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# Particle Trajectories in Linearized Irrotational Shallow Water Flows

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**Abstract**

We investigate the particle trajectories in an irrotational shallow water flow over a flat bed as periodic waves propagate on the water's free surface. Within the linear water wave theory, we show that there are no closed orbits for the water particles beneath the irrotational shallow water waves. Depending on the strength of underlying uniform current, we obtain that some particle trajectories are undulating path to the right or to the left, some are looping curves with a drift to the right and others are parabolic curves or curves which have only one loop.

## 1 Introduction

The motion of water particles under the waves which advance across the water is a very old problem. The classical description of these particle paths is obtained within the framework of linear water wave theory. After the linearization of the governing equations for water waves, the ordinary differential equations system which describes the particle motion turns out to be again nonlinear and explicit solutions are not available. In the first approximation of this nonlinear system, one obtained that all water particles trace closed, circular or elliptic, orbits (see, for example, [13], [16], [20], [21], [22], [23]), a conclusion apparently supported by photographs with long exposure ([13], [22], [23]). Consequently there is no net transfer of material particles due to the passage of the wave, at least, at this order of approximation.

While in these approximations of the nonlinear system all particle paths appear to be closed, in [10] it is shown, using phase-plane considerations, that in linear periodic gravity water waves no particles trajectory is actually closed, unless the free surface is flat. Each particle trajectory involves over a period a backward/forward movement, and the path is an elliptical arc with a forward drift; on the flat bed the particle path degenerates to a backward/forward motion.

Similar results hold for the particle trajectories in deep-water, that is, the trajectories are not closed existing a forward drift over a period, which decreases with greater depth (see [5]). These conclusions are in agreement with Stokes' observation [24]: "There is one result of a second approximation which may possible importance. It appears that the

forward motion of the particles is not altogether compensated by their backward motion; so that, in addition to their motion of oscillation, the particles have a progressive motion in the direction of the propagation of the waves. In the case in which the depth of the fluid is very great, this progressive motion decreases rapidly as the depth of the particle considered increases."
For shallow water waves, the standard results are that the orbits described by water particles beneath waves are elongated ellipses with the longer axis parallel to the flat bottom, and at the bottom the orbits are straight lines (see, for example, [18]).

Similar conclusions hold for the governing equations without linearization. Analyzing a free boundary problem for harmonic functions in a planar domain, in [4] it is shown that there are no closed orbits for Stokes waves of small or large amplitude propagating at the surface of water over a flat bed; for an extension of the investigation in [4] to deep-water Stokes waves see [15]. Within a period each particle experiences a backward/forward motion with a slight forward drift. In a very recent preprint [9], the results in [4] are recovered by a simpler approach and there are also described all possible particle trajectories beneath a Stokes wave. The particle trajectories change considerably according to whether the Stokes waves enter a still region of water or whether they interact with a favorable or adverse uniform current. Some particle trajectories are closed orbits, some are undulating paths and most are looping orbits that drift either to the right or to the left, depending on the underlying current.
Analyzing a free boundary problem for harmonic functions in an infinite planar domain, in [6] it is shown that under a solitary wave, each particle is transported in the wave direction but slower than the wave speed. As the solitary wave propagates, all particles located ahead of the wave crest are lifted while those behind have a downward motion.

Notice that there are only a few explicit solutions to the nonlinear governing equations: Gerstner's wave (see [14] and the discussion in [2])), the edge wave solution related to it (see [3]), and the capillary waves in water of infinite or finite depth (see [12], [19]). These solutions are peculiar and their special features (a specific vorticity for Gerstner's wave and its edge wave correspondent, and complete neglect of gravity in the capillary case) are not deemed relevant to sea waves.

The present paper is concerned with the particle trajectories in an irrotational shallow water flow over a flat bed as a periodic wave propagates on the water's free surface. It is natural to start this investigation for shallow water waves by simplifying the governing equations via linearization. In Section 2 we recall the governing equations for water waves. In Section 3 we present their nondimensionalisation and scaling. The linearized problem in the irrotational shallow water regime is written in Section 4. We also obtain the general solution of this problem. The next section is devoted to the description of all the possible particle trajectories beneath a linear periodic irrotational shallow water wave. We see that these particle trajectories are not closed. Depending on the strength of underlying uniform current, denoted by the constant $c_0$, we obtain that: for $c_0 > 2$ the particle trajectories are undulating path to the right, for $c_0 < -1$ the particle trajectories are undulating path to the left, for $-1 \leq c_0 < 0$ the particle trajectories are looping curves with a drift to the right and for $0 \leq c_0 \leq 2$ the particle trajectories are parabolic curves or curves which have only one loop.

## 2 The governing equations for gravity water waves

We consider a two-dimensional inviscid incompressible fluid in a constant gravitational field. For gravity water waves these are physically reasonable assumptions (see [16] and [21]). Thus, the motion of water is given by Euler's equations

$$\begin{aligned} u_t + uu_x + vu_z &= -\frac{1}{\rho}p_x \\ v_t + uv_x + vv_z &= -\frac{1}{\rho}p_z - g \end{aligned} \tag{2.1}$$

Here $(x, z)$ are the space coordinates, $(u(x, z, t), v(x, z, t))$ is the velocity field of the water, $p(x, z, t)$ denotes the pressure, $g$ is the constant gravitational acceleration in the negative $z$ direction and $\rho$ is the constant density. The assumption of incompressibility implies the equation of mass conservation

$$u_x + v_z = 0 \tag{2.2}$$

Let $h_0 > 0$ be the undisturbed depth of the fluid and let $z = h_0 + \eta(x, t)$ represent the free upper surface of the fluid (see Figure 1).

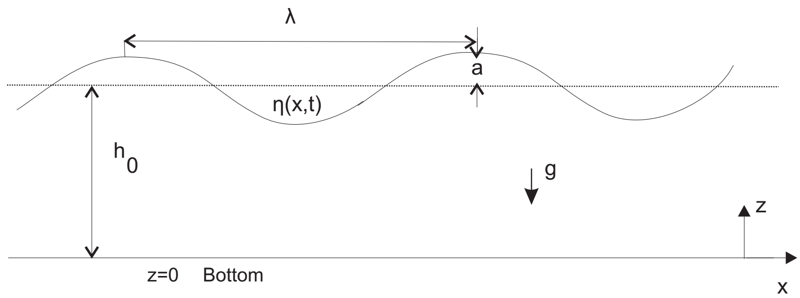


Figure 1. A periodic water wave propagating over a flat bed.

The boundary conditions at the free surface are constant pressure

$$p = p_0 \ \text{ on } z = h_0 + \eta(x, t), \tag{2.3}$$

$p_0$ being the constant atmospheric pressure, and the continuity of fluid velocity and surface velocity

$$v = \eta_t + u\eta_x \ \text{ on } \ z = h_0 + \eta(x, t) \tag{2.4}$$

On the flat bottom $z = 0$, only one condition is required for an inviscid fluid, that is,

$$v = 0 \ \text{ on } z = 0 \tag{2.5}$$

Summing up, the exact water-wave problem is given by the system (2.1)-(2.5). In respect of the well-posedness for the initial-value problem for (2.1)-(2.5) there has been significant recent progress, see [11] and the references therein.

A key quantity in fluid dynamics is the *curl* of the velocity field, called vorticity. For two-dimensional flows with the velocity field $(u(x, z, t), v(x, z, t))$, we denote the scalar vorticity of the flow by

$$\omega(x, z) = u_z - v_x \tag{2.6}$$

Vorticity is adequate for the specification of a flow: a flow which is uniform with depth is described by a zero vorticity (irrotational case), constant non-zero vorticity corresponds to a linear shear flow and non-constant vorticity indicates highly sheared flows.

The full Euler equations (2.1)-(2.5) are often too complicated to analyze directly. One can pursue for example a mathematical study of their periodic steady solutions in the irrotational case (see [1], [25]) or a study of their periodic steady solutions in the case of non-zero vorticity (see [7], [8]). But in order to reach detailed information about qualitative features of water waves, it is useful to derive approximate models which are more amenable to an in-depth analysis.

## 3 Nondimensionalisation and scaling

In order to develop a systematic approximation procedure, we need to characterize the water-wave problem (2.1)-(2.5) in terms of the sizes of various fundamental parameters. These parameters are introduced by defining a set of non-dimensional variables.
First we introduce the appropriate length scales: the undisturbed depth of water $h_0$, as the vertical scale and a typical wavelength $\lambda$ (see Figure 1), as the horizontal scale. In order to define a time scale we require a suitable velocity scale. An appropriate choice for the scale of the horizontal component of the velocity is $\sqrt{gh_0}$. Then, the corresponding time scale is $\frac{\lambda}{\sqrt{gh_0}}$ and the scale for the vertical component of the velocity is $h_0\frac{\sqrt{gh_0}}{\lambda}$. The surface wave itself leads to the introduction of a typical amplitude of the wave $a$ (see Figure 1). For more details see [16]. Thus, we define the set of non-dimensional variables

$$\begin{gathered} x \mapsto \lambda x, \quad z \mapsto h_0 z, \quad \eta \mapsto a\eta, \quad t \mapsto \frac{\lambda}{\sqrt{gh_0}}t, \\ u \mapsto \sqrt{gh_0}u, \quad v \mapsto h_0\frac{\sqrt{gh_0}}{\lambda}v \end{gathered} \tag{3.1}$$

where, to avoid new notations, we have used the same symbols for the non-dimensional variables $x$, $z$, $\eta$, $t$, $u$, $v$, on the right-hand side. The partial derivatives will be replaced by

$$\begin{aligned} &u_t \mapsto \frac{gh_0}{\lambda}u_t, \quad u_x \mapsto \frac{\sqrt{gh_0}}{\lambda}u_x, \quad u_z \mapsto \frac{\sqrt{gh_0}}{h_0}u_z, \\ &v_t \mapsto \frac{gh_0^2}{\lambda^2}v_t, \quad v_x \mapsto h_0\frac{\sqrt{gh_0}}{\lambda^2}v_x, \quad v_z \mapsto \frac{\sqrt{gh_0}}{\lambda}v_z \end{aligned} \tag{3.2}$$

Let us now define the non-dimensional pressure. If the water would be stationary, that is, $u \equiv v \equiv 0$, from the equations (2.1) and (2.3) with $\eta = 0$, we get for a non-dimensionalised $z$, the hydrostatic pressure $p_0 + \rho g h_0(1-z)$. Thus, the non-dimensional pressure is defined by

$$p \mapsto p_0 + \rho g h_0(1-z) + \rho g h_0 p \tag{3.3}$$

therefore

$$p_x \mapsto \rho\frac{gh_0}{\lambda}p_x, \quad p_z \mapsto -\rho g + \rho g p_z \tag{3.4}$$

Taking into account (3.1), (3.2), (3.3) and (3.4), the water-wave problem (2.1)-(2.5) writes in non-dimensional variables, as

$$\begin{gathered} u_t + uu_x + vu_z = -p_x \\ \delta^2(v_t + uv_x + vv_z) = -p_z \\ u_x + v_z = 0 \\ v = \epsilon(\eta_t + u\eta_x) \;\text{ and }\; p = \epsilon\eta \;\text{ on }\; z = 1 + \epsilon\eta(x,t) \\ v = 0 \;\text{ on }\; z = 0 \end{gathered} \tag{3.5}$$

where we have introduced the amplitude parameter $\epsilon = \frac{a}{h_0}$ and the shallowness parameter $\delta = \frac{h_0}{\lambda}$. In view of (3.2), the vorticity equation (2.6) writes in non-dimensional variables as

$$u_z = \delta^2 v_x + \frac{\sqrt{gh_0}}{g}\omega(x,z) \tag{3.6}$$

For zero vorticity flows (irrotational flows) this equation writes as

$$u_z = \delta^2 v_x \tag{3.7}$$

After the nondimensionalisation of the system (2.1)-(2.5) let us now proceed with the scaling transformation. First we observe that, on $z = 1+\epsilon\eta$, both $v$ and $p$ are proportional to $\epsilon$. This is consistent with the fact that as $\epsilon \to 0$ we must have $v \to 0$ and $p \to 0$, and it leads to the following scaling of the non-dimensional variables

$$p \mapsto \epsilon p, \quad (u,v) \mapsto \epsilon(u,v) \tag{3.8}$$

where we avoided again the introduction of a new notation. The problem (3.5) becomes

$$\begin{gathered} u_t + \epsilon(uu_x + vu_z) = -p_x \\ \delta^2[v_t + \epsilon(uv_x + vv_z)] = -p_z \\ u_x + v_z = 0 \\ v = \eta_t + \epsilon u\eta_x \;\text{ and }\; p = \eta \;\text{ on }\; z = 1 + \epsilon\eta(x,t) \\ v = 0 \;\text{ on }\; z = 0 \end{gathered} \tag{3.9}$$

and the equation (3.6) keeps the same form.
The system which describes our problem in the irrotational case is given by

$$\begin{gathered} u_t + \epsilon(uu_x + vu_z) = -p_x \\ \delta^2[v_t + \epsilon(uv_x + vv_z)] = -p_z \\ u_x + v_z = 0 \\ u_z = \delta^2 v_x \\ v = \eta_t + \epsilon u\eta_x \;\text{ and }\; p = \eta \;\text{ on }\; z = 1 + \epsilon\eta(x,t) \\ v = 0 \;\text{ on }\; z = 0 \end{gathered} \tag{3.10}$$

## 4 The linearized problem

The two important parameters $\epsilon$ and $\delta$ that arise in water-waves theories, are used to define various approximations of the governing equations and the boundary conditions. The scaled version (3.10) of the equations for our problem, allows immediately the identification of the linearized problem, by letting $\epsilon \to 0$, for arbitrary $\delta$. The linearized problem in the shallow water regime is obtain by letting further $\delta \to 0$. Thus, in the irrotational case, we get the following linear systems

$$\begin{aligned} u_t + p_x &= 0 \\ p_z &= 0 \\ u_x + v_z &= 0 \\ u_z &= 0 \\ v = \eta_t \ \text{ and } \ p = \eta & \ \text{ on } \ z = 1 \\ v = 0 & \ \text{ on } \ z = 0 \end{aligned} \tag{4.1}$$

From the second equation in (4.1) we get in the both cases that $p$ does not depend on $z$. Because $p = \eta(x,t)$ on $z = 1$, we have

$$p = \eta(x,t) \qquad \text{for any } \ 0 \leq z \leq 1 \tag{4.2}$$

Therefore, using the first equation and the fourth equation in (4.1), we obtain in the irrotational case

$$u = -\int_0^t \eta_x(x,s)ds + \mathcal{F}(x) \tag{4.3}$$

where $\mathcal{F}$ is an arbitrary function such that

$$\mathcal{F}(x) = u(x,0) \tag{4.4}$$

Differentiating (4.3) with respect to $x$ and using the third equation in (4.1) we get, after an integration against $z$,

$$v = -zu_x = z\left(\int_0^t \eta_{xx}(x,s)ds - \mathcal{F}'(x)\right) \tag{4.5}$$

In view of the fifth equation in (4.1) we get after a differentiation with respect to $t$, that $\eta$ has to satisfy the equation

$$\eta_{tt} - \eta_{xx} = 0 \tag{4.6}$$

The general solution of this equation is $\eta(x,t) = f(x-t) + g(x+t)$, where $f$ and $g$ are differentiable functions. It is convenient first to restrict ourselves to waves which propagate in only one direction, thus, we choose

$$\eta(x,t) = f(x-t) \tag{4.7}$$

From (4.5), (4.7) and the condition $v = \eta_t$ on $z = 1$, we obtain

$$\mathcal{F}(x) = f(x) + c_0 \tag{4.8}$$

where $c_0$ is constant.
Therefore, in the irrotational case, taking into account (4.2), (4.3), (4.5), (4.7) and (4.8), the solution of the linear system (4.1) is given by

$$\begin{aligned} &\eta(x,t) = f(x-t) \\ &p(x,t) = f(x-t) \\ &u(x,z,t) = f(x-t) + c_0 \\ &v(x,z,t) = -zf'(x-t) = -zu_x \end{aligned} \tag{4.9}$$

# 5 Particles trajectories in the irrotational case

Let $(x(t), z(t))$ be the path of a particle in the fluid domain, with location $(x(0), z(0))$ at time $t = 0$. The motion of the particle is described by the differential system

$$\begin{cases} \frac{dx}{dt} = u(x, z, t) \\ \frac{dz}{dt} = v(x, z, t) \end{cases} \tag{5.1}$$

with the initial data $(x(0), z(0)) := (x_0, z_0)$.
Making the *Ansatz*

$$f(x - t) = \cos(2\pi(x - t)) \tag{5.2}$$

from (4.9), the differential system (5.1) becomes

$$\begin{cases} \frac{dx}{dt} = \cos(2\pi(x - t)) + c_0 \\ \\ \frac{dz}{dt} = 2\pi z \sin(2\pi(x - t)) \end{cases} \tag{5.3}$$

Notice that the constant $c_0$ is the average of the horizontal fluid velocity over any horizontal segment of length 1, that is,

$$c_0 = \frac{1}{1} \int_x^{x+1} u(s, z, t) ds, \tag{5.4}$$

representing therefore the strength of the underlying uniform current. Thus, $c_0 = 0$ will correspond to a region of still water with no underlying current, $c_0 > 0$ will characterize a favorable uniform current and $c_0 < 0$ will characterize an adverse uniform current.

The right-hand side of the differential system (5.3) is smooth and bounded, therefore, the unique solution of the Cauchy problem with initial data $(x_0, z_0)$ is defined globally in time.

To study the exact solution of the system (5.3) it is more convenient to re-write it in the following moving frame

$$X = 2\pi(x - t), \quad Z = z \tag{5.5}$$

This transformation yields

$$\begin{cases} \frac{dX}{dt} = 2\pi \cos(X) + 2\pi(c_0 - 1) \\ \\ \frac{dZ}{dt} = 2\pi Z \sin(X) \end{cases} \tag{5.6}$$

Let us now investigate the differential system (5.6).

## 5.1 The case $\mathbf{c_0 = 0}$

The horizontal component of the velocity $u$ in (4.9), with $f(x - t)$ given by (5.2), has in the moving frame (5.5), the following expression

$$u(X, Z, t) = \cos(X) + c_0 \tag{5.7}$$

Thus, the case $c_0 = 0$ is obtained for

$$\int_0^{2\pi} u(X, Z, t) dX = 0 \tag{5.8}$$

This is the Stokes condition for irrotational flows, that is, the horizontal velocity has a vanishing mean over a period.
In the considered case, we write the first equation of the system (5.6) into the form

$$\int \frac{dX}{\cos(X) - 1} = 2\pi t \tag{5.9}$$

We use the following substitution (see [17], I.76, page 308)

$$\sin(X) = \frac{2y}{y^2 + 1}, \quad \cos(X) = \frac{y^2 - 1}{y^2 + 1}, \quad dX = -\frac{2}{y^2 + 1} dy \tag{5.10}$$

In the new variable, (5.9) integrates at

$$y = 2\pi t + k \tag{5.11}$$

k being an integration constant. Hence,

$$X(t) = 2\text{arccot } (2\pi t + k) \tag{5.12}$$

Taking into account (5.10), (5.11), we obtain

$$\sin(X(t)) = \frac{2(2\pi t + k)}{1 + (2\pi t + k)^2} \tag{5.13}$$

Therefore, the second equation in (5.6) yields

$$Z(t) = Z(0) \exp\left( \int_0^t 2\pi \sin(X(s))\, ds \right) = Z(0) \exp\left( \ln \left[ \frac{1 + (2\pi t + k)^2}{1 + k^2} \right] \right) \tag{5.14}$$

From (5.5), (5.12) and (5.14), we obtain that the solution of the system (5.3), with the initial data $(x_0, z_0)$, has the following expression

$$\begin{cases} x(t) = t + \frac{1}{\pi}\text{arccot } (2\pi t + k) \\ \\ z(t) = \frac{z_0}{1+k^2}[1 + (2\pi t + k)^2] \end{cases} \tag{5.15}$$

From the initial conditions, we get $k := \cot\ (\pi x_0)$.
The derivatives of $x(t)$ and $z(t)$ with respect to $t$, have the expressions

$$x'(t) = \frac{(2\pi t+k)^2 - 1}{1+(2\pi t+k)^2}$$

$$z'(t) = \frac{4\pi z_0}{1+k^2}(2\pi t + k)$$

Therefore

$$x'(t) > 0 \iff |2\pi t + k| > 1$$
$$z'(t) > 0 \iff (2\pi t + k) > 0$$

the flat bottom being at $z = 0$, we have $z_0 > 0$.
Thus, for $t$ in the intervals $(-\infty, \frac{-1-k}{2\pi})$, $(\frac{-1-k}{2\pi}, -\frac{k}{2\pi})$, $(-\frac{k}{2\pi}, \frac{1-k}{2\pi})$ and $(\frac{1-k}{2\pi}, \infty)$, the derivatives $x'(t)$, $z'(t)$, have the following signs

$$\begin{array}{c|c|c|c|c} t: & \frac{-1-k}{2\pi} & -\frac{k}{2\pi} & \frac{1-k}{2\pi} & \\ \hline x'(t) > 0 & x'(t) < 0 & x'(t) < 0 & x'(t) > 0 & \\ z'(t) < 0 & z'(t) < 0 & z'(t) > 0 & z'(t) > 0 & \end{array} \tag{5.16}$$

The limits of $x(t)$, $z(t)$ and $\frac{z(t)}{x(t)}$ for $t \to -\infty$ and $t \to \infty$ are

$$\begin{aligned} &\lim_{t\to-\infty} x(t) = -\infty, \quad \lim_{t\to\infty} x(t) = \infty \\ &\lim_{t\to-\infty} z(t) = \infty, \quad \lim_{t\to\infty} z(t) = \infty \\ &\lim_{t\to-\infty} \frac{z(t)}{x(t)} = -\infty, \quad \lim_{t\to\infty} \frac{z(t)}{x(t)} = \infty \end{aligned} \tag{5.17}$$

Thus, taking into account (5.16) and (5.17), we sketch below the graph of the parametric curve (5.15)

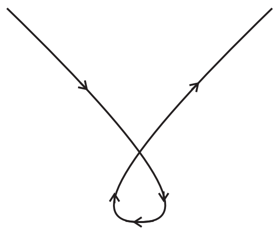

Figure 2. Particle trajectory in the case $c_0$=0

Thus, we get:

**Theorem 1.** *In the case of no underlying current, the particle trajectories beneath the irrotational shallow water waves are curves which have only one loop like in Figure 2.*

## 5.2 The case $\mathbf{c_0(c_0-2) > 0}$

In this case, we write the first equation of the system (5.6) into the form

$$\int \frac{dX}{\cos(X) + c_0 - 1} = 2\pi t \tag{5.18}$$

We use the same substitution (5.10). In the new variable, (5.18) becomes

$$-\frac{2}{c_0} \int \frac{dy}{y^2 + \frac{c_0-2}{c_0}} = 2\pi t \tag{5.19}$$

which integrates at

$$-\frac{2}{c_0}\sqrt{\frac{c_0}{c_0-2}}\arctan\left(\sqrt{\frac{c_0}{c_0-2}}\,y\right)=2\pi t+k \tag{5.20}$$

k being an integration constant. Further, we obtain

$$y=\mathfrak{C}_0\tan(\alpha(t)), \tag{5.21}$$

with

$$\mathfrak{C}_0:=\sqrt{\frac{c_0-2}{c_0}} \tag{5.22}$$

$$\alpha(t):=-\frac{c_0\mathfrak{C}_0}{2}(2\pi t+k) \tag{5.23}$$

Hence, returning to the variable $X$, we get

$$X(t)=2\mathrm{arccot}\left[\mathfrak{C}_0\tan\left(\alpha(t)\right)\right] \tag{5.24}$$

Taking into account (5.10), (5.21), we obtain

$$\sin(X(t))=\frac{2\mathfrak{C}_0\tan\left(\alpha(t)\right)}{1+\left[\mathfrak{C}_0\tan\left(\alpha(t)\right)\right]^2} \tag{5.25}$$

The second equation in (5.6) yields

$$Z(t)=Z(0)\exp\left(\int_0^t 2\pi\sin(X(s))ds\right) \tag{5.26}$$

From (5.5), (5.24) and (5.26), we obtain that the solution of the system (5.3), with the initial data $(x_0,z_0)$, $z_0>0$, has the following expression

$$\begin{cases} x(t)=t+\frac{1}{\pi}\mathrm{arccot}\left[\mathfrak{C}_0\tan\left(\alpha(t)\right)\right] \\ z(t)=z_0\exp\left(\int_0^t\frac{4\pi\mathfrak{C}_0\tan(\alpha(s))}{1+\left[\mathfrak{C}_0\tan(\alpha(s))\right]^2}ds\right) \end{cases} \tag{5.27}$$

The derivatives of $x(t)$ and $z(t)$ with respect to $t$, have the expressions

$$\begin{aligned} x'(t)&=\frac{\left(\mathfrak{C}_0^2-1\right)\sin^2(\alpha(t))+c_0-1}{\cos^2(\alpha(t))\left[1+\mathfrak{C}_0^2\tan^2(\alpha(t))\right]}=\left[\frac{2\left(\sin^2(\alpha(t))-\frac{c_0^2-c_0}{2}\right)}{(-c_0)}\right]\cdot\frac{1}{\cos^2(\alpha(t))\left[1+\mathfrak{C}_0^2\tan^2(\alpha(t))\right]} \\ z'(t)&=z_0\frac{4\pi\mathfrak{C}_0\tan(\alpha(t))}{1+\left[\mathfrak{C}_0\tan(\alpha(t))\right]^2}\exp\left(\int_0^t\frac{4\pi\left[\mathfrak{C}_0\tan(\alpha(s))\right]}{1+\left[\mathfrak{C}_0\tan(\alpha(s))\right]^2}ds\right)\end{aligned} \tag{5.28}$$

Let us now study the signs of the derivatives in (5.28). We are in the case $\mathbf{c_0(c_0-2)>0}$, that is, $c_0 \in (-\infty, 0) \cup (2, \infty)$.

**(a)** If $\mathbf{c_0 < -1}$, then $\frac{c_0^2-c_0}{2} > 1$. Therefore, $\sin^2(\alpha(t)) - \frac{c_0^2-c_0}{2} < 0$. Thus, we obtain that $x'(t) < 0$, for all $t$.
The sign of $z'(t)$ will depend on $\alpha(t)$. For $\alpha(t)$ in intervals of the form $\alpha(t) \in \left(-\frac{\pi}{2} + l\pi, l\pi\right)$, $l \in \mathbb{Z}$, we get $z'(t) < 0$, and for $\alpha(t) \in \left(l\pi, l\pi + \frac{\pi}{2}\right)$, $l \in \mathbb{Z}$, we get $z'(t) > 0$.

We sketch below the particle trajectory in this case:

Figure 3. Particle trajectory in the case $c_0 < -1$

**(b)** If $\mathbf{-1 \leq c_0 < 0}$, then $\frac{c_0^2-c_0}{2} \leq 1$. Thus,
for $\alpha(t) < -\arcsin\left(\sqrt{\frac{c_0^2-c_0}{2}}\right) + l\pi$, we get $x'(t) > 0$, $z'(t) < 0$,
for $\alpha(t) \in \left(-\arcsin\left(\sqrt{\frac{c_0^2-c_0}{2}}\right) + l\pi,\, l\pi\right)$, we get $x'(t) < 0$, $z'(t) < 0$,
for $\alpha(t) \in \left(l\pi,\, \arcsin\left(\sqrt{\frac{c_0^2-c_0}{2}}\right) + l\pi\right)$ we get $x'(t) < 0$, $z'(t) > 0$,
for $\alpha(t) > \arcsin\left(\sqrt{\frac{c_0^2-c_0}{2}}\right) + l\pi$, we get $x'(t) > 0$, $z'(t) > 0$,
where $l \in \mathbb{Z}$. We sketch below the particle trajectory in this case:

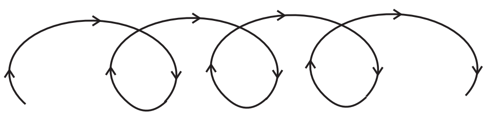

Figure 4. Particle trajectory in the case $-1 \leq c_0 < 0$

**(c)** If $\mathbf{c_0 > 2}$, then $\frac{c_0^2-c_0}{2} > 1$. Therefore, $\sin^2(\alpha(t)) - \frac{c_0^2-c_0}{2} < 0$. Thus, we obtain that $x'(t) > 0$, for all $t$.
The sign of $z'(t)$ will depend on $\alpha(t)$. $z'(t) < 0$ for $\alpha(t) \in \left(-\frac{\pi}{2} + l\pi, l\pi\right)$, $l \in \mathbb{Z}$, and $z'(t) > 0$ for $\alpha(t) \in \left(l\pi, l\pi + \frac{\pi}{2}\right)$, $l \in \mathbb{Z}$.

We sketch below the particle trajectory in this case:

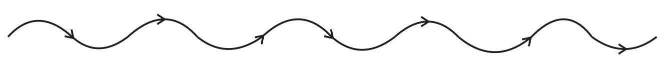

Figure 5. Particle trajectory in the case $c_0 > 2$

Therefore, we proved:

**Theorem 2.** *In the case that the underlying uniform current is moving in the same direction as an irrotational shallow water wave and the strength of the current is bigger than 2, then the particles trajectories beneath the wave are undulating paths to the right (see Figure 5).*

*In the case that the underlying uniform current is moving in the opposite direction as an irrotational shallow water wave and the strength of the current is smaller than -1, then the particles trajectories beneath the wave are undulating paths to the left (see Figure 3). If the strength of the adverse current is bigger than -1, then the particle trajectories are loops with positive drift (see Figure 4).*

### 5.3 The case $\mathbf{c_0} \in (\mathbf{0}, \mathbf{2}]$

In this case, we write the first equation of the system (5.6) into the form

$$\int \frac{dX}{\cos(X) + c_0 - 1} = 2\pi t \tag{5.29}$$

We use the same substitution (5.10). In the new variable, (5.29) becomes

$$-\frac{2}{c_0} \int \frac{dy}{y^2 - \frac{2-c_0}{c_0}} = 2\pi t \tag{5.30}$$

which integrates at

$$-\frac{1}{c_0}\sqrt{\frac{c_0}{2-c_0}} \ln \left| \frac{y - \sqrt{\frac{2-c_0}{c_0}}}{y + \sqrt{\frac{2-c_0}{c_0}}} \right| = 2\pi t + k \tag{5.31}$$

k being an integration constant. Further, we obtain

$$\begin{aligned} y &= \mathfrak{K}_0 \frac{\exp(2\beta(t)) + 1}{\exp(2\beta(t)) - 1} \quad \text{if} \quad |y| > \mathfrak{K}_0\,, \\ y &= \mathfrak{K}_0 \frac{\exp(2\beta(t)) - 1}{\exp(2\beta(t)) + 1} \quad \text{if} \quad |y| < \mathfrak{K}_0\,, \end{aligned} \tag{5.32}$$

where

$$\mathfrak{K}_0 := \sqrt{\frac{2-c_0}{c_0}} \tag{5.33}$$

$$\beta(t) := \frac{c_0 \mathfrak{K}_0}{2}(2\pi t + k) \tag{5.34}$$

Hence, returning to the variable $X$, we get

$$X(t) = 2\text{arccot}\left[\mathfrak{K}_0 \coth\left(\beta(t)\right)\right] \quad \text{or} \quad X(t) = 2\text{arccot}\left[\mathfrak{K}_0 \tanh\left(\beta(t)\right)\right] \tag{5.35}$$

if $|\cot\left(\frac{X}{2}\right)| > \mathfrak{K}_0$, respectively, $|\cot\left(\frac{X}{2}\right)| < \mathfrak{K}_0$.
Taking into account (5.10), (5.32), we obtain

$$\sin(X(t)) = \frac{2\mathfrak{K}_0 \coth(\beta(t))}{1+\left[\mathfrak{K}_0 \coth(\beta(t))\right]^2} \quad \text{or} \quad \sin(X(t)) = \frac{2\mathfrak{K}_0 \tanh(\beta(t))}{1+\left[\mathfrak{K}_0 \tanh(\beta(t))\right]^2} \tag{5.36}$$

Thus, the solution of the system (5.3), with the initial data $(x_0, z_0)$, $z_0 > 0$, has in this case the following expressions

$$\begin{cases} x(t) = t + \frac{1}{\pi}\text{arccot}\left[\mathfrak{K}_0 \coth(\beta(t))\right] \\[2ex] z(t) = z_0 \exp\left(\int_0^t \frac{4\pi\mathfrak{K}_0 \coth(\beta(s))}{1+\left[\mathfrak{K}_0 \coth(\beta(s))\right]^2} ds\right) \end{cases} \tag{5.37}$$

or

$$\begin{cases} x(t) = t + \frac{1}{\pi}\text{arccot}\left[\mathfrak{K}_0 \tanh(\beta(t))\right] \\[2ex] z(t) = z_0 \exp\left(\int_0^t \frac{4\pi\mathfrak{K}_0 \tanh(\beta(s))}{1+\left[\mathfrak{K}_0 \tanh(\beta(s))\right]^2} ds\right) \end{cases} \tag{5.38}$$

We derive $x(t)$ and $z(t)$ from (5.37) with respect to $t$ and we get

$$\begin{cases} x'(t) = 1 + \frac{2-c_0}{\sinh^2(\beta(t))\left[1+\mathfrak{K}_0^2 \coth^2(\beta(t))\right]} \\[2ex] z'(t) = z_0 \frac{4\pi\mathfrak{K}_0 \coth(\beta(t))}{1+\left[\mathfrak{K}_0 \coth(\beta(t))\right]^2} \exp\left(\int_0^t \frac{4\pi\mathfrak{K}_0 \coth(\beta(s))}{1+\left[\mathfrak{K}_0 \coth(\beta(s))\right]^2} ds\right) \end{cases} \tag{5.39}$$

Because we are in the case $c_0 \in (0, 2]$, we have $2 - c_0 > 0$. Thus, the derivative $x'(t) > 0$ for all $t$. The sign of $z'(t)$ depends on the sign of $\beta(t)$, that is, for $\beta(t) < 0$ we have $z'(t) < 0$ and for $\beta(t) > 0$ we have $z'(t) > 0$. Then, the particle trajectory in this case is like in Figure 6 (a).

For the second alternative (5.38), we get

$$\begin{cases} x'(t) = \frac{\left(\mathfrak{K}_0^2+1\right)\sinh^2(\beta(t))+c_0-1}{\cosh^2(\beta(t))\left[1+\mathfrak{K}_0^2 \tanh^2(\beta(t))\right]} = \left[\frac{2\left(\sinh^2(\beta(t))-\frac{c_0-c_0^2}{2}\right)}{c_0}\right] \frac{1}{\cosh^2(\beta(t))\left[1+\mathfrak{K}_0^2 \tanh^2(\beta(t))\right]} \\[2ex] z'(t) = z_0 \frac{4\pi\mathfrak{K}_0 \tanh(\beta(s))}{1+\left[\mathfrak{K}_0 \tanh(\beta(s))\right]^2} \exp\left(\int_0^t \frac{4\pi\mathfrak{K}_0 \tanh(\beta(s))}{1+\left[\mathfrak{K}_0 \tanh(\beta(s))\right]^2} ds\right) \end{cases} \tag{5.40}$$

**(a)** If $\mathbf{1} < \mathbf{c_0} \leq \mathbf{2}$, then we get $x'(t) > 0$, for all $t$.
The sign of $z'(t)$ will depend on $\beta(t)$. For $\beta(t) < 0$, we get $z'(t) < 0$, and for $\beta(t) > 0$, we

get $z'(t) > 0$.
The particle trajectory in this case is like in Figure 6 (a).

**(b)** If $\mathbf{0 < c_0 \leq 1}$, then
for $\sinh(\beta(t)) < -\sqrt{\frac{c_0 - c_0^2}{2}}$, we get $x'(t) > 0$, $z'(t) < 0$,
for $\sinh(\beta(t)) \in \left(-\sqrt{\frac{c_0 - c_0^2}{2}}, 0\right)$, we get $x'(t) < 0$, $z'(t) < 0$,
for $\sinh(\beta(t)) \in \left(0, \sqrt{\frac{c_0 - c_0^2}{2}}\right)$ we get $x'(t) < 0$, $z'(t) > 0$,
for $\sinh(\beta(t)) > \sqrt{\frac{c_0 - c_0^2}{2}}$, we get $x'(t) > 0$, $z'(t) > 0$,
Thus, the particle trajectory in this case is sketched in Figure 6 (b).

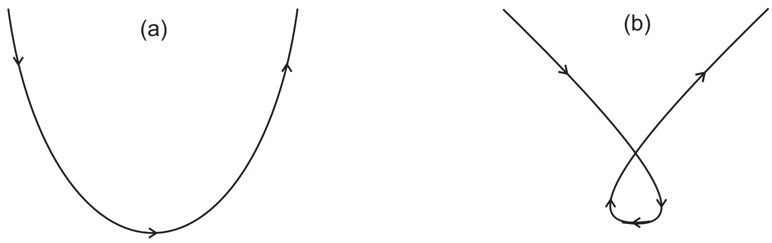


Figure 6. Particle trajectory in the case $0 < c_0 \leq 2$

We thus have:

**Theorem 3.** *In the case that the underlying uniform current is moving in the same direction as an irrotational shallow water wave and the strength of the current is smaller than 2, then the particles trajectories beneath the wave are parabolic curves or curves which have only one loop like in Figure 6.*

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