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Leading order integrability conditions for differential-difference equations

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Abstract

A necessary condition for the existence of conserved densities, ρ, and fluxes of a differential-difference equation which depend on q shifts, for q sufficiently large, is presented. This condition depends on the eigenvalues of the leading terms in the differential-difference equation. It also gives, explicitly, the leading integrability conditions on the density in terms of second derivatives of ρ.

1 Introduction

Consider a nonlinear (autonomous) differential-difference equation (DDE) of the form

$$\dot{w}_n = f(w_{n-l},\ w_{n-l+1},\ \ldots,\ w_n,\ \ldots,\ w_{n+m-1},\ w_{n+m}) \tag{1.1}$$

where $\dot{w}_n$ is the time derivative of w_n,

$$\frac{\partial f}{\partial w_{n-l}} \neq 0, \qquad \frac{\partial f}{\partial w_{n+m}} \neq 0$$

and n is an arbitrary integer. In general, f is a vector-valued function of a finite number of dynamical variables and each w_k is a vector-valued function of t.

The index n may lie in $\mathbf{Z}$ or the w_k may be periodic, $w_k = w_{k+M}$. The integers l and m measure the degree of non-locality in (1.1). If $l = m = 0$ then the equation is local and reduces to a system of ordinary differential equations.

This class of equations arise in a number of areas including the modelling of many physical interesting phenomena and in numerical simulation of nonlinear partial differential equations. Their integrability properties are also of interest in their own right. See, for example, [14, 15]. Of particular interest are conservation laws for (1.1) which depend on arbitrarily many shifts of the dynamical varible, w_q. The existence of such laws is an indicator of the complete integrability of (1.1).

In this paper, we derive a necessary condition for the existence of conserved densities, ρ, that depend on q shifts. This condition (3.12) involves the eigenvalues of

$$\frac{\partial f}{\partial w_{n-l}}, \qquad \frac{\partial f}{\partial w_{n+m}}$$

and, moreover, will give, explicitly, the leading integrability conditions in terms of second order derivatives of ρ.

We begin with establishing our notation and enumerating some basic facts about DDEs. In this section we use the classic example of the completely integrable Toda lattice to illustrate the concepts concretely. In the following two sections, we derive the leading integrablity conditions. In section 3, the necessary condition itself is obtained. In the penultimate section, this condition is applied to a number of examples from the literature. Finally we discuss applications and of this result to the computation of conserved densities.

2 Differential-Difference Equations

The shift operator D is defined by $\mathrm{D}w_k = w_{k+1}$. Following [7], we use of the shift operatorto generate (1.1) from the single equation

$$\dot{w}_0 = f(w_{-l},\ w_{-l+1},\ \ldots,\ w_0,\ \ldots,\ w_{m-1},\ w_m)$$

via

$$\dot{w}_n = \mathrm{D}^n \dot{w}_0 = \mathrm{D}^n f.$$

In the case where w_k is vector-valued, notation can quickly become cumbersome. We adopt the convention that w denotes the zero-shifted dependent variable. Shifts of w (w_k in the above discussion) will be denoted by $\mathrm{D}^k w$ and we will reserve w_α to denote the α component of w. Thus $\mathrm{D}^k w_\alpha$ will be the α component shifted k times. With these conventions we have

$$\dot{w} = f(\mathrm{D}^{-l}w,\ \mathrm{D}^{-l+1}w,\ \ldots,\ w,\ \ldots,\ \mathrm{D}^{m-1}w,\ \mathrm{D}^m w). \tag{2.1}$$

The case $l = m$ will be called symmetric.

A classic example of a (vector-valued) DDE is provided by the completely integrable Toda lattice [16].

$$\begin{aligned} \dot{u}_n &= v_{n-1} - v_n \\ \dot{v}_n &= v_n\,(u_n - u_{n+1}) \end{aligned}$$

for $n \in \mathbf{Z}$. The shift operator allows the system to be generated by

$$\begin{aligned} \dot{u}_0 &= v_{-1} - v_0 \\ \dot{v}_0 &= v_0\,(u_0 - u_1) \end{aligned}$$

or, with the convention adopted above,

$$\begin{aligned} \dot{u} &= \mathrm{D}^{-1}v - v \\ \dot{v} &= v\,(u - \mathrm{D}u). \end{aligned} \tag{2.2}$$

This equation has the form of (2.1) with

$$w = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{and} \quad f(\mathrm{D}^{-1}w,\ w,\ \mathrm{D}w) = \begin{bmatrix} \mathrm{D}^{-1}w_2 - w_2 \\ w_2\,(w_1 - \mathrm{D}w_1) \end{bmatrix}.$$

The total time derivative $D_t\, g$ of a function $g = g(\mathrm{D}^p w,\ \mathrm{D}^{p+1} w,\ \ldots,\ \mathrm{D}^q w)$ is the time derivative along solutions of (2.1); that is

$$\begin{aligned} D_t\, g &= \sum_{k=p}^{q} \frac{\partial g}{\partial \mathrm{D}^k w}\, \mathrm{D}^k \dot{w} = \sum_{k=p}^{q} \frac{\partial g}{\partial \mathrm{D}^k w}\, \mathrm{D}^k f \\ &= \sum_{k=p}^{q} \left(\mathrm{D}^k (f \, \frac{\partial}{\partial w}) \right) g = \sum_{k=p}^{q} (\mathrm{D}^k F)\, g \end{aligned}$$

where

$$F \equiv f\, \frac{\partial}{\partial w} = \sum_{\alpha} f_\alpha \, \frac{\partial}{\partial w_\alpha}.$$

Note that

$$\mathrm{D}^k F\, g = \mathrm{D}^k (F\, \mathrm{D}^{-k} g). \tag{2.3}$$

The total time derivative commutes with the shift operator

$$D_t\, \mathrm{D} g = \sum_{k=p+1}^{q+1} (\mathrm{D}^k F)\, \mathrm{D} g = \mathrm{D} \sum_{k=p}^{q} (\mathrm{D}^k F)\, g = \mathrm{D}\, D_t\, g. \tag{2.4}$$

Returning to the Toda lattice, the operator F is given by

$$F = \dot{u}\, \frac{\partial}{\partial u} + \dot{v}\, \frac{\partial}{\partial v} = (\mathrm{D}^{-1} v - v)\, \frac{\partial}{\partial u} + v\, (u - \mathrm{D} u)\, \frac{\partial}{\partial v}.$$

The difference operator, $\Delta = \mathrm{D} - \mathrm{I}$, takes the role of a spatial derivative on the shifted variables as many examples of DDEs arise from discretization of a PDE in $(1+1)$ variables [12].

A (scalar) function $\rho = \rho(\mathrm{D}^p w,\ \mathrm{D}^{p+1} w,\ \ldots,\ \mathrm{D}^q w)$ is a (conserved) *density* if there exists J, called the (associated) *flux*, such that

$$D_t\, \rho + \Delta\, J = 0. \tag{2.5}$$

Equation (2.5) is a local conservation law and, with appropriate boundary conditions, will give conserved quantity. If

$$\rho = \Delta\, \psi$$

then ρ is trivially a density. Also note that if ρ is a density then, by (2.4) $\mathrm{D}^k \rho$ is also density. Thus, without loss of generality, we may assume that a density that depends on q shifts has canonical form $\rho(w,\ Dw,\ \ldots,\ \mathrm{D}^q w)$.

For example,

$$\rho = \tfrac{1}{3}\, u^3 + u(\mathrm{D}^{-1} v + v)$$

is a density for the Toda lattice since

$$
\begin{aligned}
D_t\,\rho &= (\mathrm{D}^{-1}F + F)\,\rho \\
&= (\mathrm{D}^{-1}v - v)\,(u^2 + \mathrm{D}^{-1}v + v) + u\mathrm{D}^{-1}(v(u - \mathrm{D}u)) + uv(u - \mathrm{D}u) \\
&= u^2\,(\mathrm{D}^{-1}v - v) + (\mathrm{D}^{-1}v)^2 - v^2 + u\mathrm{D}^{-1}v\,(\mathrm{D}^{-1}u - u) + u^2\,v - uv\mathrm{D}u \\
&= (\mathrm{D}^{-1}v)^2 - v^2 + u\mathrm{D}^{-1}v\,\mathrm{D}^{-1}u - uv\mathrm{D}u \qquad (2.6)\\
&= \;-\Delta\,\left((\mathrm{D}^{-1}v)^2\right) - \Delta\left(u\mathrm{D}^{-1}v\,\mathrm{D}^{-1}u\right)
\end{aligned}
$$

and so the associated flux is

$$J = (\mathrm{D}^{-1}v)^2 + u\mathrm{D}^{-1}v\,\mathrm{D}^{-1}u.$$

The canonical form of this density is

$$\mathrm{D}\rho = \tfrac{1}{3}\,(\mathrm{D}u)^3 + \mathrm{D}u(v + \mathrm{D}v)$$

with flux

$$\mathrm{D}J = v^2 + uv\mathrm{D}u.$$

A necessary and sufficient condition for a function $g(\mathrm{D}^p w,\ \mathrm{D}^{p+1}w,\ \ldots,\ \mathrm{D}^q w) = \Delta\,h$, a total difference, is [1, 7, 13]

$$\mathrm{E}(g) = 0 \tag{2.7}$$

where E is the *discrete Euler operator* (variational derivative)

$$\mathrm{E}(g) = \frac{\partial}{\partial w}\left(\sum_{k=p}^{q}\mathrm{D}^{-k}\,g\right); \tag{2.8}$$

that is

$$\mathrm{E}_\alpha(g) \equiv \frac{\partial}{\partial w_\alpha}\left(\sum_{k=p}^{q}\mathrm{D}^{-k}\,g\right) = 0$$

for each α.

For the Toda lattice, the Euler operator has two components

$$E = \begin{bmatrix} E_u \\ E_v \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial u}\sum \mathrm{D}^{-k} \\ \dfrac{\partial}{\partial v}\sum \mathrm{D}^{-k} \end{bmatrix}.$$

Applying this operator to the above example (2.6), we find

$$\begin{aligned}
E_u(D_t\,\rho) &= \frac{\partial}{\partial u}\left(\mathrm{D}^{-1}+\mathrm{I}+\mathrm{D}\right)D_t\,\rho\\
&= \frac{\partial}{\partial u}\left(-\,\mathrm{D}^{-1}u\mathrm{D}^{-1}vu+u\mathrm{D}^{-1}v\,\mathrm{D}^{-1}u-uv\mathrm{D}u+uv\mathrm{D}u\right)\\
&= 0\\
E_v(D_t\,\rho) &= \frac{\partial}{\partial v}\left(\mathrm{D}^{-1}+\mathrm{I}+\mathrm{D}\right)D_t\,\rho\\
&= \frac{\partial}{\partial v}\left(v^2-v^2+uv\mathrm{D}u-uv\mathrm{D}u\right)\\
&= 0.
\end{aligned}$$

The computation of the total time derivative and the action of the Euler operator are tasks well suited for a computer algebra system such as Maple or Mathematica.

A necessary condition [7] for (2.7) is

$$\frac{\partial^2 g}{\partial \mathrm{D}^p w\,\partial \mathrm{D}^q w}=0. \tag{2.9}$$

Note that, in the system case, this equation is a "matrix" equation. If p_α is the least shift and q_α is the greatest shift with which the component u_α occurs in g, then this necessary condition reads

$$\frac{\partial^2 g}{\partial \mathrm{D}^{p_\alpha} w_\alpha\,\partial \mathrm{D}^{q_\beta} w_\beta}=0$$

for each α and β.

3 Necessary Conditions for a Density

In this section we will obtain the result:

Theorem 1. *Consider the differential-difference equation*

$$\dot w = f(\mathrm{D}^{-l}w,\ \mathrm{D}^{-l+1}w,\ \ldots,\ w,\ \ldots,\ \mathrm{D}^{m-1}w,\ \mathrm{D}^m w)$$

for a vector-valued variable w. Let $L=\max(l,m)$ and λ_i, μ_i be the eigenvalues of

$$\frac{\partial f}{\partial \mathrm{D}^{-L}w} \qquad \text{and} \qquad \frac{\partial f}{\partial \mathrm{D}^{L}w}$$

respectively. A necessary condition for the differential-difference equation to have a conserved density depending on $q=pL+r>L$ shifts is that

$$\zeta\,\mathrm{D}^r\mu_j = -\,\lambda_i\,\mathrm{D}^L\zeta$$

has a non-zero solution ζ for some λ_i and μ_j. In particular, if w is a scalar then such densities can only occur when $l=m$.

Our method of attack is to first remove terms in our candidate density ρ that contribute directly to the flux. Rather than applying the Euler operator on the remaining terms in ρ, we use the necessary condition (2.9) to obtain a system of equations for the the terms that depend on the maximal shift, $\mathrm{D}^q w$, in ρ. This system is rewritten as a matrix equation. Solutions (or lack of solutions) to this system will give us the above result.

3.1 The Initial Split

Lemma 2. *Consider the differential-difference equation*

$$\dot{w} = f(\mathrm{D}^{-l}w,\ \mathrm{D}^{-l+1}w,\ \ldots,\ w,\ \ldots,\ \mathrm{D}^{m-1}w,\ \mathrm{D}^{m}w)$$

for a vector-valued variable w. $\rho = \rho(w,\ \mathrm{D}w,\ \ldots,\ \mathrm{D}^q w)$ is a density for this differential-difference equation if and only if

$$\sigma = \mathrm{D}^l \left(f \frac{\partial}{\partial w} \right) \sum_{j=0}^{l} \mathrm{D}^j \rho + \sum_{j=l+1}^{q} \mathrm{D}^j \left(f \frac{\partial}{\partial w} \right) \rho$$

is a total difference.

Proof. We begin by splitting the identity operator

$$\mathrm{I} = (\mathrm{D} - \mathrm{I} + \mathrm{I})\,\mathrm{D}^{-1} = \Delta\,\mathrm{D}^{-1} + \mathrm{D}^{-1}. \tag{3.1}$$

This split is applied to terms in the candidate density ρ that do not depend on the lowest order shifted variables, ρ^* say. The first term $\Delta\,\mathrm{D}^{-1}\,\rho^*$ contributes to the flux while the second term $\mathrm{D}^{-1}\,\rho^*$ has a strictly lower shift. Applying this split repeatly we obtain

$$\mathrm{I} = (\mathrm{D}^k - \mathrm{I} + \mathrm{I})\,\mathrm{D}^{-k} = \Delta\,\left(\mathrm{D}^{k-1} + \mathrm{D}^{k-2} + \cdots + \mathrm{D} + \mathrm{I}\right)\mathrm{D}^{-k} + \mathrm{D}^{-k} \tag{3.2}$$

where, again, the first term contributes to the flux and the second term has strictly lower shift.

This decomposition is repeatedly applied to terms that do not involve the lowest order shifted variables. Any terms that remain will involve the lowest order shifted variable. These terms yield the constraints on the undetermined coefficients or unknown functions in the density ρ.

In more detail, assume $\rho = \rho(w,\ \mathrm{D}w,\ \ldots,\ \mathrm{D}^q w)$. If

$$\frac{\partial^2 \rho}{\partial \mathrm{D}^q w\,\partial w} = 0$$

then

$$\begin{aligned}
\rho(w, \ldots, \mathrm{D}^q w) &= g_1(w,\ \ldots,\ \mathrm{D}^{q-1}w) + g_2(\mathrm{D}w,\ \ldots,\ \mathrm{D}^q w) \\
&= \rho^{(1)}(w,\ \ldots,\ \mathrm{D}^{q-1}w) + \Delta\,\rho^{(2)}(w,\ \ldots,\ \mathrm{D}^{q-1}w)
\end{aligned}$$

with

$$\begin{aligned}
\rho^{(1)} &= g_1 + \mathrm{D}^{-1} g_2 \\
\rho^{(2)} &= \mathrm{D}^{-1} g_2.
\end{aligned}$$

Thus ρ is, at most, a non-trivial density depending on $q-1$ shifts. Therefore, without loss of generality, we may assume

$$\frac{\partial^2 \rho}{\partial \mathrm{D}^q w\,\partial w} \neq 0. \tag{3.3}$$

In order to apply the split (3.1) to $D_t\,\rho$ we must identify terms in $D_t\,\rho$ that do not depend on the lowest order shifted variable. Now

$$D_t\rho = \sum_{k=0}^{q}(\mathrm{D}^k F)\,\rho$$

and the term with the lowest shift is $F\,\rho$ which depends on $\mathrm{D}^{-l}w$. The other terms, $(\mathrm{D}^k F)\,\rho$, will depend on $\mathrm{D}^{k-l}w$ or w (if $k > l$) and higher shifted variables. Therefore, applying (3.2) to each of these terms, we obtain

$$\begin{aligned}\sum_{k=1}^{l}(\mathrm{D}^k F)\,\rho &= \sum_{k=1}^{l}(\mathrm{D}^k - \mathrm{I} + \mathrm{I})\,\mathrm{D}^{-k}((\mathrm{D}^k F)\,\rho)\\ &= \sum_{k=1}^{l}\left(\Delta\left(\mathrm{D}^{k-1} + \mathrm{D}^{k-2} + \cdots + \mathrm{D} + \mathrm{I}\right)\mathrm{D}^{-k} + \mathrm{D}^{-k}\right)(\mathrm{D}^k F)\,\rho\\ &= F\sum_{k=1}^{l}\mathrm{D}^{-k}\rho + \Delta\left(\sum_{k=1}^{l}\sum_{j=0}^{k-1}(\mathrm{D}^j F)\,\mathrm{D}^{j-k}\rho\right)\\ &= F\sum_{j=1}^{l}\mathrm{D}^{-j}\rho + \Delta\left(\sum_{j=1}^{l}\sum_{k=0}^{l-j}\left(\mathrm{D}^k F\right)\mathrm{D}^{-j}\rho\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{k=l+1}^{q}(\mathrm{D}^k F)\,\rho &= (\mathrm{D}^l - \mathrm{I} + \mathrm{I})\,\mathrm{D}^{-l}\sum_{k=l+1}^{q}(\mathrm{D}^k F)\,\rho\\ &= \sum_{k=l+1}^{q}\Delta\left(\sum_{j=0}^{l-1}\left(\mathrm{D}^{j-l+k}F\right)\mathrm{D}^{j-l}\rho\right) + \sum_{k=l+1}^{q}(\mathrm{D}^{k-l}F)\,\mathrm{D}^{-l}\rho\\ &= \sum_{k=1}^{q-l}(\mathrm{D}^k F)\,\mathrm{D}^{-l}\rho + \Delta\left(\sum_{j=1}^{l}\sum_{k=l+1-j}^{q-j}\left(\mathrm{D}^k F\right)\mathrm{D}^{-j}\rho\right).\end{aligned}$$

using (2.3). Combining these pieces we obtain

$$\begin{aligned}\mathrm{D}_t\,\rho &= F\,\rho + \sum_{k=1}^{l}(\mathrm{D}^k F)\,\rho + \sum_{k=l+1}^{q}(\mathrm{D}^k F)\,\rho\\ &= F\sum_{j=0}^{l}\mathrm{D}^{-j}\rho + \sum_{k=1}^{q-l}(\mathrm{D}^k F)\,\mathrm{D}^{-l}\rho + \Delta\left(\sum_{j=1}^{l}\sum_{k=0}^{q-j}\left(\mathrm{D}^k F\right)\mathrm{D}^{-j}\rho\right).\end{aligned}$$

The third sum contributes directly to the flux. Thus it is the first two sums that may be the obstruction to ρ being a density. For convenience, we shift this expression to give

$$\begin{aligned}\mathrm{D}^l D_t\,\rho &= (\mathrm{D}^l F)\sum_{j=0}^{l}\mathrm{D}^j\rho + \sum_{j=l+1}^{q}(\mathrm{D}^j F)\,\rho + \Delta\left(\sum_{j=0}^{l-1}\sum_{k=l}^{q+l-j}(\mathrm{D}^k F)\,\mathrm{D}^j\rho\right)\\ &= \sigma + \Delta\,J\end{aligned}$$

say. The variable of lowest shift is now w. ■

3.2 Leading Integrability Conditions

At this stage J will form part of the flux. Rather than applying the Euler operator (2.8) to σ, we use the necessary condition (2.9). The resulting equations are most easily expressed in terms of the Kronecker sum [10, Chapter 13].

Definition. Let R be $m \times n$ matrix and S be an arbitrary matrix. The Kronecker (or direct or tensor) product of R and S is the matrix given by

$$R \otimes S \equiv \begin{bmatrix} R_{11}\, S & R_{12}\, S & \cdots & R_{1n}\, S \\ R_{21}\, S & R_{22}\, S & \cdots & R_{2n}\, S \\ \vdots & \vdots & \ddots & \vdots \\ R_{m1}\, S & R_{m2}\, S & \cdots & R_{mn}\, S \end{bmatrix}.$$

If R and S are square matrices, Kronecker sum of R and S is given by

$$R \oplus S \equiv R \otimes \mathrm{I} + \mathrm{I} \otimes S$$

where I is the appropriately sized identity matrix.

The leading integrabilty conditions which follow from (2.9) are the conditions that must hold on terms in the candidate density that depend on both w and $\mathrm{D}^q w$. However, in the initial split, all terms that depend on $\mathrm{D}^q w$ but *not* w were shifted. Thus these leading integrability conditions give *all* conditions that involve the highest shift variable $\mathrm{D}^q w$.

Theorem 3. *Consider the differential-difference equation*

$$\dot{w} = f(\mathrm{D}^{-l} w,\ \mathrm{D}^{-l+1} w,\ \ldots,\ w,\ \ldots,\ \mathrm{D}^{m-1} w,\ \mathrm{D}^m w)$$

for a vector-valued variable $w = \begin{bmatrix} w_1 & w_2 & \cdots & w_N \end{bmatrix}^T$*. Let* $L = \max(l, m)$*. For a density* ρ *that depends on* $q > L$ *shifts, the leading integrability conditions are*

$$\mathcal{S} X \equiv \left[\left(\mathrm{D}^l \left(\frac{\partial f}{\partial \mathrm{D}^{-L} w} \right)^T \mathrm{D}^l \right) \oplus \mathrm{D}^q \left(\frac{\partial f}{\partial \mathrm{D}^{L} w} \right)^T \right] X = 0 \tag{3.4}$$

where X *is a vector with* N^2 *components given by*

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \qquad \textit{with} \qquad X_j = \frac{\partial^2 \rho}{\partial w_j\, \partial \mathrm{D}^q w} = \begin{bmatrix} \dfrac{\partial^2 \rho}{\partial w_j\, \partial \mathrm{D}^q w_1} \\ \dfrac{\partial^2 \rho}{\partial w_j\, \partial \mathrm{D}^q w_2} \\ \vdots \\ \dfrac{\partial^2 \rho}{\partial w_j\, \partial \mathrm{D}^q w_N} \end{bmatrix}.$$

Proof. From Lemma 2, the obstruction for ρ to be a density is that

$$\sigma = \mathrm{D}^l \left(f \frac{\partial}{\partial w}\right) \sum_{j=0}^{l} \mathrm{D}^j \rho + \sum_{j=l+1}^{q} \mathrm{D}^j \left(f \frac{\partial}{\partial w}\right) \rho$$

be a total difference. All terms in both sums depend on w which is the variable of lowest shift. The term with maximum shift in first sum is $j = l$ which depends on $\mathrm{D}^{q+l} w$ (assuming $q > m$) and the term with maximum shift in second sum occurs when $j = q$ which depends on $\mathrm{D}^{q+m} w$. Now

$$\begin{aligned}\frac{\partial \sigma}{\partial w} &= \left(\frac{\partial}{\partial w} \mathrm{D}^l f\right) \frac{\partial}{\partial \mathrm{D}^l w} \sum_{j=0}^{l} \mathrm{D}^j \rho + \mathrm{D}^l f \frac{\partial^2}{\partial w\, \partial \mathrm{D}^l w} \sum_{j=0}^{l} \mathrm{D}^j \rho + \sum_{j=l+1}^{q} (\mathrm{D}^j f) \frac{\partial^2 \rho}{\partial w\, \partial \mathrm{D}^j w} \\ &= \left(\frac{\partial}{\partial w} \mathrm{D}^l f\right) \frac{\partial}{\partial \mathrm{D}^l w} \sum_{j=0}^{l} \mathrm{D}^j \rho + \sum_{j=l}^{q} (\mathrm{D}^j f) \frac{\partial^2 \rho}{\partial w\, \partial \mathrm{D}^j w}.\end{aligned}$$

In terms of components, this equation reads

$$\frac{\partial \sigma}{\partial w_\beta} = \sum_\alpha \left((\frac{\partial}{\partial w_\beta} \mathrm{D}^l f_\alpha) \frac{\partial}{\partial \mathrm{D}^l w_\alpha} \sum_{j=0}^{l} \mathrm{D}^j \rho + \sum_{j=l}^{q} \frac{\partial^2 \rho}{\partial w_\beta\, \partial \mathrm{D}^j w_\alpha} \mathrm{D}^j f_\alpha \right).$$

In the asymmetric case $l > m$, (2.9) gives

$$\frac{\partial^2 \sigma}{\partial \mathrm{D}^{q+l} w\, \partial w} = \mathrm{D}^l \left(\frac{\partial^2 \rho}{\partial \mathrm{D}^q w\, \partial w} \frac{\partial f}{\partial \mathrm{D}^{-l} w} \right) = 0$$

with

$$\frac{\partial f}{\partial \mathrm{D}^{-l} w} \equiv \begin{bmatrix} \dfrac{\partial f_1}{\partial \mathrm{D}^{-l} w_1} & \dfrac{\partial f_1}{\partial \mathrm{D}^{-l} w_2} & \cdots & \dfrac{\partial f_1}{\partial \mathrm{D}^{-l} w_N} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_N}{\partial \mathrm{D}^{-l} w_1} & \dfrac{\partial f_N}{\partial \mathrm{D}^{-l} w_2} & \cdots & \dfrac{\partial f_N}{\partial \mathrm{D}^{-l} w_N} \end{bmatrix}$$

and

$$\frac{\partial^2 \rho}{\partial \mathrm{D}^q w\, \partial w} \equiv \begin{bmatrix} \dfrac{\partial^2 \rho}{\partial \mathrm{D}^q w_1\, \partial w_1} & \dfrac{\partial^2 \rho}{\partial \mathrm{D}^q w_1\, \partial w_2} & \cdots & \dfrac{\partial \rho}{\partial \mathrm{D}^q w_1\, \partial w_N} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 \rho}{\partial \mathrm{D}^q w_N\, \partial w_1} & \dfrac{\partial^2 \rho}{\partial \mathrm{D}^q w_N\, \partial w_2} & \cdots & \dfrac{\partial^2 \rho}{\partial \mathrm{D}^q w_N\, \partial w_N} \end{bmatrix}$$

where N is the number of components of w. Therefore, if

$$\frac{\partial f}{\partial \mathrm{D}^{-l} w}$$

has full rank, (3.3) implies that there are no non-trivial densities depending on $q > l$ shifts.

Similiarly, for the case $l < m$, (2.9) gives

$$\frac{\partial^2\sigma}{\partial \mathrm{D}^{q+m}w\,\partial w} = \frac{\partial^2\rho}{\partial w\,\partial \mathrm{D}^q w}\,\mathrm{D}^q\frac{\partial f}{\partial \mathrm{D}^m w} = 0.$$

Again, if

$$\frac{\partial f}{\partial \mathrm{D}^m w}$$

has full rank, (3.3) implies that there are no non-trivial densities depending on $q > m$ shifts. In particular, the scalar case will have no non-trivial densities for $q > \max(l,\ m)$ unless it is symmetric.

For the symmetric case, we have

$$\frac{\partial^2\sigma}{\partial \mathrm{D}^{q+l}w\,\partial w} = \mathrm{D}^l\left(\frac{\partial^2\rho}{\partial \mathrm{D}^q w\,\partial w}\,\frac{\partial f}{\partial \mathrm{D}^{-l}w}\right) + \mathrm{D}^q\left(\frac{\partial f}{\partial \mathrm{D}^l w}\right)^T\frac{\partial^2\rho}{\partial \mathrm{D}^q w\,\partial w}$$

and so (2.9) gives (after a transpose)

$$\mathrm{D}^l\left(\frac{\partial f}{\partial \mathrm{D}^{-l}w}\right)^T\mathrm{D}^l\,\frac{\partial^2\rho}{\partial w\,\partial \mathrm{D}^q w} + \frac{\partial^2\rho}{\partial w\,\partial \mathrm{D}^q w}\,\mathrm{D}^q(\frac{\partial f}{\partial \mathrm{D}^l w}) = 0. \tag{3.5}$$

This system may be rewritten as a linear system for the vector unknown X given in Theorem 3. X^T is formed by the concatenation of the rows of $\dfrac{\partial^2\rho}{\partial w\,\partial \mathrm{D}^q w}$. The system (3.5) becomes [10]

$$\left[\left(\mathrm{D}^l\left(\frac{\partial f}{\partial \mathrm{D}^{-l}w}\right)^T\mathrm{D}^l\right)\oplus \mathrm{D}^q\left(\frac{\partial f}{\partial \mathrm{D}^l w}\right)^T\right]X = 0.$$

Note that this also covers the asymmetric cases when one of the factors is 0. ∎

For the Toda lattice (2.2), we have

$$\frac{\partial f}{\partial \mathrm{D}^{-1}w} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \frac{\partial f}{\partial \mathrm{D}w} = \begin{bmatrix} 0 & 0 \\ -\,v & 0 \end{bmatrix}.$$

and so the leading integrability condition (3.4) is

$$\begin{bmatrix} 0 & -\,\mathrm{D}^q v & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{D} & 0 & 0 & -\,\mathrm{D}^q v \\ 0 & \mathrm{D} & 0 & 0 \end{bmatrix}\begin{bmatrix} \dfrac{\partial^2\rho}{\partial u\,\partial \mathrm{D}^q u} \\ \dfrac{\partial^2\rho}{\partial u\,\partial \mathrm{D}^q v} \\ \dfrac{\partial^2\rho}{\partial v\,\partial \mathrm{D}^q u} \\ \dfrac{\partial^2\rho}{\partial v\,\partial \mathrm{D}^q v} \end{bmatrix} = 0.$$

Thus

$$\begin{aligned}
\frac{\partial^2 \rho}{\partial u\,\partial \mathrm{D}^q u} &= \mathrm{D}^{q-1} v\, c \\
\frac{\partial^2 \rho}{\partial u\,\partial \mathrm{D}^q v} &= 0 \\
\frac{\partial^2 \rho}{\partial v\,\partial \mathrm{D}^q v} &= \mathrm{D} c
\end{aligned} \tag{3.6}$$

where $c = c(w, \mathrm{D}w, \ldots, \mathrm{D}^{q-1}w)$ with the remaining derivative arbitrary.

3.3 Necessary Conditions

It is clear that if the coefficient matrix, $\mathcal{S}$, in (3.4) has a zero eigenvalue then the eigenspace associated with this eigenvalue will be a solution to (3.4). However, since the shift operator appears in $\mathcal{S}$, eigenvalues may depend on D. In this case the eigenvalue may have a non-trivial kernel which will lead to other solutions (in the case of the Toda lattice, there are no other solutions).

Lemma 4. *Let A and B be square matrices with eigenvalues λ_i, μ_j respectively. Then the eigenvalues of $A \oplus B$ are given by $\lambda_i + \mu_j$. Furthermore, let*

$$\begin{aligned}
\mathcal{A}_i &= A - \lambda_i \,\mathrm{I} \\
\mathcal{B}_j &= B - \mu_j \,\mathrm{I}
\end{aligned}$$

and $\tilde{x}$, $\tilde{y}$ be non-zero solutions of

$$\mathcal{A}_i^2\, \tilde{x} = 0 \qquad \text{and} \qquad \mathcal{B}_j^2\, \tilde{y} = 0.$$

Then the eigenvectors of $A \oplus B$ associated with $\lambda_i + \mu_j$ are

$$\tilde{x} \otimes \tilde{y}$$

if $\tilde{x}$ is an eigenvector of A and $\tilde{y}$ is an eigenvector of B and

$$z = \mathcal{A}_i\, \tilde{x} \otimes \tilde{y} - \tilde{x} \otimes \mathcal{B}_j\, \tilde{y}. \tag{3.7}$$

if neither $\tilde{x}$ nor $\tilde{y}$ are eigenvectors.

Proof. Suppose the eigenvalues of A are λ_i with associated eigenvectors $x^{(i)}$ and the eigenvalues of B are μ_i with associated eigenvectors $y^{(i)}$. Since $\otimes$ is a tensor product, we have

$$[A \oplus B]\, x^{(i)} \otimes y^{(j)} = A\, x^{(i)} \otimes y^{(j)} + x^{(i)} \otimes B\, y^{(j)} = (\lambda_i + \mu_j)\, x^{(i)} \otimes y^{(j)}.$$

Thus the eigenvalues of $A \oplus B$ are $\mu_i + \lambda_j$ with associated eigenvectors $x^{(i)} \otimes y^{(j)}$. If neither A nor B is defective, this gives the complete set of eigenvectors.

Suppose A is defective and that $\tilde{x}$ is a generalized eigenvector associated with the defective eigenvalue λ_i; that is

$$\mathcal{A}_i^k\, \tilde{x} = 0 \qquad \text{with} \qquad \mathcal{A}_i^{k-1}\, \tilde{x} \neq 0$$

for some integer $k > 1$. Note that

$$[A \oplus B - (\lambda_i + \mu_j)\,\mathrm{I} \otimes \mathrm{I}]^k = [\mathcal{A}_i \otimes \mathrm{I} + \mathrm{I} \otimes \mathcal{B}_j]^k = \sum_{m=0}^{k} \binom{k}{m} \mathcal{A}_i^m \otimes \mathcal{B}_j^{k-m}$$

since

$$(A \otimes \mathrm{I})\,(\mathrm{I} \otimes B) = A \otimes B = (\mathrm{I} \otimes B)\,(A \otimes \mathrm{I})$$

for any matrices A and B. Therefore

$$[A \oplus B - (\lambda_i + \mu_j)\,\mathrm{I} \otimes \mathrm{I}]^k\, \tilde{x} \otimes y^{(j)} = 0$$

and so $\tilde{x} \otimes y^{(j)}$ is a generalized eigenvector of $A \oplus B$. Similiarly, if B is defective with a generalized eigenvector $\tilde{y}$ then $x^{(i)} \otimes \tilde{y}$ will be generalized eigenvectors of $A \oplus B$. Note that these generalized eigenvectors can never become true eigenvectors.

If both A and B are defective with generalized eigenvectors

$$\mathcal{A}_i^{k_1}\, \tilde{x} = 0 \qquad \text{and} \qquad \mathcal{B}_j^{k_2}\, \tilde{y} = 0$$

then

$$[A \oplus B - (\lambda_i + \mu_j)\,\mathrm{I} \otimes \mathrm{I}]^{k_1+k_2-1}\, \tilde{x} \otimes \tilde{y} = 0$$

since the second factor will vanish for terms in the sum with $m < k_1$ ($k_1 + k_2 - 1 - m \geq k_2$) and the first factor will vanish for the remaining terms. Therefore $\tilde{x} \otimes \tilde{y}$ is a generalized eigenvector. However, in this case, it may be possible to construct some genuine eigenvectors. Note that $(A - \lambda_i\,\mathrm{I})^m\, \tilde{x}$ is a generalized eigenvector for each $m = 0, \ldots, (k_1 - 2)$ and, in fact, is an eigenvector for $m = k_1 - 1$. Consider the vector

$$z = \mathcal{A}_i^{k_1-1}\, \tilde{x} \otimes \mathcal{B}_j^{k_2-2}\, \tilde{y} - \mathcal{A}_i^{k_1-2}\, \tilde{x} \otimes \mathcal{B}_j^{k_2-1}\, \tilde{y}. \tag{3.8}$$

We have

$$\begin{aligned}
[A \oplus B - (\lambda_i + \mu_j)\,\mathrm{I} \otimes \mathrm{I}]\, z &= [\mathcal{A}_i \oplus \mathcal{B}_j]\, z \\
&= 0 \otimes \mathcal{B}_j^{k_2-2}\, \tilde{y} + \mathcal{A}_i^{k_1-1}\, \tilde{x} \otimes \mathcal{B}_j^{k_2-1}\, \tilde{y} \\
&\qquad - \mathcal{A}_i^{k_1-1}\, \tilde{x} \otimes \mathcal{B}_j^{k_2-1}\, \tilde{y} - \mathcal{A}_i^{k_1-2}\, \tilde{x} \otimes 0 \\
&= 0.
\end{aligned}$$

Therefore z is an eigenvector. Note that, if $k_1 > 2$ then $\mathcal{A}_i\, \tilde{x}$ is also a generalized eigenvector. However the eigenvector generated by it will be the same as that generated by $\tilde{x}$. Therefore the linearly independent eigenvectors will be generated by the solutions of

$$\mathcal{A}_i^2\, \tilde{x} = 0 \qquad \text{and} \qquad \mathcal{B}_j^2\, \tilde{y} = 0$$

which are not eigenvectors. With these choices, (3.8) becomes (3.7). ■

Lemma 5. *Let A be a square matrix with eigenvalues λ_i and associated eigenvectors $x^{(i)}$. Then the matrix $\mathrm{D}^q A$ has eigenvalues $\mathrm{D}^q\lambda_i$ with associated eigenvectors $\mathrm{D}^q x^{(i)}$ and the matrix $\mathrm{D}^l A\,\mathrm{D}^l$ has eigenvalues $\mathrm{D}^l\lambda_i\,\mathrm{D}^l$ with associated eigenvectors $x^{(i)}$. Moreover the eigenvectors* (3.7) *of $\mathrm{D}^l\lambda_i\,\mathrm{D}^l \oplus \mathrm{D}^q B$ are*

$$z = \mathrm{D}^l\,(\mathcal{A}_i\,\tilde{x}) \otimes \mathrm{D}^q\tilde{y} - \tilde{x}\,\mathrm{D}^{-l} \otimes \mathrm{D}^q\mathcal{B}_j\,\mathrm{D}^q\tilde{y}. \tag{3.9}$$

Proof. We have

$$(\mathrm{D}^q A)\,\mathrm{D}^q x^{(i)} = \mathrm{D}^q(A\,x^{(i)}) = \mathrm{D}^q\lambda_i\,\mathrm{D}^q x^{(i)}$$

and so the eigenvalues of $\mathrm{D}^q A$ are $\mathrm{D}^q\lambda_i$ with associated eigenvectors $\mathrm{D}^q x^{(i)}$. Furthermore

$$\left(\mathrm{D}^l(A)\,\mathrm{D}^l\right)x^{(i)} = \mathrm{D}^l\left(A\,x^{(i)}\right) = \mathrm{D}^l\left(\lambda_i\,x^{(i)}\right) = \left(\mathrm{D}^l\lambda_i\,\mathrm{D}^l\right)x^{(i)}$$

and so the eigenvalues of $\mathrm{D}^l(A)\,\mathrm{D}^l$ are $\mathrm{D}^l\lambda_i\,\mathrm{D}^l$ with associated eigenvectors $x^{(i)}$.

Finally note that

$$[\mathrm{D}^l\mathcal{A}_i\,\mathrm{D}^l]\,\mathrm{D}^l\,(\mathcal{A}_i\,\tilde{x}) = 0$$

since $\mathrm{D}^l\,(\mathcal{A}_i\,\tilde{x})$ is an eigenvector of $\mathrm{D}^l\mathcal{A}_i\,\mathrm{D}^l$ and

$$\mathrm{D}^q\mathcal{B}_j\,\mathrm{D}^q\mathcal{B}_j\,\mathrm{D}^q\tilde{y} = \mathrm{D}^q(\mathcal{B}_j^2\,\tilde{y}) = 0.$$

Therefore

$$[\mathrm{D}^l\mathcal{A}_i\,\mathrm{D}^l \oplus \mathrm{D}^q\mathcal{B}_j]\,z = \mathrm{D}^l(\mathcal{A}_i\,\tilde{x}) \otimes \mathrm{D}^q\mathcal{B}_j\,\mathrm{D}^q\tilde{y} - \mathrm{D}^l(\mathcal{A}_i\,\tilde{x}) \otimes \mathrm{D}^q\mathcal{B}_j\,\mathrm{D}^q\tilde{y} = 0.$$

■

For the Toda lattice, the eigenvalues are all zero. The eigenvectors are $\begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and the generalized eignvectors are $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$, $\begin{bmatrix} 0 & 1 \end{bmatrix}^T$ for

$$\mathcal{A} = \left(\frac{\partial f}{\partial \mathrm{D}^{-1}w}\right)^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and

$$\mathcal{B} = \mathrm{D}^q\left(\frac{\partial f}{\partial \mathrm{D}w}\right)^T = \begin{bmatrix} 0 & -\mathrm{D}^q v \\ 0 & 0 \end{bmatrix}$$

respectively. The eigenvalue 0 of $\mathcal{S}$ will have two eigenvectors

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and, from (3.9),

$$\mathrm{D}\left(\mathcal{A}\begin{bmatrix} c \\ 0 \end{bmatrix}\right)\otimes\begin{bmatrix} 0 \\ 1 \end{bmatrix}-\begin{bmatrix} c \\ 0 \end{bmatrix}\mathrm{D}^{-1}\otimes\mathcal{B}\begin{bmatrix} 0 \\ 1 \end{bmatrix}=\begin{bmatrix} c\,\mathrm{D}^{q-1}v \\ 0 \\ 0 \\ \mathrm{D}c \end{bmatrix}.$$

Note that the presence of the shift operator in the first factor necessiates that the "constant" c, which could be a function of w and its shifts, be included explicitly in the generalized eigenvector $\tilde{x}$. The resultant integrability conditions are clearly the same as (3.6).

We are now in a position to complete the proof of Theorem 1. Returning to (3.4), let λ_i, $x^{(i)}$ and μ_i, $y^{(i)}$ be the eigenvalues and associated eigenvectors of $\left(\dfrac{\partial f}{\partial \mathrm{D}^{-L}w}\right)^T$ and $\left(\dfrac{\partial f}{\partial \mathrm{D}^{L}w}\right)^T$ respectively. The eigenvalues of $\mathcal{S}$ are

$$\mathrm{D}^L\lambda_i\,\mathrm{D}^L+\mathrm{D}^q\mu_j. \tag{3.10}$$

For (3.4) to have a non-trivial solution, at least one of the eigenvalues (3.10) must have a non-trivial kernel in which case

$$X=c\,z \tag{3.11}$$

where z is an eigenvector associated with the eigenvalue and

$$\left[\mathrm{D}^L\lambda_i\,\mathrm{D}^L+\mathrm{D}^q\mu_j\right]c=0$$

will be a solution to (3.4).

Note that $\mathcal{S}$ will have a zero eigenvalue if and only if *both* $\dfrac{\partial f}{\partial \mathrm{D}^{-L}u}$ and $\dfrac{\partial f}{\partial \mathrm{D}^{L}u}$ have zero eigenvalues. In this case (3.10) is trivial and (3.11) will be a solution for arbitrary c. On the other hand, if either λ_i or μ_j is zero (but not both) then (3.10) has a trivial kernel.

Suppose neither λ_i nor μ_j are zero and $q=pL+r$ with $p,\ r$ integers, $0\le r\le L-1$. Let

$$c=\left(\prod_{k=1}^{p-1}\mathrm{D}^{kL}\lambda_i\right)\mathrm{D}^{pL}\zeta$$

and so

$$\begin{aligned}\left[\mathrm{D}^L\lambda_i\,\mathrm{D}^L+\mathrm{D}^q\mu_j\right]c&=\mathrm{D}^L\lambda_i\left(\prod_{k=2}^{p}\mathrm{D}^{kL}\lambda_i\right)\mathrm{D}^{(p+1)L}\zeta+\mathrm{D}^{pL+r}\mu_j\left(\prod_{k=1}^{p-1}\mathrm{D}^{kL}\lambda_i\right)\mathrm{D}^{pL}\zeta\\&=\prod_{k=1}^{p-1}\mathrm{D}^{kL}\lambda_i\left(\mathrm{D}^{pL}\lambda_i\,\mathrm{D}^{(p+1)L}\zeta+\mathrm{D}^{pL+r}\mu_j\,\mathrm{D}^{pL}\zeta\right)\\&=\prod_{k=1}^{p-1}\mathrm{D}^{kL}\lambda_i\,\mathrm{D}^{pL}\left(\lambda_i\,\mathrm{D}^L\zeta+\zeta\,\mathrm{D}^r\mu_j\right).\end{aligned}$$

Therefore c will be in the kernel of (3.10) if and only if

$$\zeta\, \mathrm{D}^r \mu_j = \; - \lambda_i\, \mathrm{D}^L \zeta. \tag{3.12}$$

This completes to proof of Theorem 1.

Corollary 6. *Suppose that*

$$\zeta\, \mathrm{D}^r \mu_j = \; - \lambda_i\, \mathrm{D}^L \zeta$$

has a non-zero solution, ζ. Then

$$\mathrm{D}^L \lambda_j\, \mathrm{D}^L + \mathrm{D}^{mL+r} \mu_i$$

will have a one dimensional kernel generated by

$$c = \left(\prod_{k=1}^{m-1} \mathrm{D}^{kL} \lambda_j \right) \mathrm{D}^{mL} \zeta. \tag{3.13}$$

for each $m = 0,\ 1,\ 2, \ldots$.

Proof. Note that if (3.12) has a non-zero solution for some r then, by the proof of Theorem 1, (3.13) will lie in the kernel of

$$\mathrm{D}^L \lambda_j\, \mathrm{D}^L + \mathrm{D}^q \mu_i$$

for any $q = mL + r$, $m = 0,\ 1,\ 2, \ldots$. It remains to show that the kernel is one-dimensional. Let $\lambda_i = \alpha\, \zeta$ for some α. Then $\mathrm{D}^r \mu_j = -\alpha\, \mathrm{D}^L \zeta$. Moreover, since the eigenvalues are functions of $\mathrm{D}^{-L} w,\ \ldots,\ \mathrm{D}^L w$ then $\alpha = \alpha(\mathrm{D}^{-L+r} w,\ \ldots,\ \mathrm{D}^L w)$ and $\zeta = \zeta(\mathrm{D}^{-L} w,\ \ldots,\ \mathrm{D}^r w)$. Suppose ζ' is another non-zero solution of (3.12) then

$$- \zeta' \alpha\, \mathrm{D}^L \zeta = -\alpha\, \zeta\, \mathrm{D}^L \zeta'$$

and so

$$\frac{\zeta'}{\zeta} = \mathrm{D}^L \left(\frac{\zeta'}{\zeta} \right).$$

Thus $\zeta' = a\zeta$ for some constant a (since $L \neq 0$). Therefore the kernel is 1-dimensional. ■

4 Examples

Bogoyavlenskii Lattice

The Bogoyavlenskii lattice [4] is a given by [14, Eq. (17.1.2)]

$$\dot{u} = u \left(\prod_{j=1}^{p} \mathrm{D}^j u - \prod_{j=1}^{p} \mathrm{D}^{-j} u \right)$$

is a generalization of the Kac-van Moerbeke lattice [9, 8]

$$\dot{u} = u(\mathrm{D}u - \mathrm{D}^{-1}u).$$

Here $L = p$ and

$$\frac{\partial f}{\partial \mathrm{D}^{-p}u} = \lambda = -\prod_{j=0}^{p-1} \mathrm{D}^{-j}u$$

$$\frac{\partial f}{\partial \mathrm{D}^{p}u} = \mu = \prod_{j=0}^{p-1} \mathrm{D}^{j}u.$$

Therefore, for a density that depends on $q > p$ shifts, the leading integrability conditions (3.4) are given by

$$\mathcal{S}\,\frac{\partial^2 \rho}{\partial u\,\partial \mathrm{D}^q u} = 0$$

with

$$\mathcal{S} = \mathrm{D}^p \lambda\, \mathrm{D}^p + \mathrm{D}^q \mu = -\left(\prod_{j=0}^{p-1} \mathrm{D}^{p-j}u\right)\mathrm{D}^p + \prod_{j=0}^{p-1} \mathrm{D}^{q+j}u.$$

The kernel of $\mathcal{S}$ will be generated by the solution of (3.12)

$$\zeta \prod_{j=0}^{p-1} \mathrm{D}^{r+j}u = \left(\prod_{j=0}^{p-1} \mathrm{D}^{-j}u\right)\mathrm{D}^p \zeta$$

for $r = 0,\ 1,\ \ldots,\ p-1$ which is

$$\zeta = \prod_{j=-(p-1)}^{r-1} \mathrm{D}^j u.$$

Thus the kernel is generated by (with $q = mp + r$)

$$c = \left(\prod_{k=1}^{m-1} \mathrm{D}^{kp}\lambda\right)\mathrm{D}^{mp}\zeta = (-1)^{m-1}\prod_{k=1}^{q-1} \mathrm{D}^k u.$$

Therefore the density may be choosen so that

$$\frac{\partial^2 \rho}{\partial u\,\partial \mathrm{D}^q u} = \prod_{k=1}^{q-1} \mathrm{D}^k u$$

and so, if it exists, has the form

$$\rho = \prod_{k=0}^{q} \mathrm{D}^k u + \rho^{(1)}(u,\ \mathrm{D}u,\ \ldots,\ \mathrm{D}^{q-1}u).$$

For the case $p = 2$, the lowest order densities with $q > 2$ are

$$\begin{aligned}\rho_3 &= u\,\mathrm{D}u\,\mathrm{D}^2u\,\mathrm{D}^3u + u\,(\mathrm{D}u)^2\,\mathrm{D}^2u + \tfrac{1}{2}\,u^2\,(\mathrm{D}u)^2\\ \rho_5 &= u\,\mathrm{D}u\,\mathrm{D}^2u\,\mathrm{D}^3u\,\mathrm{D}^4u\,\mathrm{D}^5u + u\,(\mathrm{D}u)^2\,\mathrm{D}^2u\,\mathrm{D}^3u\,\mathrm{D}^4u + u\,\mathrm{D}u\,\mathrm{D}^2u\,(\mathrm{D}^3u)^2\,\mathrm{D}^4u\\ &\qquad + (u\,\mathrm{D}u)^2\,\mathrm{D}^2u\,\mathrm{D}^3u + u\,(\mathrm{D}u\,\mathrm{D}^2u)^2\,\mathrm{D}^3u + u\,\mathrm{D}u\,(\mathrm{D}^2u\,\mathrm{D}^3u)^2\\ &\qquad + u^2\,(\mathrm{D}u)^3\,\mathrm{D}^2u + u\,(\mathrm{D}u)^3\,(\mathrm{D}^2u)^2 + \tfrac{1}{3}\,u^3\,(\mathrm{D}u)^3\end{aligned}$$

which illustrate the general form obtained above.

Shifted Modified Volterra Lattice

Consider the modified Volterra Lattice [8] in which the right hand side has been shifted

$$\dot{u} = u^2\,(\mathrm{D}^s u - \mathrm{D}^{-s}u)$$

with $s > 0$. In this case, (3.12) is

$$-\,\zeta\mathrm{D}^r u^2 = \,-\,u^2\,\mathrm{D}^s\zeta.$$

For $r = 0$ this equation has the solution $\zeta = 1$ for all s. However a non-zero solution does not exist for any other $r < s$. Thus this DDE can only have densities for $q = ps$ shifts.

Belov-Chaltikian Lattice

The Belov-Chaltikian lattice [2, Eq. (12)] is a given by

$$\begin{aligned}\dot{u} &= u(\mathrm{D}u - \mathrm{D}^{-1}u) + \mathrm{D}^{-1}u - v\\ \dot{v} &= v(\mathrm{D}^2u - \mathrm{D}^{-1}u)\end{aligned}$$

This is an asymmetric lattice for the vector variable

$$w = \begin{bmatrix} u \\ v \end{bmatrix}.$$

The condition for a density that depends on $q > 2$ shifts to exist is that

$$\frac{\partial f}{\partial \mathrm{D}^2 w} = \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}$$

has a zero eigenvalue. This is clearly the case. The eigenvector (of the transpose) associated with 0 is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and so the eigenvectors of $\mathcal{S}$ associated with 0 are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(remember that the 2×2 zero matrix has two eigenvctors associated with 0). Thus the leading integrability conditions are

$$\frac{\partial^2 \rho}{\partial u\, \partial \mathrm{D}^q v} = \frac{\partial^2 \rho}{\partial v\, \partial \mathrm{D}^q v} = 0;$$

that is, the density must have the form

$$\rho = g(w,\ \mathrm{D}w,\ \ldots,\ \mathrm{D}^{q-1}w)\, \mathrm{D}^q u + \rho^{(1)}(w,\ \mathrm{D}w,\ \ldots,\ \mathrm{D}^{q-1}w).$$

This form is demonstrated in the rank 4 density given in [11].

Blaszak-Marciniak Three Field Lattice I

The Blaszak-Marciniak three field lattice as given in [11, Eq. (2)] is

$$\begin{aligned} \dot{x} &= \mathrm{D}z - \mathrm{D}^{-1}z \\ \dot{y} &= \mathrm{D}^{-1}x\, \mathrm{D}^{-1}z - xz \\ \dot{z} &= z\,(y - \mathrm{D}y). \end{aligned} \tag{4.1}$$

Let

$$w = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and thus

$$A = \left(\frac{\partial f}{\partial \mathrm{D}^{-1} w}\right)^T = \begin{bmatrix} 0 & \mathrm{D}^{-1}z & 0 \\ 0 & 0 & 0 \\ -1 & \mathrm{D}^{-1}x & 0 \end{bmatrix}$$

$$B = \left(\frac{\partial f}{\partial \mathrm{D} w}\right)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -z \\ 0 & 0 & 0 \end{bmatrix}.$$

A has a triple eigenvalue 0 with a single eigenvector $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. The generalized eigenvector $\tilde{x} = \begin{bmatrix} c & 0 & 0 \end{bmatrix}^T$ is the solution of $A^2 x = 0$. B has double eigenvalue 0 with only one eigenvector $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ a simple eigenvalue 1. The generalized eigenvector $\tilde{y} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ is the solution of $B^2 x = 0$. Therefore the Kronecker sum $\mathcal{S}$ has a 6-fold eigenvalue 0 and a triple eigenvalue 1. The integrability conditions will be given by the eigenvectors associated with 0. There are two such eigenvectors; one generated by the eigenvectors

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and one generated by the generalized eigenvectors via (3.9)

$$\mathrm{D}(A\,\tilde{x}) \otimes \mathrm{D}^q\tilde{y} - \tilde{x}\,\mathrm{D}^{-1} \otimes \mathrm{D}^q B\,\mathrm{D}^q\tilde{y} = \begin{bmatrix} 0 \\ 0 \\ -\mathrm{D}c \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ -\mathrm{D}^{q-1}z \\ 0 \end{bmatrix}.$$

Therefore the leading integrability conditions are

$$\begin{aligned} \frac{\partial^2\rho}{\partial x\,\partial \mathrm{D}^q y} &= -\,c_1\,\mathrm{D}^{q-1}z \\ \frac{\partial^2\rho}{\partial z\,\partial \mathrm{D}^q y} &= c_2 \\ \frac{\partial^2\rho}{\partial z\,\partial \mathrm{D}^q z} &= -\,\mathrm{D}c_1 \end{aligned}$$

with all other second order derivatives zero. The rank 4 density for this lattice [11, 18] is (after shifts)

$$\tfrac{1}{4}\,y^4 + \tfrac{1}{2}\,x^2\,z^2 + (x\,\mathrm{D}x - y - \mathrm{D}y - \mathrm{D}^2y)\,z\,\mathrm{D}z + x\,y\,\mathrm{D}y\,z + x\,(\mathrm{D}y)^2\,z + x\,y^2\,z$$

which satisfies the leading integrablity conditions with $c_1 = 0$ and $c_2 = -\,\mathrm{D}z$.

In [3, Eq. (3.23)] this lattice is given by

$$\begin{aligned} \dot{x} &= \mathrm{D}^2 z - z \\ \dot{y} &= x\,\mathrm{D}z - z\,\mathrm{D}^{-1}x \\ \dot{z} &= z\,(y - \mathrm{D}^{-1}y) \end{aligned}$$

(the form (4.1) may be obtained by $y \mapsto -y$ and $z \mapsto \mathrm{D}^{-1}z$). In this form, the lattice is asymmetric and so the leading integrability conditions are determined by

$$\frac{\partial f}{\partial \mathrm{D}^2 w} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The transpose of this matrix has a triple eigenvalue 0 with two eigenvectors. However this yields six eigenvectors associated with 0 for $\mathcal{S}$. Therefore the resultant intergrability conditions give only three zero derivatives; they are a subset of those derived for the symmetric form.

5 Conclusion

There is an extensive literature on the subject of integrable differential-difference equations. Most of this literature is focused on the construction of integrable systems. For a symmetry approach see the work of Yamilov and his coworkers [1, 12, 13, 17] and papers cited therein, [14] for an exhaustive discussion of the Hamiltonian approach and [15] for Jacobi operator approach. In contrast, this paper examines the integrability or lack of integrability of a lattice directly.

This approach is useful to compute densities and their associated flux for systems that are not completely integrable. With the initial split, not only a candidate for the density is constructed but its associated flux is simultaneously updated. This would avoid the need for the use of discrete homotopy operators [6] to compute fluxes if a density was determined purely from an application of the Euler operator. An additional advantage is the relative ease with which the leading conditions may be integrated.

Lower order integrability conditions may also be found by this approach. Integration of the leading conditions splits the candidate density

$$\rho = \tilde{\rho} + \rho^{(1)}(w,\ \mathrm{D}w,\ \ldots,\ \mathrm{D}^{q-1}w).$$

The next condition is similar to (3.5) but with a non-zero right hand side

$$\mathrm{D}^L\left(\frac{\partial f}{\partial \mathrm{D}^{-L}w}\right)^T \mathrm{D}^l\,\frac{\partial^2\rho^{(1)}}{\partial w\,\partial \mathrm{D}^{q-1}w} + \frac{\partial^2\rho^{(1)}}{\partial w\,\partial \mathrm{D}^{q-1}w}\,\mathrm{D}^{q-1}(\frac{\partial f}{\partial \mathrm{D}^l w}) = \mathcal{K}(\tilde{\rho})$$

for $q-1 > L$. The left hand side again has the structure of a Kronecker sum. Now we require that $\mathcal{K}(\tilde{\rho})$ lie in the column space of this Kronecker sum. Integration of this conditions yields a split of $\rho^{(1)}$. A similar form is shared by the integrability conditions for $q-s > L$. However the computation of the right hand sides for specific equations quickly becomes a task for an algebraic program such as Maple or Mathematica and there appears little to gain from attempting these calculations for the general case. Term explosion is polynomial in nature. Thus the resources required are on a par with Lie symmetry programs.

Another approach to the computation of densities is described in [5]. In this approach it is assumed that a scaling symmetry is present and that the densities depend polynomially on w and its shifts. Under these assumptions, they reduce the determining equations for a density to a (potentially, very large) linear algebraic system of equations. Part of the problem here is that the shifts of a variable all have the same weight under the scaling symmetry. They have no way *a priori* to limit the shifts in their candidate density of a given rank. However one can now limit the number of shifts required by computing the rank of $\tilde{\rho}$ which then gives the minimum rank that requires q shifts. This, in turn, reduces the size of the system of linear equations that determine the density.

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