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Fractional q-Calculus on a time scale

Ferhan M. Atici † *and Paul W. Eloe* ‡

† *Department of Mathematics, Western Kentucky University, Bowling Green, KY 42101-3576, USA*
E-mail: ferhan.atici@wku.edu

‡ *University of Dayton, Department of Mathematics, Dayton, Ohio 45469-2316, USA*
E-mail: Paul.Eloe@notes.udayton.edu

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Abstract

The study of fractional q-calculus in this paper serves as a bridge between the fractional q-calculus in the literature and the fractional q-calculus on a time scale $\mathbb{T}_{t_0} = \{t : t = t_0q^n, n \text{ a nonnegative integer }\} \cup \{0\}$, where $t_0 \in \mathbb{R}$ and $0 < q < 1$. By use of time scale calculus notation, we find the proof of many results more straight forward. We shall develop some properties of fractional q-calculus, we shall develop some properties of a q-Laplace transform, and then we shall employ the q-Laplace transform to solve fractional q-difference equations.

1 Introduction

In this article, we shall study fractional calculus on the specific time scale, $\mathbb{T}_{t_0} = \{t : t = t_0q^n, n \text{ a nonnegative integer }\} \cup \{0\}$, where $t_0 \in \mathbb{R}$ and $0 < q < 1$. In general, a time scale is a closed subset of the reals [8].

The purpose of the article is two-fold. First, we shall develop a q-transform method on $\mathbb{T}$. We shall then define some q-fractional difference equations on $\mathbb{T}$ and apply the q-transform method to obtain solutions. The application of a q-transform method to q-fractional difference equations is new. Second, throughout this article, we shall apply the time scales calculus notation [8]. We shall illustrate calculations using time scales calculus notation with the implication that the time scales calculus notation will make the theory and calculations more transparent. In addition to providing more transparent arguments, we do produce new properties, e.g., a useful power rule, through this application on time scales.

Much is already known about q-calculus. Early developments for q-fractional calculus can be found in the work of Al-Salam and co-authors [3], [4], or [5]. A q-Laplace transform method has been developed by Abdi ([14]) and applied to q-difference equations ([1], [2]). Moreover, there is currently much activity to reexamine and further develop the q-special functions. Notable early work includes the work of Jackson ([15], [16], [17], [18], [19]) and Hahn ([13], [14]). We also refer the reader to more recent articles by De Sole and Kac

[10], McAnally ([22], [23]), Ernst [12], and Koornwinder [20], [21] and books by Andrews, Askey, Verma [6] and Carlson [9] for current and excellent accounts of q-special functions. For further reading on the fractional calculus on a time scale $\mathbb{T} = \mathbb{R}$ the books by Miller and Ross [24] and by Podlubny [25] are excellent sources.

We intend that the article be self-contained; we shall introduce sufficient notation from the calculus on time scales so the reader does not need previous familiarity; an excellent account of the calculus on time scales is found in the monograph of Bohner and Peterson [8].

In Sections 2 and 3, we shall present the introductory definitions. We shall employ both the traditional notation employed in the references cited above and the time scales notation. Definitions will include the q-factorial function, the q-fractional integral, a version of the q-exponential function, and a q-Gamma function. Of note, we shall then obtain an integral representation of the q-Gamma function and obtain a power rule for fractional derivatives. In Section 4, we shall illustrate the Laplace transform method and define and solve several families of linear fractional q-difference equations with constant coefficients. Sections 3 and 4 are modelled after a recent development [7] for the fractional calculus of finite difference equations on $\mathbb{Z}$.

2 q-Gamma and the q-exponential functions

Let $t_0 \in \mathbb{R}$ and define

$$\mathbb{T}_{t_0} = \{t : t = t_0 q^n, n \text{ a nonnegative integer } \} \cup \{0\}, \quad 0 < q < 1.$$

If there is no confusion concerning t_0 we shall denote $\mathbb{T}_{t_0}$ by $\mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{R}$, the nabla q-derivative of f is

$$\nabla_q f(t) = \frac{f(qt) - f(t)}{(q-1)t} \tag{2.1}$$

for all $t \in \mathbb{T} \setminus \{0\}$. The nabla q-integral of f is $\int_0^t f(s)\nabla s = (1-q)t \sum_{i=0}^{\infty} q^i f(tq^i)$. The fundamental theorem of calculus applies to the nabla q-derivative and nabla q-integral; in particular,

$$\nabla_q \int_0^t f(s)\nabla s = f(t),$$

and if f is continuous at 0, then

$$\int_0^t (\nabla_q f(s))\nabla s = f(t) - f(0).$$

Change of variables is valid in times scales integration ([8, 11]). In the following theorem, we state a special case as it applies in the article. In each application throughout this article, $g(t) = at$ for some positive constant a. Thus, $\nabla_q g(t) = a$.

Theorem 1. Let $\mathbb{T}_{t_1}, \mathbb{T}_{t_2}$ denote two time scales. Let $f : \mathbb{T}_{t_1} \to \mathbb{R}$ be continuous, let $g : \mathbb{T}_{t_1} \to \mathbb{T}_{t_2}$ be nabla $q-$differentiable, strictly increasing, and $g(0) = 0$. Then for $b \in \mathbb{T}_{t_1}$,

$$\int_0^b f(t)\nabla_q g(t)\nabla t = \int_0^{g(b)} (f \circ g^{-1})(s)\nabla s.$$

Definition 1. The q-factorial function is defined in the following way. If n is a positive integer, then

$$(t-s)^{\underline{(n)}} = (t-s)(t-qs)(t-q^2 s)...(t-q^{n-1}s).$$

If ν is not a positive integer, then

$$(t-s)^{\underline{(\nu)}} = t^{\nu} \prod_{n=0}^{\infty} \frac{1-\frac{s}{t}q^n}{1-\frac{s}{t}q^{\nu+n}}.$$

We shall state several properties of the q-factorial function; each property is verified using the definition and a straightforward calculation.

Theorem 2. *(i)* $(t-s)^{\underline{(\beta+\gamma)}} = (t-s)^{\underline{(\beta)}}(t-q^{\beta}s)^{\underline{(\gamma)}}$
(ii) $(at-as)^{\underline{(\beta)}} = a^{\beta}(t-s)^{\underline{(\beta)}}$
(iii) The nabla q-derivative of the q-factorial function with respect to t is

$$\nabla_q (t-s)^{\underline{(\nu)}} = \frac{1-q^{\nu}}{1-q}(t-s)^{\underline{(\nu-1)}}$$

(iv) The nabla q-derivative of the q-factorial function with respect to s is

$$\nabla_q (t-s)^{\underline{(\nu)}} = -\frac{1-q^{\nu}}{1-q}(t-qs)^{\underline{(\nu-1)}},$$

where $\beta, \gamma \in \mathbb{R}$.

Definition 2. The q-exponential function is defined as

$$e_q(t) = \prod_{n=0}^{\infty}(1-q^n t), \quad e_q(0) = 1.$$

Note that $e_q(1) = 0$ and

$$\nabla_q(e_q(t)) = \frac{e_q(qt)}{q-1}. \tag{2.2}$$

We are now in a position to give the integral representation of the q-gamma function. Let $\alpha \in \mathbb{R} \setminus \{..., -2, -1, 0\}$. Define the q-Gamma function by

$$\Gamma_q(\alpha) = \frac{1}{1-q}\int_0^1 \left(\frac{t}{1-q}\right)^{\alpha-1} e_q(qt)\nabla t. \tag{2.3}$$

Remark 2.1. Make the change of variable $(1-q)s = t$ in the above definition and apply the change of variable theorem, Theorem 1. It is clear that the definition of the q-Gamma function given by (2.3) is equivalent to the form defined in [20] and employed in [10]:

$$\Gamma_q(\alpha) = \int_0^{\frac{1}{1-q}} s^{\alpha-1} e_q(q(1-q)s)\nabla s.$$

Lemma 1. $\Gamma_q(\alpha+1) = \dfrac{1-q^\alpha}{1-q}\Gamma_q(\alpha); \quad \Gamma_q(1) = 1$, where $\alpha \in \mathbb{R}^+$.

Proof. Integration by parts is a valid tool in time scales calculus [8, page 332], and it follows from the product rule for differentiation:

$$\nabla_q(fg)(t) = f(qt)\nabla_q g(t) + (\nabla_q f(t))g(t).$$

$$\begin{aligned}
\Gamma_q(\alpha+1) &= \frac{1}{1-q}\int_0^1 (\frac{t}{1-q})^\alpha e_q(qt)\nabla t \\
&= \frac{1}{1-q}\int_0^1 \frac{t^\alpha}{(1-q)^{\alpha-1}}[-\frac{e_q(qt)}{q-1}]\nabla t \\
&= \frac{1}{1-q}\int_0^1 \frac{t^\alpha}{(1-q)^{\alpha-1}}\nabla_q e_q(t)\nabla t \\
&= \frac{1}{1-q}\left(-\frac{t^\alpha}{(1-q)^{\alpha-1}}e_q(t)|_0^1 + \frac{1-q^\alpha}{1-q}\int_0^1 \frac{t^{\alpha-1}}{(1-q)^{\alpha-1}}e_q(qt)\nabla t\right) \\
&= \frac{1-q^\alpha}{1-q}\Gamma_q(\alpha).
\end{aligned}$$

$$\Gamma_q(1) = \frac{1}{1-q}\int_0^1 e_q(qt)\nabla t = \frac{1}{1-q}\int_0^1 \nabla_q e_q(t)\nabla t = -e_q(t)|_0^1 = 1.$$

□

For any positive integer k,

$$\Gamma_q(k+1) = \frac{(1-q^k)}{(1-q)}\frac{(1-q^{k-1})}{(1-q)}\dots\frac{(1-q^2)}{(1-q)}\frac{(1-q)}{(1-q)}.$$

This observation allows us to see $k!$ on $q^{\mathbb{N}_0}$

$$[k]! = \frac{(1-q^k)}{(1-q)}\dots\frac{(1-q^2)}{(1-q)}\frac{(1-q)}{(1-q)} = 1(1+q)(1+q+q2)\dots(1+q+q2+\dots+q^{k-1}).$$

The notation $[k]!$ has been used in [4] previously.

Remark 2.2. In [4], Al-Salam defined the q-Gamma function in the following way:

$$\Gamma_q^*(\alpha) = \frac{e_q(q)}{(1-q)^{\alpha-1}e_q(q^\alpha)}, \quad \alpha \neq 0, -1, -2, -3, \dots. \tag{2.4}$$

To see that $\Gamma_q(\alpha) = \Gamma_q^*(\alpha)$, we use the following formula given by Hahn in [13]:

$$\int_0^1 s^{\alpha-1}e_q(qs)\nabla s = (1-q)\prod_{i=0}^{\infty}\frac{1-q^{i+1}}{1-q^{\alpha+i}}.$$

Hence the q-Gamma function satisfies the functional equation in Lemma 1 for any α other than negative integers.

3 Fractional q-integral

The fractional q derivative and the fractional q-integral have been defined in earlier work [3, 4, 5]. We shall employ the following definition of the fractional q- integral:

$$\nabla_q{}^{-\nu}(f(t)) = \frac{1}{\Gamma_q(\nu)} \int_0^t (t-qs)^{\underline{(\nu-1)}} f(s)\nabla s. \tag{3.1}$$

The first elementary property we derive is a power rule. We shall employ the q- beta function

$$B_q(t,s) = \int_0^1 x^{t-1}(1-qx)^{\underline{(s-1)}}\nabla x,$$

recently defined in [10].

Lemma 2.

$$\nabla_q^{-\nu} t^{\mu} = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu+\nu+1)} t^{\mu+\nu}.$$

Proof. Begin with the left hand side of the equality

$$\nabla_q^{-\nu} t^{\mu} = \frac{1}{\Gamma_q(\nu)} \int_0^t (t-qs)^{\underline{(\nu-1)}} s^{\mu}\nabla s.$$

Set $tr = s$ and apply Theorem 1. Then

$$\nabla_q^{-\nu} t^{\mu} = \frac{1}{\Gamma_q(\nu)} \int_0^1 (t-qtr)^{\underline{(\nu-1)}} (tr)^{\mu} t\nabla r.$$

Apply Theorem 2 (ii) and

$$\begin{aligned}\nabla_q^{-\nu} t^{\mu} &= \frac{1}{\Gamma_q(\nu)} t^{\nu+\mu} \int_0^1 r^{\mu}(1-qr)^{\underline{(\nu-1)}}\nabla r \\ &= \frac{1}{\Gamma_q(\nu)} t^{\nu+\mu} B_q(\mu+1,\nu).\end{aligned}$$

Employ the identity $B_q(\mu+1,\nu) = \dfrac{\Gamma_q(\mu+1)\Gamma_q(\nu)}{\Gamma_q(\mu+\nu+1)}$ proved in [10], and the power rule follows.

□

Lemma 3. If $f(t)$ is defined and finite, then for $0 < \nu < 1$

$$\nabla_q^{\nu} f(t) = \nabla_q \nabla_q^{-(1-\nu)} f(t),$$

where $t \in [q^a, q^b] \subset \mathbb{T}$ with $a, b \in \mathbb{N}_0$, $b < a$.

Proof. The following equality [8, page 333] is crucial for the proof:

$$\nabla_q[\int_a^t f(t,s)\nabla s] = \int_a^t \nabla_q(f(t,s))\nabla s + f(qt,t).$$

Using this identity and Remark 2.2 and since $(t-qs)^{\underline{(-\nu-1)}}$ is continuous as a function of t on $[q^a, q^b]$, we have

$$\begin{aligned}\nabla_q\nabla_q^{-(1-\nu)}(f(t)) &= \nabla_q[\frac{1}{\Gamma_q(1-\nu)}\int_0^t (t-qs)^{\underline{(-\nu)}} f(s)\nabla s] \\ &= \frac{1}{\Gamma_q(1-\nu)}\int_0^t \frac{1-q^{-\nu}}{1-q}(t-qs)^{\underline{(-\nu-1)}} f(s)\nabla s + (qt-qt)^{\underline{(-\nu)}} f(t) \\ &= \frac{1}{\Gamma_q(-\nu)}\int_0^t (t-qs)^{\underline{(-\nu-1)}} f(s)\nabla s = \nabla_q^{\nu} f(t).\end{aligned}$$

Remark 3.1. If $\mu > 0$ and $m-1 < \mu < m$ for a positive integer m, we extend the idea of the proof of Lemma 3 and write

$$\nabla_q{}^{\mu} f(t) = \nabla_q{}^{m-(m-\mu)} f(t) = \nabla_q{}^{m}(\nabla_q{}^{-(m-\mu)} f(t))$$

on $[q^a, q^b]$ and treat the operator $\nabla_q{}^m$ as an iterated higher order q-difference as defined by (2.1).

4 The q-Laplace transform

We shall employ the q-Laplace transform defined by W. Hahn in 1949 [14].

$$L_q\{f(t)\}(s) = \frac{1}{1-q}\int_0^{\frac{1}{s}} f(t)e_q(qst)\nabla t$$

Lemma 4. For any $\alpha \in \mathbb{R} \setminus \{..., -2, -1, 0\}$,

$$L_q\{\frac{t^{\alpha-1}}{(1-q)^{\alpha-1}}\}(s) = \frac{\Gamma_q(\alpha)}{s^{\alpha}}.$$

Proof. We again illustrate the integration methods of time scale calculus.

$$L_q\{\frac{t^{\alpha-1}}{(1-q)^{\alpha-1}}\}(s) = \frac{1}{1-q}\int_0^{\frac{1}{s}} \frac{t^{\alpha-1}}{(1-q)^{\alpha-1}} e_q(qst)\nabla t.$$

Set $r = st$.

$$L_q\{\frac{t^{\alpha-1}}{(1-q)^{\alpha-1}}\}(s) = \frac{1}{1-q}\int_0^{1} \frac{r^{\alpha-1}}{(s(1-q))^{\alpha-1}} e_q(qr)\frac{\nabla r}{s} = \frac{\Gamma_q(\alpha)}{s^{\alpha}}.$$

□

We now turn our attention to a shift theorem for the q-Laplace transform. First note the following identity.

$$\nabla_q(e_q^{-1}(t)) = \frac{1}{1-q} e_q^{-1}(t).$$

Lemma 5. Let a be any real number. Then

$$L_q\{e_q^{-1}(at)\}(s) = \frac{1}{s-a}.$$

Let n denote a positive integer. Then

$$L_q\{t^n e_q^{-1}(at)\}(s) = \frac{1}{s-a}\prod_{j=1}^{n}\frac{1-q^j}{s-q^j a} = \frac{(1-q)^n[n]!}{(s-a)^{\underline{(n+1)}}}. \tag{4.1}$$

Proof. Note that

$$\nabla_q(e_q^{-1}(at)e_q(st)) = \frac{s-a}{q-1}e_q^{-1}(at)e_q(qst).$$

So for $n=0$,

$$L_q\{e_q^{-1}(at)\}(s) = \frac{1}{s-a}.$$

The proof proceeds by induction. Let $n \geq 1$ be an integer. Note that

$$\nabla_q(t^n e_q^{-1}(at)e_q(st)) = t^n e_q^{-1}(at)\nabla_q e_q(st) + \nabla_q(t^n e_q^{-1}(at))e_q(qst)$$

$$= \frac{s-q^n a}{q-1}t^n e_q^{-1}(at)e_q(qst) + \frac{q^n-1}{q-1}t^{n-1}e_q^{-1}(at)e_q(qst).$$

Integrate this identity from 0 to $\frac{1}{s}$. The left hand side vanishes and the right hand side yields

$$L_q\{t^n e_q^{-1}(at)\}(s) = \frac{1-q^n}{s-q^n a}L_q\{t^{n-1}e_q^{-1}(at)\}(s).$$

□

Remarkably, Hahn [14] obtained a convolution theorem for specific classes of functions. In particular, set $F_1(t) = t^\mu$ and set $F_2(t) = t^{\nu-1}$. Define $F_2[t] = (t-qrt)^{\underline{(\nu-1)}}$. Define the convolution

$$(F_1 * F_2)(t) = \frac{t}{1-q}\int_0^1 F_2[t]F_1(rt)\nabla r = \frac{t}{1-q}\int_0^1 (t-qrt)^{\underline{(\nu-1)}}F_1(rt)\nabla r,$$

where $\nu \in \mathbb{R}\setminus\{..., -2, -1, 0\}$.

We can obtain Hahn's [14] result directly as an application of the power rule and Lemma 4. Note that with the change of variable $s = rt$,

$$(F_1 * F_2)(t) = \frac{t}{1-q}\int_0^1 (t-qrt)^{\underline{(\nu-1)}}F_1(rt)\nabla r$$

$$= \frac{\Gamma_q(\nu)}{1-q}\frac{1}{\Gamma_q(\nu)}\int_0^t (t-qs)^{\underline{(\nu-1)}}F_1(s)\nabla s = \frac{\Gamma_q(\nu)}{1-q}\nabla_q{}^{-\nu}(F_1(t)).$$

Hence, we calculate $F_1 * F_2$ for $F_1(t) = t^\mu$ and $F_2(t) = t^{\nu-1}$. By the power rule,

$$(F_1 * F_2)(t) = \frac{\Gamma_q(\nu)\Gamma_q(\mu+1)}{(1-q)\Gamma_q(\mu+\nu+1)}t^{\mu+\nu}.$$

Now simply apply Lemma 4 to each of $F_1, F_2, F_1 * F_2$ and obtain a conolution theorem,

$$L_q\{(F_1 * F_2)(t)\}(s) = L_q\{F_1(t)\}(s)L_q\{F_2(t)\}(s).$$

Note then, as Hahn [14] noted, that the convolution theorem will be valid for functions F_1 representing linear sums of functions of the form t^{μ}. Clearly, μ is not necessarily an integer. We do not state a general theorem, but we do state a corollary which will be applied in Example 3.

Corollary 1. Let F_1 be an analytic function and let $F_2(t) = t^{\nu-1}$ on $\mathbb{T} \setminus \{0\}$. Then

$$L_q\{(F_1 * F_2)(t)\}(s) = L_q\{F_1(t)\}(s) L_q\{F_2(t)\}(s). \tag{4.2}$$

We now obtain some of the standard properties for the L_q-transform.

Lemma 6. Assume $f = F_1$ is of the type such that (4.2) is valid. Then

$$L_q\{\nabla_q^{-\nu} f(t)\}(s) = \frac{(1-q)^{\nu}}{s^{\nu}} L_q\{f(t)\}(s).$$

Proof. First note that

$$\begin{aligned}
\nabla_q{}^{-\nu} f(t) &= \frac{1}{\Gamma_q(\nu)} \int_0^t (t-qs)^{\underline{(\nu-1)}} f(s) \nabla s \\
&= \frac{1}{\Gamma_q(\nu)} \int_0^1 (t-qrt)^{\underline{(\nu-1)}} f(rt) t \nabla r \\
&= \frac{t}{\Gamma_q(\nu)} \int_0^1 (t-qrt)^{\underline{(\nu-1)}} f(rt) \nabla r \\
&= \frac{1-q}{\Gamma_q(\nu)} (f * F_2)(t),
\end{aligned}$$

where $F_2(t) = t^{\nu-1}$. Thus,

$$\begin{aligned}
L_q\{\nabla_q{}^{-\nu} f(t)\}(s) &= \frac{(1-q)}{\Gamma_q(\nu)} L_q\{(f * F_2)(t)\}(s) \\
&= \frac{(1-q)}{\Gamma_q(\nu)} L_q\{f(t)\}(s) L_q\{F_2(t)\}(s) \\
&= \frac{(1-q)}{\Gamma_q(\nu)} L_q\{f(t)\}(s) (1-q)^{\nu-1} \frac{\Gamma_q(\nu)}{s^{\nu}} \\
&= \frac{(1-q)^{\nu}}{s^{\nu}} L_q\{f(t)\}(s).
\end{aligned}$$

□

For the next set of properties, first note that

$$\nabla_q(e_q(st)f(t)) = (\nabla_q e_q(st)) f(t) + (\nabla_q f(t)) e_q(qst).$$

Thus,

$$L_q\{\nabla_q f(t)\}(s) = \frac{1}{1-q}(s L_q\{f(t)\}(s) - f(0)).$$

It follows by induction that if m denotes a positive integer, then

$$L_q\{\nabla_q{}^m f(t)\}(s) = \frac{s^m}{(1-q)^m} L_q\{f(t)\}(s) - \sum_{i=0}^{m-1} \frac{s^{m-1-i}}{(1-q)^{m-i}} \nabla_q{}^i f(0). \tag{4.3}$$

Lemma 7. If f is analytic function on $\mathbb{T}\setminus\{0\}$, then we have

$$L_q\{\nabla_q^m\nabla_q^\nu f(t)\}(s)=\frac{s^{m+\nu}}{(1-q)^{m+\nu}}L_q\{f(t)\}(s)-\sum_{i=0}^{m-1}\frac{s^{m-1-i}}{(1-q)^{m-i}}\nabla_q^i\nabla_q^\nu(f(t))|_{t=0}$$

.

Proof. The proof easily follows from the equality (4.3) and Lemma 6. □

By $\nabla_q^i\nabla_q^\nu(f(t))|_{t=0}$ we mean $\lim_{t\to 0^+}\nabla_q^i\nabla_q^\nu(f(t))$.

Example 1. Consider the following fractional q-difference equations:

a) $\nabla_q^{3/2}y(t)=0$ for $\mathbb{T}\setminus\{0\}$.

b) $\nabla_q\nabla_q^{1/2}y(t)=0$ for $\mathbb{T}\setminus\{0\}$. Assume that $\nabla_q^{1/2}y(0)$ is defined and finite.

c) $\nabla_q^2\nabla_q^{-1/2}y(t)=0$ for $\mathbb{T}\setminus\{0\}$. Assume that $\nabla_q^{-1/2}y(0)$ is defined and finite.

Note that the operators on the left side of each equation (i.e $\nabla_q^{3/2}$, $\nabla_q\nabla_q^{1/2}$ and $\nabla_q^2\nabla_q^{-1/2}$) are equivalent on $[q^a,q^b]\subset\mathbb{T}$. Since our method requires knowledge of the fractional derivatives of the solution defined at zero as well, the equations given in each of (a), (b), and (c) are not equivalent.

We search for analytic solution(s) on $\mathbb{T}\setminus\{0\}$ for each equation by use of q-Laplace transform.

For part (a), if we take the L_q-transform of the each side of the equation, then by use of Lemma 6 we have $y(t)=0$.

For part (b), we take the L_q-transform of the each side of the equation then we use the properties of the L_q transform and obtain

$$L_q\{y(t)\}=\frac{(1-q)^{1/2}}{s^{3/2}}\nabla_q^{1/2}y(t)|_{t=0}.$$

By use of Lemma 4 we have the solution

$$y(t)=\frac{\nabla_q^{1/2}y(t)|_{t=0}}{\Gamma_q(3/2)}t^{1/2}.$$

For part (c), we take the L_q-transform of the each side of the equation then we use the properties of the L_q transform and obtain

$$L_q\{y(t)\}=\frac{1}{s^{1/2}(1-q)^{1/2}}\nabla_q^{-1/2}y(t)|_{t=0}+\frac{(1-q)^{1/2}}{s^{3/2}}\nabla_q\nabla_q^{-1/2}y(t)|_{t=0}.$$

By use of Lemma 4 we have the solution

$$y(t)=\frac{\nabla_q^{-1/2}y(t)|_{t=0}}{\Gamma_q(1/2)}t^{-1/2}+\frac{\nabla_q\nabla_q^{-1/2}y(t)|_{t=0}}{\Gamma_q(3/2)}t^{1/2}.$$

Example 2. Consider the problem $\nabla_q\nabla_q^{-2/3}y(t)=t^\mu$ for $\mathbb{T}\setminus\{0\}$.

Applying L_q-transform for the each side of the given equation, we have

$$s^{1/3}(1-q)^{-1/3}L_q(y(t))-(1-q)^{-1}\nabla_q^{-2/3}y(t)|_{t=0}=\frac{(1-q)^\mu\Gamma_q(\mu+1)}{s^{\mu+1}}.$$

$$L_q\{y(t)\}=\frac{\nabla_q^{-2/3}y(t)|_{t=0}}{s^{1/3}(1-q)^{2/3}}+\frac{(1-q)^{\mu+1/3}\Gamma_q(\mu+1)}{s^{\mu+4/3}}.$$

Then the solution is $y(t) = \dfrac{\nabla_q^{-2/3} y(t)|_{t=0}}{\Gamma_q(1/3)} t^{-2/3} + \dfrac{\Gamma_q(\mu+1)}{\Gamma_q(\mu+4/3)} t^{\mu+1/3}$.

The above calculation makes clear that if $f(t)$ is a linear sum of terms of the form t^{μ_i}, $f(t) = \sum_i a_{\mu_i} t^{\mu_i}$, then a solution of $\nabla_q \nabla_q^{-2/3} y(t) = f(t)$ for $t \in \mathbb{T} \setminus \{0\}$ has the form

$$y(t) = \frac{\nabla_q^{-2/3} y(t)|_{t=0}}{\Gamma_q(1/3)} t^{-2/3} + \sum_i a_{\mu_i} \frac{\Gamma_q(\mu_i+1)}{\Gamma_q(\mu_i+4/3)} t^{\mu_i+1/3},$$

where $\mu_i \in \mathbb{R} \setminus (\{..., -2, -1\} \cup \{..., -2-4/3, -1-4/3, -4/3\})$.

Example 3. Consider the problem $\nabla_q^2 \nabla_q^{-1/3} y(t) + \alpha \nabla_q \nabla_q^{-1/3} y(t) = t^2 e_q^{-1}(q(q-1)\alpha t)$ for $\mathbb{T} \setminus \{0\}$.

Applying L_q-transform to each side of the equation, we have

$$L_q\{y(t)\} = \frac{s^{1/3}}{(1-q)^{1/3}(s+(1-q)\alpha)} \nabla_q^{-1/3} y(t)|_{t=0}$$
$$+ \frac{1}{s^{2/3}(1-q)^{-2/3}(s+(1-q)\alpha)} (\nabla_q \nabla_q^{-1/3} y(t)|_{t=0} + \alpha \nabla_q^{-1/3} y(t)|_{t=0}) + \frac{(1-q)^{11/3}[2]!}{s^{2/3}(s+(1-q)\alpha)^{\underline{(4)}}}.$$

As a result of Corollary 1 and Lemma 5, we have the solution of the fractional q-difference equation

$$y(t) = \frac{(1-q)\nabla_q^{-1/3} y(t)|_{t=0}}{\Gamma_q(-1/3)} e_q^{-1}((q-1)\alpha t) * t^{-4/3}$$
$$+ \frac{(1-q)(\nabla_q \nabla_q^{-1/3} y(t)|_{t=0} + \alpha \nabla_q^{-1/3} y(t)|_{t=0})}{\Gamma_q(2/3)} e_q^{-1}((q-1)\alpha t) * t^{-1/3}$$
$$+ \frac{(1-q)}{(1+q+q^2)\Gamma_q(2/3)} (t^3 e_q^{-1}((q-1)\alpha t)) * t^{-1/3}.$$

We close by illustrating a second form of solution in Example 3. As a corollary, we exhibit a method to calculate $\nabla_q^{-\nu}(e_q^{-1}(t))$.

Lemma 8. [2, 14] *(i)* If $L_q\{f(t)\} = F(s)$, then $L_q\{f(at)\} = \dfrac{1}{a} F(\dfrac{s}{a})$

(ii) $L_q\{t^{\beta-1}{}_2\varphi_1(q^\alpha, 0; q^\beta; t)\} = \dfrac{(1-q)^{\underline{(\beta-1)}}}{s^\beta} \dfrac{1}{(1-\frac{1}{s})^{\underline{(\alpha)}}}$,

where ${}_2\varphi_1(a, 0; b; t) = \displaystyle\sum_{r=0}^{\infty} \frac{(1-a)^{\underline{(r)}}}{(1-b)^{\underline{(r)}}(1-q)^{\underline{(r)}}} t^r$ is q-hypergeometric function.

Remark 4.1. One can easily verify the following using Corollary 1 and Lemma 8 (ii) for the case $\alpha = 1$:

$L_q\{\nabla_q^{-\nu}(e_q^{-1}(t))\} = \dfrac{1-q}{\Gamma_q(\nu)} L_q(t^{\nu-1} * e_q^{-1}(t)) = \dfrac{(1-q)^\nu}{s^\nu(s-1)}$

and

$L_q\{\dfrac{(1-q)^\nu}{(1-q)^{\underline{(\nu)}}} t^\nu {}_2\varphi_1(q, 0; q^{\nu+1}; t)\} = \dfrac{(1-q)^\nu}{s^\nu(s-1)}$.

The uniqueness of q-Laplace transform [14] implies that

$$\nabla_q^{-\nu}(e_q^{-1}(t)) = \frac{(1-q)^\nu}{(1-q)^{\underline{(\nu)}}} t^\nu {}_2\varphi_1(q, 0; q^{\nu+1}; t)).$$

After observing all these nice relations, we have another way to write the solution of the Example 3:

$$
\begin{aligned}
y(t) = & \frac{(1-q)\nabla_q^{-1/3} y(t)|_{t=0}}{(1-q)^{\underline{(-1/3)}}}((q-1)t)^{-1/3}{}_2\varphi_1(q,0;q^{2/3};(q-1)\alpha t) \\
& + \frac{(1-q)(\nabla_q \nabla_q^{-1/3} y(t)|_{t=0} + \alpha \nabla_q^{-1/3} y(t)|_{t=0})}{(1-q)^{\underline{(2/3)}}}((q-1)t)^{2/3}{}_2\varphi_1(q,0;q^{5/3};(q-1)\alpha t) \\
& + \frac{(1-q)}{(1+q+q^2)\Gamma_q(2/3)}(t^3 e_q^{-1}((q-1)\alpha t)) * t^{-1/3}.
\end{aligned}
$$

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