar{P}_2^2. \tag{7.2}$$

Proof: In the present case, A is flat (Theorem 4.1). Hence each P_j is essentially self-adjoint on $C_0^\infty(M; \boldsymbol{C}^p)$ (Theorem 5.3). Moreover, $\bar{P}_1$ and $\bar{P}_2$ strongly commute (Theorem 3.3). Hence, applying [8, Theorem 3.4], we conclude that $\sigma_1 \otimes \bar{P}_1$ and $\sigma_2 \otimes \bar{P}_2$ strongly anticommute[5]. Hence $\not{D}$ is self-adjoint and (7.2) holds [29]. It follows from (7.1) that $\not{D}$ is an extension of $\not{D}_{\min}$. □

Remark : The part of self-adjointness of $\not{D}$ in Theorem 7.2 can also be proven by employing the fact that, under the present condition, the representation $\{q_j, \bar{P}_j\}_{j=1}^2$ of CCR is a Schrödinger 2-system (Theorem 3.3).

7.2 Zero-energy states

The zero-energy states of the Dirac operators $\not{D}_j, j = 1, 2$, are particularly interesting. We first note the following fact:

Theorem 7.3. *Suppose that the gauge flux is locally quantized. Then* $\ker \not{D} = \{0\}$.

Proof: Similar to the Abelian case [4, Theorem 4.2]. □

Remark: In the Abelian case, Theorem 7.3 implies that the Aharonov-Casher theorem [2], which relates the number of the zero-energy states of the Dirac-Weyl operator with a *regular* gauge potential to the total magnetic flux, does not hold in the present singular case.

In the case where the gauge flux is not necessarily quantized, we proceed as follows.

Lemma 7.4.

$$\ker \bar{D}_\pm = \{0\}.$$

Proof: Similar to the proof of [4, Lemma 4.3].

It follows from Lemma 7.4 that

$$\ker \not{D}_1 = \{0\} \bigoplus \ker D_+^*, \quad \ker \not{D}_2 = \ker D_-^* \bigoplus \{0\}.$$

Thus we need only to identify $\ker D_\pm^*$. As usual, we denote by $z = x + iy$ the complex number corresponding to the point $\boldsymbol{r} = (x, y)$. We set

$$a_n = a_{n1} + ia_{n2}, \quad n = 1, \cdots, N.$$

Theorem 7.5. *Let A be of the form (6.2) with g as a $U(p)$-valued, twice continuously differentiable function on M and $\{T_n\}_{n=1}^N$ be a family of commuting $p \times p$ Hermitian matrices. Let*

$$\Omega_{f,g;T_1,\cdots,T_n}(\boldsymbol{r}) = g(\boldsymbol{r}) e^{\sum_{n=1}^N T_n \log|\boldsymbol{r}-\boldsymbol{a}_n|} f(\boldsymbol{r}),$$

where f is a $\boldsymbol{C}^p$-valued function on M. Then

$$\begin{aligned}
\ker D_+^* &= \{\Omega_{f,g;T_1,\cdots,T_n} | f^* \textit{ is a meromorphic function on } \boldsymbol{C} \setminus \{a_n\}_{n=1}^N \\
&\qquad \textit{with a polynomial order at infinity, } e^{\sum_{n=1}^N T_n \log|\boldsymbol{r}-\boldsymbol{a}_n|} f \in L^2(\boldsymbol{R}^2; \boldsymbol{C}^p)\} \\
\ker D_-^* &= \{\Omega_{f,g;-T_1,\cdots,-T_n} | f \textit{ is a meromorphic function on } \boldsymbol{C} \setminus \{a_n\}_{n=1}^N \\
&\qquad \textit{with a polynomial order at infinity, } e^{-\sum_{n=1}^N T_n \log|\boldsymbol{r}-\boldsymbol{a}_n|} f \in L^2(\boldsymbol{R}^2; \boldsymbol{C}^p)\}
\end{aligned}$$

[5] Two self-adjoint operators S and T on a Hilbert space are said to *strongly anticommute* if, for all $t \in \mathbf{R}$, $e^{itS} T \subset T e^{-itS}$. See [8, 9, 10, 20, 24, 29] for the general theory of strongly anticommuting self-adjoint operators and [11] for applications to Dirac-type operators.

In particular, if (i) $N = 1$ or (ii) $N \geq 2$ and the gauge flux is locally quantized, then $\ker D_{\pm}^{*} = \{0\}$.

This theorem is a generalization of [4, Theorem 7] in an Abelian case. The method of proof is similar to that of the cited theorem. Theorem 7.5 shows that, in the case where $N \geq 2$ and the gauge flux is not locally quantized, the zero-energy state of the Dirac operators $\ker\, \not{D}_j$ may be degenerate. See [4] for a detailed anaysis on the degenerate zero-energy states in the Abelian case.

7.3 Supersymmetric structure

It is easy to see that $\Gamma := \sigma_3 \otimes I$ leaves $D(\not{D}_j)$ invariant and

$$\Gamma\, \not{D}_j + \not{D}_j \Gamma = 0 \quad \text{on } D(\not{D}_j).$$

Let

$$H_j = \not{D}_j^2, \quad j = 1, 2.$$

Then each quadruple $\{\boldsymbol{C}^2 \otimes L^2(\boldsymbol{R}^2; \boldsymbol{C}^p), H_j, \not{D}_j, \Gamma\}$ is a model of supersymmetric quantum mechanics ([12], [28, Chapt.5]). By Theorem 7.5, the supersymmetry breaking in these models depends on whether the gauge flux is quantized or not and hence has an interrelation to the Aharonov-Bohm effect.

7.4 Strong coupling limit of the zero-energy-state density

In this subsection, we restrict our attention to the Abelian case $p = 1$. For a constant $q > 0$, we define $\not{D}_j(q)$ (resp. $\not{D}(q)$) to be the operator $\not{D}_j$ (resp. $\not{D}$) with A replaced by qA. The zero-energy-state density (ZESD) of $\not{D}_j(q)$ is defined by

$$\varrho_q^{(j)}(\boldsymbol{r}) = \sum_{k=1}^{\dim\ker\, \not{D}_j(q)} ||\psi_k^{(j)}(\boldsymbol{r})||^2_{\boldsymbol{C}^2}, \tag{7.3}$$

where $\{\psi_k^{(j)}\}_{k=1}^{\dim\ker\, \not{D}_j(q)}$ is a complete orthonormal system of $\ker\, \not{D}_j(q)$. The right–hand side of (7.3) is independent of the choice of complete orthonormal systems of $\ker\, \not{D}_j(q)$.

As for the ZESD $P_q(\boldsymbol{r})$ of a self-adjoint extension of $\not{D}(q)|C_0^\infty(\boldsymbol{R}^2; \boldsymbol{C}^2)$ with a *regular* magnetic field

$$B := D_x A_2 - D_y A_1$$

on $\boldsymbol{R}^2$, the following result is known [15]: for any sequence $\{q_n\}_{n=1}^\infty$ with $q_n \to \infty$ $(n \to \infty)$,

$$\lim_{q_n \to \infty} \frac{P_{q_n}(\boldsymbol{r})}{q_n} = \frac{1}{2\pi} B(\boldsymbol{r}), \ \text{a.e.} \tag{7.4}$$

This result, which means that the magnetic field is recovered as a strong coupling limit of the ZESD, may be regarded as a local form of the Aharonov-Casher result on the degeneracy of zero-energy states [2]. As already remarked, in the case of singular gauge potentials considered in the present paper, the Aharonov-Casher result does not hold. Hence, in such a singular case, we can not expect that (7.4) holds. But it is interesting to see how the ZESD behaves in the strong coupling limit $q \to \infty$ in that case.

In the Abelian case, (6.2) is the general form of flat gauge potentials on M. In this case, we can write $g(\boldsymbol{r}) = e^{i\phi(\boldsymbol{r})}$ with $\phi(\boldsymbol{r})$ a real-valued function on M and the quantity $2\pi T_n$ physically means the magnetic flux at $\boldsymbol{a}_n$.

In what follows, we consider only the simplest (but nontrivial) case $N = 2$ and present results only for the strong coupling limit of $\varrho_q^{(2)}(\boldsymbol{r})$ (the other case $\varrho_q^{(2)}$ can be treated similarly). We assume that

$$T_\nu > 0, \quad \nu = 1, 2,$$

and set

$$\varepsilon_\nu(q) = qT_\nu - [qT_\nu], \quad \nu = 1, 2,$$

where $[x]$ denotes the largest integer less than or equal to x. It follows that $0 \leq \varepsilon_\nu(q) < 1$. Let

$$\Omega_q(\boldsymbol{r}) = e^{iq\phi(\boldsymbol{r})} \left(\prod_{\nu=1}^{2} |\boldsymbol{r} - \boldsymbol{a}_\nu|^{-qT_\nu} (z - a_\nu)^{[qT_\nu]} \right).$$

Lemma 7.6. $\dim \ker \widetilde{\not{D}}_2(q) = 1$ *if and only if*

$$\varepsilon_1(q) + \varepsilon_2(q) > 1. \tag{7.5}$$

In that case, the zero-energy state of $\widetilde{\not{D}}_2(q)$ *is given by* $\begin{pmatrix} \Omega_q(\boldsymbol{r}) \\ 0 \end{pmatrix}$ *(up to constant multiples).*

Proof: This is a special case of Theorem 7.5 (the case $N = 2$). $\square$

Remark: Under condition (7.5), qT_1 and qT_2 are not integers, i.e, the magnetic flux is not locally quantized.

Henceforth we consider only the case where (7.5) is satisfied. Under this condition, we have

$$\varrho_q(\boldsymbol{r}) := \varrho_q^{(2)}(\boldsymbol{r}) = \frac{|\Omega_q(\boldsymbol{r})|^2}{||\Omega_q||_{L^2}^2} = \frac{|\boldsymbol{r} - \boldsymbol{a}_1|^{-2\varepsilon_1(q)} |\boldsymbol{r} - \boldsymbol{a}_2|^{-2\varepsilon_2(q)}}{\int_{\boldsymbol{R}^2} |\boldsymbol{r} - \boldsymbol{a}_1|^{-2\varepsilon_1(q)} |\boldsymbol{r} - \boldsymbol{a}_2|^{-2\varepsilon_2(q)} d\boldsymbol{r}}. \tag{7.6}$$

In the present case, the magnetic field is not a function, but a distribution. Hence it is natural to take the strong coupling limit of $\varrho_{\boldsymbol{r}}$ in the distribution sense. For this purpose, for each $\mu, \lambda \in (0, 1)$ satisfying $\mu + \lambda > 1$, we define a functional $\Phi_{\mu,\lambda}$ on $L^\infty(\boldsymbol{R}^2)$ (the Banach space of essentially bounded functions on $\boldsymbol{R}^2$) by

$$\Phi_{\mu,\lambda}(f) = \int_{\boldsymbol{R}^2} \frac{f(\boldsymbol{r})}{|\boldsymbol{r} - \boldsymbol{a}_1|^{2\mu} |\boldsymbol{r} - \boldsymbol{a}_2|^{2\lambda}} d\boldsymbol{r}, \quad f \in L^\infty(\boldsymbol{R}^2).$$

In terms of this functional, the zero-energy-state functional

$$\varrho_q(f) := \int_{\boldsymbol{R}^2} \varrho_q(\boldsymbol{r}) f(\boldsymbol{r}) d\boldsymbol{r}$$

is written as

$$\varrho_q(f) = \frac{\Phi_{\varepsilon_1(q), \varepsilon_2(q)}(f)}{\Phi_{\varepsilon_1(q), \varepsilon_2(q)}(1)}.$$

As (7.6) shows, the q-dependence of ϱ_q comes only from the factors $\varepsilon_\nu(q), \nu = 1, 2$. It is obvious that $\lim_{q\to\infty} \varepsilon_\nu(q)$ does not exist. But, for suitable monotone increasing sequences $\{q_n\}_{n=1}^\infty$ of positive numbers satisfying

$$q_n \to \infty \ (n \to \infty), \quad \varepsilon_1(q_n) + \varepsilon_2(q_n) > 1, \quad \geq 1, \tag{7.7}$$

the limits

$$\lambda_\nu := \lim_{n\to\infty} \varepsilon_\nu(q_n), \quad \nu = 1, 2, \tag{7.8}$$

may exist, depending on the choice of $\{q_n\}_{n=1}^\infty$. For this reason, we discuss the strong coupling limit of the ZESD according to the magnitude of $\lambda_\nu, \nu = 1, 2$.

We denote by $\mathcal{B}(\boldsymbol{R}^2)$ the set of bounded continuous functions on $\boldsymbol{R}^2$. Limiting behaviors of the functional $\Phi_{\mu,\lambda}$ in μ and λ are given in the following lemma.

Lemma 7.7.
(i) Let $\mu_0, \lambda_0 \in (0, 1)$ such that $\mu_0 + \lambda_0 > 1$. Then, for all $f \in L^\infty(\boldsymbol{R}^2)$,

$$\lim_{\mu\to\mu_0, \lambda\to\lambda_0} \Phi_{\mu,\lambda}(f) = \Phi_{\mu_0,\lambda_0}(f).$$

(ii) Let $\lambda_0 \in (0, 1)$. Then, for all $f \in \mathcal{B}(\boldsymbol{R}^2)$,

$$\lim_{\mu\to 1, \lambda\to\lambda_0} (1 - \mu)\Phi_{\mu,\lambda}(f) = \frac{\pi f(\boldsymbol{a}_1)}{|\boldsymbol{a}_1 - \boldsymbol{a}_2|^{2\lambda_0}}.$$

(iii) Let $\tau > 0$. Then, for all $f \in \mathcal{B}(\boldsymbol{R}^2)$

$$\lim_{\mu\to 1} (1 - \mu)\Phi_{\mu,(\tau-1+\mu)/\tau}(f) = \frac{\pi}{|\boldsymbol{a}_1 - \boldsymbol{a}_2|^2}(f(\boldsymbol{a}_1) + \tau f(\boldsymbol{a}_2)).$$

Proof. See [6]. □

A simple application of Lemma 7.7 to $\varrho_q(f)$ gives the following result.

Theorem 7.8. *Let $\{q_n\}_{n=1}^\infty$ be a sequence satisfying (7.7) and (7.8).*
(i) Suppose that $\lambda_1, \lambda_2 \in (0, 1)$ such that $\lambda_1 + \lambda_2 > 1$. Then, for all $f \in L^\infty(\boldsymbol{R}^2)$,

$$\lim_{n\to\infty} \varrho_{q_n}(f) = \frac{\Phi_{\lambda_1,\lambda_2}(f)}{\Phi_{\lambda_1,\lambda_2}(1)}.$$

(ii) Suppose that $\lambda_1 = 1, \lambda_2 \in (0, 1)$. Then, for all $f \in \mathcal{B}(\boldsymbol{R}^2)$,

$$\lim_{n\to\infty} \varrho_{q_n}(f) = f(\boldsymbol{a}_1).$$

(iii) Let $\tau > 0$ and suppose that, for all sufficiently large n, $1 - \varepsilon_1(q_n) = \tau(1 - \varepsilon_2(q_n))$ and $\lambda_1 = 1$ (hence $\lambda_2 = 1$). Then, for all $f \in \mathcal{B}(\boldsymbol{R}^2)$,

$$\lim_{n\to\infty} \varrho_{q_n}(f) = \frac{f(\boldsymbol{a}_1) + \tau f(\boldsymbol{a}_2)}{1 + \tau}.$$

Remark. One can eaily find examples of $\{q_n\}_{n=1}^\infty$ for each case in Theorem 7.8.

As a corollary, we obtain the folloiwng:

Corollary 7.9. *Let $\{q_n\}_{n=1}^{\infty}$ be a sequence satisfying (7.7) and (7.8). Then, for all cases (i)–(iii) in Theorem 7.8,*

$$\lim_{n\to\infty} \frac{\varrho_{q_n}(f)}{q_n} = 0, \quad f \in \mathcal{B}(\boldsymbol{R}^2).$$

This result shows, as expected, that the Erdös' theorem [15] does not hold in the present case.

In a special case of part (iii) of Theorem 7.8, a strong coupling limit of the ZESD recovers the magnetic field :

Corollary 7.10. *Let $\{q_n\}_{n=1}^{\infty}$ be a sequence satisfying (7.7) and (7.8) with $\lambda_\nu = 1, \nu = 1, 2$. Suppose that, for all sufficiently large n, $T_1(1 - \varepsilon_1(q_n)) = T_2(1 - \varepsilon_2(q_n))$. Then,*

$$\lim_{n\to\infty} (T_1 + T_2)\varrho_{q_n}(\boldsymbol{r}) = T_1\delta(\boldsymbol{r} - \boldsymbol{a}_1) + T_2\delta(\boldsymbol{r} - \boldsymbol{a}_2) \tag{7.9}$$

in the distribution sense. In particular, if ϕ is a twice continuously differentiable function on $\boldsymbol{R}^2$, then

$$\lim_{n\to\infty} (T_1 + T_2)\varrho_{q_n}(\boldsymbol{r}) = \frac{1}{2\pi} B(\boldsymbol{r}) \tag{7.10}$$

in the distribution sense.

Proof. Formula (7.9) follows from a simple application of part (iii) of Theorem 7.8 with $\tau = T_2/T_1$. If ϕ satisfies the assumption, then

$$B = 2\pi[T_1\delta(\boldsymbol{r} - \boldsymbol{a}_1) + T_2\delta(\boldsymbol{r} - \boldsymbol{a}_2)].$$

Hence (7.10) follows. □

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References

[1] Aharonov Y. and Bohm D., Significance of electromagnetic potentials in the quantum theory, *Phys.Rev.*, 1959, V.115, 485–491.

[2] Aharonov Y. and Casher A, Ground state of a spin-$\frac{1}{2}$ charged particle in a two-dimensional magnetic field, *Phys.Rev.*, 1979, V.19A, 2461–2462.

[3] Arai A., Momentum operators with gauge potentials, local quantization of magnetic flux, and representation canonical commutation relations, *J. Math. Phys.*, 1992, V.33, 3374–3378.

[4] Arai A., Properties of the Dirac-Weyl operator with a strongly singular gauge potential, *J. Math. Phys.*, 1993, V.34, 915–935.

[5] Arai A., Strong coupling limit of the zero-energy-state density of the Dirac-Weyl operator with a singular vector potential, Hokkaido University Preprint Series in Math. No.286, 1995.

[6] Arai A., Gauge theory on a non-simply connected domain and representations of canonical commutation relations, *J. Math. Phys.*, 1995, V.36, 2569–2580.

[7] Arai A. and Ogurisu O., Meromorphic $N = 2$ Wess-Zumino supersymmetric quantum mechanics, *J. Math. Phys.*, 1991, V.32, 2427–2434.

[8] Arai A., Characterization of anticommutativity of self-adjoint operators in connection with Clifford algebra and applications, *Integr. Equat. Oper. Th.*, 1993, V.17, 451–463.

[9] Arai A., Commutation properties of anticommuting self-adjoint operators, spin representation and Dirac operators, *Integr. Equat. Oper. Th.*, 1993, V.16, 38–63.

[10] Arai A., Analysis on anticommuting self–adjoint operators, *Adv. Stud. Pure Math.*, 1994, V.23, 1–15.

[11] Arai A., Scaling limit of anticommuting self-adjoint operators and applications to Dirac operators, *Integr. Equat. Oper. Th.*, 1995, V.21, 139–173.

[12] Arai A., Some remarks on scattering theory in supersymmetric quantum mechanics, *J. Math. Phys.*, 1987, V.28, 472–476.

[13] Dorfmeister G. and Dorfmeister J., Classification of certain pairs of operators (P, Q) satisfying $[P, Q] = -iId$, *J. Funct. Anal.*, 1984, V.57, 301–328.

[14] Dollard J.D. and Friedman C.N., *Product Integration*, Encyclopedia of Mathematics and Its Applications Vol.10: Analysis, Addison-Wesley, Reading, Massachusetts, 1979.

[15] Erdös L., Ground-state density of the Pauli operator in the large field limit, *Lett. Math. Phys.*, 1993, V.29, 219–240.

[16] Fuglede B., On the relation $PQ - QP = -iI$, *Math. Scand.*, 1967, V.20,79–88.

[17] Goldin G.A., Menikoff R. and Sharp D.H., Representations of a local current algebra in nonsimply connected space and the Aharonov-Bohm effect, *J. Math. Phys.*, 1981, V.22, 1664–1668.

[18] Kurose H. and Nakazato H., Geometric construction of $*$-representation of the Weyl algebra with degree 2, preprint, 1994.

[19] von Neumann J., Die Eindeutigkeit der Schrödingerschen Operatoren, *Math. Ann.*, 1931, V.104, 570–578.

[20] Pedersen S., Anticommuting self-adjoint operators, *J. Funct. Anal.*, 1990, V.89, 428–443.

[21] Putnam C. R., *Commutation Properties of Hilbert Space Operators*, Springer, Berlin, 1967.

[22] Reed M. and Simon B., Methods of Modern Mathematical Physics Vol.I, Academic Press, New York, 1972.

[23] Reeh H., A remark concerning canonical commutation relations, *J. Math. Phys.*, 1988, V.29, 1535–1536.

[24] Samoilenko Y.S., Spectral Theory of Families of Self-Adjoint Operators, Kluwer Academic Publishers, Dordrecht, 1991.

[25] Schlesinger L., Weitere Beiträge zum Infinitesimalkalkul der Matrin, *Math. Zeit.*, 1932, V.35, 485–501.

[26] Schmüdgen K., On the Heisenberg commutation relation. I, *J. Funct. Anal.*, 1983, V.50, 8–49.

[27] Schmüdgen K., On the Heisenberg commutation relation. II, *Publ. RIMS, Kyoto Univ.*, 1983, V.19, 601–671.

[28] B. Thaller, The Dirac Equation, Springer, Berlin, Heidelberg, 1992.

[29] Vasilescu F. H., Anticommuting self-adjoint operators, *Rev. Roum. Math. Pures et Appl.*, 1983, V.XXVIII, 77–91.