Journal of Nonlinear Mathematical Physics Volume 15, Supplement 3 (2008), 334–344 ARTICLE

The Klein-Gordon Equation on the Half Line: a Riemann-Hilbert Approach

Beatrice Pelloni and Dimitrios A Pinotsis

Department of Mathematics, University of Reading, Reading RG6 6AX, UK
E-mail: b.pelloni@reading.ac.uk, d.pinotsis@reading.ac.uk

Abstract

We solve an initial-boundary problem for the Klein-Gordon equation on the half line using the Riemann-Hilbert approach to solving linear boundary value problems advocated by Fokas. The approach we present can be also used to solve more complicated boundary value problems for this equation, such as problems posed on time-dependent domains. Furthermore, it can be extended to treat integrable nonlinearisations of the Klein-Gordon equation. In this respect, we briefly discuss how our results could motivate a novel treatment of the sine-Gordon equation.

1 Introduction

In this paper we consider two important equations of mathematical physics, the Klein-Gordon equation in one space dimension

$$q_{tt}(x,t) - q_{xx}(x,t) + q(x,t) = 0, \tag{1.1}$$

and an integrable nonlinearisation of this equation known as the sine-Gordon equation

$$q_{tt}(x,t) - q_{xx}(x,t) + \sin q(x,t) = 0. \tag{1.2}$$

We consider these equations posed on the half line $x > 0$, and solve the initial-boundary value problem obtained by prescribing the following set of initial and boundary conditions:

$$q(x,0) = q_0(x), \quad q_t(x,0) = q_1(x), \quad x > 0, \tag{1.3}$$

$$q(0,t) = f_0(t), \quad t > 0, \quad q(x,t) \to_{t\to\infty} 0. \tag{1.4}$$

To avoid any technical issue not of immediate interest in this paper, we assume that all prescribed functions belong to the Schwarz class.
In principle, the Klein-Gordon equation could be solved by separation of variables and an application of an appropriate Fourier transform. However, we present a solution of the Dirichlet boundary value problem posed on the half line using a different approach. This approach follows a general method for solving boundary value problems for linear PDEs in two dimensions, first proposed by Fokas (see e.g. [2, 3]).

The Fokas transform method has two important advantages over the classical approaches. Firstly, it can be used to solve complicated boundary value problems, such as problems posed on time dependent domains. Secondly, it nonlinearises, in the sense that the same steps used to analyse the linear equation can be used for the analysis of any nonlinear *integrable* version of the same equation. This is similar to the celebrated inverse scattering transform for the solution of the Cauchy problem for integrable nonlinear PDEs.

Using this approach, we prove the following result.

Theorem 1.1. *Consider the Klein-Gordon equation (1.1), for $x > 0$ and $t > 0$. Assume that there exists a unique solution of the boundary value problem obtained by prescribing the initial conditions (1.3) and the boundary conditions (1.4).*
Then this solution is given by the following expression:

$$q(x,t) = \frac{1}{4\pi i}\int_{\mathbb{R}} \mathrm{e}^{ik_-x}\left\{\mathrm{e}^{ik_+t}[\hat{q}_1(k_-)+ik_+\hat{q}_0(k_-)] - \mathrm{e}^{-ik_+t}[\hat{q}_1(k_-)-ik_+\hat{q}_0(k_-)]\right\}\frac{kdk}{1+k^2}, \quad t<x,$$

$$\begin{aligned} q(x,t) &= \frac{1}{4\pi i}\int_{\mathbb{R}}\left\{\mathrm{e}^{ik_-x+ik_+t}[\hat{q}_1(k_-)+ik_+\hat{q}_0(k_-)] - \mathrm{e}^{ik_-x-ik_+t}[\hat{q}_1(k_-)-ik_+\hat{q}_0(k_-)]\right\}\frac{kdk}{1+k^2} \\ &+ \frac{1}{4\pi i}\int_{\Gamma_1}\mathrm{e}^{ik_-x-ik_+t}[\hat{f}_1(-k_+)+ik_-\hat{f}_0(-k_+)]\frac{kdk}{1+k^2} \\ &+ \frac{1}{4\pi i}\int_{\Gamma_2}\mathrm{e}^{ik_-x+ik_+t}[\hat{f}_1(k_+)+ik_-\hat{f}_0(k_+)]\frac{kdk}{1+k^2}, \quad x<t, \end{aligned} \tag{1.5}$$

where the function $f_1(t)$ is given by

$$\begin{aligned} f_1(t) &= \frac{1}{4\pi}\int_{\mathbb{R}}\left\{\mathrm{e}^{ik_+t}[\hat{q}_1(k_-)+ik_+\hat{q}_0(k_-)] + \mathrm{e}^{-ik_+t}[\hat{q}_1(k_-)-ik_+\hat{q}_0(k_-)]\right\}dk \\ &+ \frac{1}{4\pi}\int_{\Gamma_2}\mathrm{e}^{ik_+t}ik_-\hat{f}_0(k_+)dk - \frac{1}{4\pi}\int_{\Gamma_1}\mathrm{e}^{-ik_+t}ik_-\hat{f}_0(-k_+)dk - 2f_0'(t). \end{aligned} \tag{1.6}$$

In these expressions, $\widehat{g}(k)$ denotes the usual Fourier transform of the function $g(x)$,

$$g(k) = \int_0^\infty e^{-ikx}g(x)dx,$$

k_-, k_+ are functions of the complex parameter k defined by,

$$k_- = \frac{1}{2}\left(k-\frac{1}{k}\right), \quad k_+ = \frac{1}{2}\left(k+\frac{1}{k}\right), \tag{1.7}$$

and Γ_1 and Γ_2 are the oriented contours

$$\Gamma_1 = (-\infty,-1]\cup\{|k|=1, Im(k)>0\}\cup[1,\infty) \qquad \textit{(left to right)}, \tag{1.8}$$

$$\Gamma_2 = -[-1,1]\cup\{|k|=1, Im(k)>0\} \qquad \textit{(clockwise)}. \tag{1.9}$$

Note that the function $f_1(t)$ denotes the unknown boundary value of the solution at $x=0$, namely $f_1(t) = q_x(0,t)$.

The analogous boundary value problem for the sine-Gordon equation (1.2) has been considered in [4, 5, 11, 14]. At the end of this paper, we propose an alternative way of solving this nonlinear equation, motivated by the solution of the corresponding linearised problem presented here.

The Fokas transform method is based on the formulation of a linear or nonlinear PDE as the compatibility condition of a pair of linear eigenvalue equations, called the *Lax pair* [6]. The spectral analysis of this pair yields a *Riemann-Hilbert problem*, which is scalar in the linear case and matrix-valued in the nonlinear case. The solution of this Riemann-Hilbert problem yields a formal representation of the solution of the boundary value problem.
In the case of evolution and elliptic PDEs, the derivation of an *effective* solution of a boundary value problem (i.e. a representation of the solution in terms only of the given initial and bondary conditions) involves not only the analysis of the Lax pair, but also the analysis of a relation coupling all initial and boundary values, called the *global relation* [3].
In this paper, we present the application of this method to the case of linear *hyperbolic* PDEs. It appears that, for hyperbolic equations, an effective solution of boundary value problems in time-independent domains can be obtained by analysing the Lax pair only, as long as the Lax pair selected is of second order. Indeed, this second order Lax pairs gives rise to two first-order pairs. The analysis of these two pairs can be combined to yield an effective representation of the solution, without resorting to the global relation. The same idea can be extended to the analysis of the sine-Gordon equation. The Dirichlet problem on the half line for this nonlinear integrable equation is usually solved using just one Lax pair and the global relation. The Lax pair used is the nonlinear version of one of the two first-order Lax pairs of the linear problem. We propose instead to use the nonlinear version of both first-order Lax pairs.

This paper is organised as follows. In section 2, we derive a Lax pair formulation of the Klein-Gordon equation as well as a variant of the Fourier transform which we will need later. In section 3, we prove the theorem 1.1. Finally, in section 4, we discuss the implications of our results for the case of the sine-Gordon equation.

2 Lax pairs and variants of the Fourier transform

2.1 The Lax pair formulation

Any linear PDE with constant coefficients in two variables can be written as the compatibility condition of a pair of ODEs, called a Lax pair of the PDE. The approach for the construction of a Lax pair proposed in [3] assumes that the ODE in the variable x is of the form

$$\mu_x(x,t,k) - ik\mu(x,t,k) = q(x,t), \quad k \in \mathbb{C},$$

where $q(x,t)$ denotes the solution of the PDE and then yields algorithmically an ODE in t. Using this approach and performing an appropriate change of the spectral parameter k we derive the following Lax pair of the Klein-Gordon equation:

$$\mu_x(x,t,k) - \frac{i}{2}\left(k - \frac{1}{k}\right)\mu(x,t,k) = q(x,t), \tag{2.1}$$

$$\mu_{tt}(x,t,k) + \frac{i}{4}\left(k + \frac{1}{k}\right)^2 \mu(x,t,k) = q_x(x,t) + \frac{i}{2}\left(k - \frac{1}{k}\right) q(x,t), \tag{2.2}$$

where $\mu(x,t,k)$ is a real function.
Writing (2.2) as a first order differential system and diagonalising the t-part of this system, we obtain

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}_t + \begin{pmatrix} \frac{i}{2}\left(k+\frac{1}{k}\right) & 0 \\ 0 & -\frac{i}{2}\left(k+\frac{1}{k}\right) \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} q_x + \frac{i}{2}\left(k-\frac{1}{k}\right) q \\ q_x + \frac{i}{2}\left(k-\frac{1}{k}\right) q \end{pmatrix} \tag{2.3}$$

where

$$\begin{cases} \nu_1 = \mu_t - \frac{i}{2}\left(k+\frac{1}{k}\right)\mu, \\ \nu_2 = \mu_t + \frac{i}{2}\left(k+\frac{1}{k}\right)\mu \end{cases} \quad \text{so that } \mu(x,t,k) = \frac{\nu_2(x,t,k)-\nu_1(x,t,k)}{i\left(k+\frac{1}{k}\right)}. \tag{2.4}$$

Computing the x-derivative of ν_1, ν_2, we find that each of these two functions satisfy a Lax pair, namely

$$\begin{cases} (\nu_1)_x - \frac{i}{2}\left(k-\frac{1}{k}\right)\nu_1 = q_t - \frac{i}{2}\left(k+\frac{1}{k}\right)q \\ (\nu_1)_t + \frac{i}{2}\left(k+\frac{1}{k}\right)\nu_1 = q_x + \frac{i}{2}\left(k-\frac{1}{k}\right)q, \end{cases} \tag{2.5}$$

and

$$\begin{cases} (\nu_2)_x - \frac{i}{2}\left(k-\frac{1}{k}\right)\nu_2 = q_t + \frac{i}{2}\left(k+\frac{1}{k}\right)q \\ (\nu_2)_t - \frac{i}{2}\left(k+\frac{1}{k}\right)\nu_2 = q_x + \frac{i}{2}\left(k-\frac{1}{k}\right)q. \end{cases} \tag{2.6}$$

Note that the Klein-Gordon equation can be obtained as the compatibility condition of either of the two Lax pairs (2.5) and (2.6). Hence one possible way to find the solution is to analyse either of these Lax pairs. This procedure yields two integral representations for $q(x,t)$. These representations involve both boundary values $q(0,t)$ and $q_x(0,t)$ of the solution. Since only one boundary condition can be prescribed at $x=0$, these representations are not effective. To determine the unknown boundary value each of these representations must be supplemented by the global relation. Alternatively, we show below that by *combining both* these representations, it is possible to compute explicitly the unknown boundary value *without* invoking the global relation.

Remark 2.1. Following [13], the Lax pair for the sine-Gordon equation that usually appears in the literature (see equation (4.1)) can be obtained in an algorithmic way starting from the Lax pair (2.6). In section 4, we discuss the implications of this fact for the solution of the sine-Gordon equation on the half line.

2.2 Variants of the Fourier transform

To derive an effective representation of the solution of the boundary value problem for the Klein-Gordon equation on the half line, we use a variant of the Fourier transform. This variant can be obtained by a change of variable in the Fourier inversion theorem and an appropriate manipulation of the contours of integration. We present however an alternative, direct derivation of this variant of the Fourier transform by means of the spectral analysis of an appropriate ODE. This approach offers a systematic way of deriving precisely the necessary transform for the solution of the Klein-Gordon equation. In addition, it generalises to more complicated boundary value problems for which the Fourier transform would not suffice, for example it can be used to analyse boundary value problems posed on time-dependent domains [7].

Proposition 2.1. *Let $f(t) \in \mathscr{S}[0,\infty)$, and define the function $F(k)$, $k \in C$, by*

$$F(k) = \int_0^\infty e^{-\frac{i}{2}(k+\frac{1}{k})s} f(s)ds. \tag{2.7}$$

Then

$$f(t) = \frac{1}{4\pi}\int_{\Gamma_1} e^{-\frac{i}{2}(k+\frac{1}{k})t} F(-k)dk - \frac{1}{4\pi}\int_{\Gamma_2} e^{\frac{i}{2}(k+\frac{1}{k})t} F(k)dk, \tag{2.8}$$

where Γ_1 and Γ_2 are the oriented contours given by (1.8)-(1.9).

In this statement, $\mathscr{S}$ denotes the Schwarz space of infinitely differentiable functions, rapidly decaying as $t \to \infty$.

Proof (sketch): We follow the approach reviewed in e.g. [12]. We derive the transform pair (2.7)-(2.8) by considering the ODE

$$\varphi_t(t,k) + \frac{i}{2}\left(k + \frac{1}{k}\right)\varphi(t,k) = f(t), \quad k \in \mathbb{C}.$$

and seeking a solution $\varphi(t,k)$ of it that is sectionally analytic, and bounded for all $k \in \mathbb{C}$. This solution can be found by solving a scalar Riemann-Hilbert problem on the countour determined by the equation

$$Im(k + \frac{1}{k}) = 0.$$

Solving this Riemann-Hilbert problem we obtain the following integral representation for $\varphi(t,k)$:

$$\varphi(t,\lambda) = -\frac{1}{2\pi i}\int_{\Gamma_1} e^{-\frac{i}{2}\left(k+\frac{1}{k}\right)t} F(-k)\frac{dk}{k-\lambda} + \frac{1}{2\pi i}\int_{\gamma_2} e^{-\frac{i}{2}\left(k+\frac{1}{k}\right)t} F(-k)\frac{dk}{k-\lambda},$$

where Γ_1 is given by (1.8) and

$$\gamma_2 = -[-1,1] \cup \{|k| = 1, Im(k) < 0\} \qquad \text{(clockwise)} \tag{2.9}$$

Computing $\varphi_t + \frac{i}{2}\left(\lambda + \frac{1}{\lambda}\right)\varphi$ we obtain for $f(t)$ the expression

$$f(t) = \frac{1}{4\pi}\int_{\Gamma_1} e^{-\frac{i}{2}\left(k+\frac{1}{k}\right)t} F(-k)dk - \frac{1}{4\pi}\int_{\gamma_2} e^{-\frac{i}{2}\left(k+\frac{1}{k}\right)t} F(-k)dk.$$

Equation (2.8) follows after letting $k \to -k$ in the integral along γ_2.
QED

Remark 2.2. Similarly to the above analysis or by a change of variable in the Fourier inversion theorem we derive the following result: Let $f(t) \in \mathscr{S}[0,\infty)$, and define the function $F_-(k)$, $k \in C$, by

$$F_-(k) = \int_0^\infty e^{-\frac{i}{2}(k-\frac{1}{k})s} f(s)ds. \tag{2.10}$$

The inversion formula for this transform is

$$f(x) = \frac{1}{4\pi}\int_{\mathbb{R}} e^{\frac{i}{2}(k-\frac{1}{k})x} F_-(k)dk. \tag{2.11}$$

3 The Dirichlet problem for the Klein-Gordon equation

In this section, we prove theorem 1.1.
Proof of theorem 1.1: We consider the two Lax pairs (2.5)-(2.6).
Any solution of the Lax pair (2.5) is of the form

$$v_1(x,t,k) = \int_{x_*}^{x} e^{ik_-(x-y)}(q_t - ik_+q)(y,t)dy + e^{ik_-x}\int_{t_*}^{t} e^{-ik_+(t-s)}(q_x + ik_-q)(x_0,s)ds$$

and its asymptotic behaviour as $k \to 0$ and $k \to \infty$ is given by

$$
\begin{aligned}
v_1 &= -q - 2i(q_x + q_t)k + O(k^2), \quad k \to 0, \\
v_1 &= q + O\left(\frac{1}{k}\right), \quad k \to \infty.
\end{aligned} \tag{3.1}
$$

The general choice of x_*, t_* that yields data determining an appropriate Riemann-Hilbert problem is given in [3]. To determine such a problem one needs solutions v^j with prescribed decay at infinity, each of which is analytic and bounded in k in a subdomain D_j of the complex plane $\mathbb{C}$, such that the domains D_j do not overlap and $\bigcup_j D_j = \mathbb{C}$. In the present case, we obtain the three particular solutions of (2.5)

$$
\begin{aligned}
v_1^{1,+} &= \int_0^x \mathrm{e}^{ik_-(x-y)}(q_t - ik_+ q)(y,t)dy - \mathrm{e}^{ik_- x}\int_t^\infty \mathrm{e}^{-ik_+(t-s)}(q_x + ik_- q)(0,s)ds, \\
v_1^{2,+} &= \int_0^x \mathrm{e}^{ik_-(x-y)}(q_t - ik_+ q)(y,t)dy + \mathrm{e}^{ik_- x}\int_0^t \mathrm{e}^{-ik_+(t-s)}(q_x + ik_- q)(0,s)ds, \\
v_1^- &= -\int_x^\infty \mathrm{e}^{ik_-(x-y)}(q_t - ik_+ q)(y,t)dy.
\end{aligned}
$$

Indeed, the function $v_1^{1,+}$, considered as a function of the complex variable k, is bounded and analytic in $\mathbb{C}^+ \setminus B_1$, the function $v_1^{2,+}$ is analytic in $\mathbb{C}^+ \cap B_1$ while v_1^- is bounded and analytic in $\mathbb{C}^-$, where B_1 is the unit disk in $\mathbb{C}$.
Similarly, for the Lax pair (2.6) we obtain the following particular solutions

$$
\begin{aligned}
v_2^{1,+} &= \int_0^x \mathrm{e}^{ik_-(x-y)}(q_t + ik_+ q)(y,t)dy + \mathrm{e}^{ik_- x}\int_0^t \mathrm{e}^{ik_+(t-s)}(q_x + ik_- q)(0,s)ds, \quad k \in \mathbb{C}^+ \setminus B_1, \\
v_2^{2,+} &= \int_0^x \mathrm{e}^{ik_-(x-y)}(q_t + ik_+ q)(y,t)dy - \mathrm{e}^{ik_- x}\int_t^\infty \mathrm{e}^{ik_+(t-s)}(q_x + ik_- q)(0,s)ds, \quad k \in \mathbb{C}^+ \cap B_1, \\
v_2^- &= -\int_x^\infty \mathrm{e}^{ik_-(x-y)}(q_t + ik_+ q)(y,t)dy, \quad k \in \mathbb{C}^-.
\end{aligned}
$$

We now define the three functions

$$
\begin{aligned}
\mu^{j,+}(x,t,k) &= \frac{v_2^{j,+}(x,t,k) - v_1^{j,+}(x,t,k)}{2ik_+}, \quad j = 1,2 \\
\mu^-(x,t,k) &= \frac{v_2^-(x,t,k) - v_1^-(x,t,k)}{2ik_+}.
\end{aligned}
$$

Equation (2.4) implies that the above formulae define three particular solutions of the system (2.2) which are bounded and analytic in $\mathbb{C}^+ \setminus B_1$, $\mathbb{C}^+ \cap B_1$ and $\mathbb{C}^-$ respectively, except for a simple pole at $k = i$. In addition the functions $\mu^{1,+}$ and μ^- are of order $O\left(\frac{1}{k}\right)$ as $k \to \infty$ in the respective half planes. These functions determine a Riemann-Hilbert problem with jumps along $\mathbb{R}$ and ∂B_1. These jumps are given by

$$
\begin{aligned}
\mu^{1,+}(x,t,k) - \mu^{2,+}(x,t,k) &= \frac{\mathrm{e}^{ik_- x + ik_+ t}}{2ik_+}[\hat{f}_1(k_+) + ik_- \hat{f}_0(k_+)] \\
&+ \frac{\mathrm{e}^{ik_- x - ik_+ t}}{2ik_+}[\hat{f}_1(-k_+) + ik_- \hat{f}_0(-k_+)], \quad |k| = 1,\ k \in \mathbb{C}^+,
\end{aligned}
$$

$$\begin{aligned}\mu^{1,+}(x,t,k)-\mu^{-}(x,t,k) &= \frac{e^{ik_-x+ik_+t}}{2ik_+}[\hat{q}_1(k_-)+ik_+\hat{q}_0(k_-)]\\ &- \frac{e^{ik_-x-ik_+t}}{2ik_+}[\hat{q}_1(k_-)-ik_+\hat{q}_0(k_-)]\\ &+ \frac{e^{ik_-x-ik_+t}}{2ik_+}[\hat{f}_1(-k_+)+ik_-\hat{f}_0(-k_+)], \quad |k|\geq 1,\ k\in\mathbb{R},\end{aligned}$$

$$\begin{aligned}\mu^{2,+}(x,t,k)-\mu^{-}(x,t,k) &= \frac{e^{ik_-x+ik_+t}}{2ik_+}[\hat{q}_1(k_-)+ik_+\hat{q}_0(k_-)]\\ &- \frac{e^{ik_-x-ik_+t}}{2ik_+}[\hat{q}_1(k_-)-ik_+\hat{q}_0(k_-)]\\ &- \frac{e^{ik_-x+ik_+t}}{2ik_+}[\hat{f}_1(k_+)+ik_-\hat{f}_0(k_+)], \quad -1<k<1.\end{aligned}$$

The unique solution of this Riemann-Hilbert problem which is also a bounded solution of (2.2), is given by

$$\begin{aligned}\mu(x,t,\lambda) &= \frac{1}{2\pi i}\int_{\mathbb{R}}\left\{\frac{e^{ik_-x+ik_+t}}{2ik_+}[\hat{q}_1(k_-)+ik_+\hat{q}_0(k_-)]-\frac{e^{ik_-x-ik_+t}}{2ik_+}[\hat{q}_1(k_-)-ik_+\hat{q}_0(k_-)]\right\}\frac{dk}{k-\lambda}\\ &+\frac{1}{2\pi i}\int_{\Gamma_1}\frac{e^{ik_-x-ik_+t}}{2ik_+}[\hat{f}_1(-k_+)+ik_-\hat{f}_0(-k_+)]\frac{dk}{k-\lambda}\\ &+\frac{1}{2\pi i}\int_{\Gamma_2}\frac{e^{ik_-x+ik_+t}}{2ik_+}[\hat{f}_1(k_+)+ik_-\hat{f}_0(k_+)]\frac{dk}{k-\lambda},\end{aligned} \tag{3.2}$$

where the contours $\Gamma_j,\ j=1,2$ are given by (1.8) and (1.9). Using (2.1), we finally obtain an integral representation for $q(x,t)$:

$$\begin{aligned}q(x,t) &= \frac{1}{4\pi i}\int_{\mathbb{R}}\left\{e^{ik_-x+ik_+t}[\hat{q}_1(k_-)+ik_+\hat{q}_0(k_-)]-e^{ik_-x-ik_+t}[\hat{q}_1(k_-)-ik_+\hat{q}_0(k_-)]\right\}\frac{dk}{2k_+}\\ &+\frac{1}{4\pi i}\int_{\Gamma_1}e^{ik_-x-ik_+t}[\hat{f}_1(-k_+)+ik_-\hat{f}_0(-k_+)]\frac{dk}{2k_+}\\ &+\frac{1}{4\pi i}\int_{\Gamma_2}e^{ik_-x+ik_+t}[\hat{f}_1(k_+)+ik_-\hat{f}_0(k_+)]\frac{dk}{2k_+}.\end{aligned} \tag{3.3}$$

This formal representation contains the unknown function $f_1(t)$. We now show how to evaluate this function explicitly.
We distinguish two cases: $t<x$ and $t>x$.

(a): $\mathbf{x-t>0}$

In this case the exponential $e^{ik_-x-ik_+t}e^{ik_+s}$, $s\geq 0$, is bounded and analytic in the region of $\mathbb{C}^+$ above the contour Γ_1. Indeed, this is the case for e^{ik_+s}, while

$$e^{ik_-x-ik_+t}=e^{i\frac{k}{2}(x-t)-\frac{i}{2k}(x+t)} \Longrightarrow |e^{ik_-x-ik_+t}|=e^{-\frac{k_2}{2}(x-t)}e^{\frac{k_2}{2|k|^2}(x+t)}$$

so that the latter exponential is bounded if $k_2>0$ and k is away from $k=0$. It follows by Jordan's lemma that

$$\lim_{R\to\infty}\int_{C_R}e^{ik_-x-ik_+t}[\hat{f}_1(-k_+)+ik_-\hat{f}_0(-k_+)]\frac{dk}{k_+}=0 \tag{3.4}$$

where $C_R = \{k \in \mathbb{C}^+ : \ |k| = R\}$ with counterclockwise orientation. The integrand of the integral along Γ_1 appearing in (3.3) has a simple pole at $k = i$. Computing the residue at this pole, and using (3.4), we obtain

$$\frac{1}{4\pi i}\int_{\Gamma_1} \mathrm{e}^{ik_- x - ik_+ t}[\hat{f}_1(-k_+) + ik_- \hat{f}_0(-k_+)]\frac{dk}{2k_+} = \frac{1}{2}\mathrm{e}^{-2x}[\hat{f}_1(0) - 2\hat{f}_0(0)].$$

Similarly, the exponential $\mathrm{e}^{ik_- x + ik_+ t}\mathrm{e}^{-ik_+ s}$, $s \geq 0$, is bounded and analytic in the region bounded by the contour Γ_2, where the integrand has a simple pole (at $k = i$). Hence,

$$\frac{1}{4\pi i}\int_{\Gamma_2} \mathrm{e}^{ik_- x + ik_+ t}[\hat{f}_1(k_+) + ik_- \hat{f}_0(k_+)]\frac{dk}{2k_+} = -\frac{1}{2}\mathrm{e}^{-2x}[\hat{f}_1(0) - 2\hat{f}_0(0)].$$

It follows that the contribution of the last two integrals in (3.3) cancels and we obtain

$$q(x,t) = \frac{1}{4\pi i}\int_{\mathbb{R}} \left\{\mathrm{e}^{ik_- x + ik_+ t}[\hat{q}_1(k_-) + ik_+\hat{q}_0(k_-)] - \mathrm{e}^{ik_- x - ik_+ t}[\hat{q}_1(k_-) - ik_+\hat{q}_0(k_-)]\right\}\frac{dk}{2k_+}. \quad (3.5)$$

(b): $\mathbf{t - x > 0}$

In this case, the contribution of the terms involving the boundary values is not zero. To evaluate this contribution explicitly, we take the derivative of the expression (3.3) for $q(x,t)$ with respect to t and evaluate it at $x = 0$. Thus we obtain

$$\begin{aligned} 2q_t(0,t) \quad = \quad & \frac{1}{4\pi}\int_{\mathbb{R}} \left\{\mathrm{e}^{ik_+ t}[\hat{q}_1(k_-) + ik_+\hat{q}_0(k_-)] + \mathrm{e}^{-ik_+ t}[\hat{q}_1(k_-) - ik_+\hat{q}_0(k_-)]\right\} dk \\ & -\frac{1}{4\pi}\int_{\Gamma_1} \mathrm{e}^{-ik_+ t}[\hat{f}_1(-k_+) + ik_- \hat{f}_0(-k_+)]dk \\ + \quad & \frac{1}{4\pi}\int_{\Gamma_2} \mathrm{e}^{ik_+ t}[\hat{f}_1(k_+) + ik_- \hat{f}_0(k_+)]dk. \end{aligned} \quad (3.6)$$

In expression (3.6), the left hand side is $f_0'(t)$. The term involving the unknown boundary value $f_1(t)$ is

$$\frac{1}{4\pi}\int_{\Gamma_2} \mathrm{e}^{ik_+ t}\hat{f}_1(k_+)dk - \frac{1}{4\pi}\int_{\Gamma_1} \mathrm{e}^{-ik_+ t}\hat{f}_1(-k_+)dk, \quad \hat{f}_1(-k_+) = \int_0^\infty \mathrm{e}^{ik_+ s} f_1(s)ds.$$

Using the inversion formula (2.8), the above term is equal to $-f_1(t)$. Hence we obtain the following explicit expression for the unknown function $f_1(t)$ in terms of the given initial and boundary conditions:

$$\begin{aligned} & f_1(t) = \frac{1}{4\pi}\int_{\mathbb{R}} \left\{\mathrm{e}^{ik_+ t}[\hat{q}_1(k_-) + ik_+\hat{q}_0(k_-)] + \mathrm{e}^{-ik_+ t}[\hat{q}_1(k_-) - ik_+\hat{q}_0(k_-)]\right\} dk \\ & + \frac{1}{4\pi}\int_{\Gamma_2} \mathrm{e}^{ik_+ t} ik_- \hat{f}_0(k_+)dk - \frac{1}{4\pi}\int_{\Gamma_1} \mathrm{e}^{-ik_+ t} ik_- \hat{f}_0(-k_+)dk - 2f_0'(t). \end{aligned} \quad (3.7)$$

QED

Remark 3.1. The integral representations (3.5) and (3.3) are obtained in theorem 1.1 under the assumption of existence. However, we can now *define* the function $f_1(t)$, $t > 0$, by the expression (3.7), as well as the function $q(x,t)$ by the expression (3.5) when $t < x$ and by the expression (3.3) when $t > x$. By construction, $q(x,t)$ satisfies the boundary condition $q(0,t) = f_0(t)$. In addition, using (2.11), we verify that it satifies the given initial conditions. Indeed, letting $t = 0$ in (3.5), we find

$$q(x,0) = \frac{1}{4\pi}\int_{\mathbb{R}} e^{ik_- x}\hat{q}_0(k_-)dk = q_0(x).$$

Similarly, $q_t(x,0) = q_1(x)$.

Remark 3.2. It is possible to derive the solution of the pure initial value problem posed on the real line as well as the solution (3.5) of the Dirichlet problem far away from the boundary using the representation obtained from only one of the two Lax pairs (2.5) and (2.6) and a change of variables. However, it is not possible to characterise the unknown boundary value without deriving an additional relation.

4 Remarks on the half-line problem for the sine-Gordon equation

The sine-Gordon equation is a nonlinear integrable equation in one space dimension. In [9] this equation was formulated as the compatibility condition of the Lax pair

$$\begin{aligned} &\mu_x + \frac{i}{2}k_-\sigma_3\mu = \widetilde{Q}(x,t,k)\mu, \\ &\mu_t + \frac{i}{2}k_+\sigma_3\mu = \widetilde{Q}(x,t,-k)\mu, \end{aligned} \tag{4.1}$$

where $\mu(x,t,k)$ is a 2×2 matrix and

$$\widetilde{Q}(x,t,k) = \frac{1}{4}\begin{pmatrix} \frac{i}{k}(\cos q - 1) & -i(q_x+q_t) - \frac{\sin q}{k} \\ -i(q_x+q_t) + \frac{\sin q}{k} & -\frac{i}{k}(\cos q - 1) \end{pmatrix}, \tag{4.2}$$

Using this Lax pair, the Cauchy problem for the sine-Gordon equation was solved by the inverse scattering transform.

Recently, Fokas solved the boundary value problem for the sine-Gordon equation on the half-line using his generalised transform method [4, 5]. In these papers, the global relation plays a crucial role in deriving an effective representation of the solution of this problem because it is the analysis of this relation that yields the unknown boundary value $f_1(t)$ in terms of the given boundary conditions.

In section 3, we presented an approach for the solution of the Klein-Gordon equation which is a linearisation, around $q = 0$, of the sine-Gordon equation. For evolution equations, the linearised problem can be taken as a guideline for the solution of the corresponding integrable nonlinear problem. We expect this to be the case also in the present problem. Our approach for the solution of the Klein-Gordon equation motivates an analogous treatment for the sine-Gordon equation, which does *not involve* the global relation.

A systematic approach for the derivation of *nonlinear integrable* PDEs starting from the corresponding *linear* PDEs was introduced in [13]. This approach is based on the algorithmic construction of a Lax pair of the nonlinear equation starting from the Lax pair of the corresponding linear equation.

Since the solution of the linearised equation is based on the spectral analysis of *both* Lax pairs (2.5) and (2.6), we propose to study the sine-Gordon equation by considering the Lax pair (4.1) which is the nonlinear analogue of (2.6) as well as a second Lax pair, corresponding to (2.5).
Following the approach of [13] and starting from (2.5), we obtain the following Lax pair of the sine-Gordon equation:

$$\begin{aligned} &\mu_x - \frac{i}{2}k_-\sigma_3\mu = Q^{(1)}(x,t,k)\mu \\ &\mu_t + \frac{i}{2}k_+\sigma_3\mu = Q^{(2)}(x,t,k)\mu, \end{aligned} \tag{4.3}$$

where the matrices $Q^{(j)}(x,t,k)$ $j = 1,2$ are defined by

$$Q^{(1)}(x,t,k) = \frac{1}{4}\begin{pmatrix} \frac{i}{k}(1-\cos q) & i(q_x - q_t) - \frac{\sin q}{k} \\ i(q_x - q_t) + \frac{\sin q}{k} & \frac{i}{k}(\cos q - 1) \end{pmatrix}, \tag{4.4}$$

$$Q^{(2)}(x,t,k) = \frac{1}{4}\begin{pmatrix} \frac{i}{k}(1-\cos q) & -i(q_x - q_t) - \frac{\sin q}{k} \\ -i(q_x - q_t) + \frac{\sin q}{k} & \frac{i}{k}(\cos q - 1) \end{pmatrix}. \tag{4.5}$$

A third Lax pair of the sine-Gordon equation was introduced in [1]. This alternative Lax pair was used in [5] to motivate the role of some special boundary conditions, first proposed in [14]. The relation between these results and the approach we suggest here is currently under investigation and will be presented elsewhere.

Acknowledgments. The authors gratefully acknowledge the support of EPSRC through grant EP/E022960/1. They also wish to thank A S Fokas for many useful discussions.

References

[1] ABLOWITZ M J, KAUP D J, NEWELL A C and SEGUR H, Method for solving the sine-Gordon equation, *Phys. Rev. Lett.* **30** (1973), 1262–1264.

[2] FOKAS A S, A unified transform method for solving linear and certain nonlinear PDE's, *Proc. Royal Soc. Series A* **453** (1997), 1411–1443.

[3] FOKAS A S, Two dimensional linear PDE's in a convex polygon, *Proc. Royal Soc. Lond. A*, **457** (2001), 371–393.

[4] FOKAS A S, Linearizable initial-boundary value problems for the sine-Gordon equation on the half-line, *Nonlinearity* **17** (2004), 1521–1534.

[5] FOKAS A S, The generalised Dirichlet to Neumann map for certain nonlinear evolution PDEs, *Comm. Pure Appl. Math.* **LVIII** (2005), 639–670.

[6] FOKAS A S and GELFAND I M, Integrability of linear and nonlinear evolution equations and the associated nonlinear Fourier transform, *Lett. Math. Phys.* **3** (1994), 189–210.

[7] FOKAS A S and PELLONI B, A generalised Dirichlet-to-Neumann map for evolutionary moving boundary value problems, *J. Math. Phys.* **48** (1) published online, 9 January 2007.

[8] FOKAS A S and SUNG L Y, Generalized Fourier transforms, their nonlinearization and the imaging of the brain, *Notices AMS* **52** (2005), 1178–1192.

[9] LAX P D, Integrals of nonlinear evolution equations and solitary waves, *Comm. Pur. Appl. Math.* **21** (1968), 467–490.

[10] PELLONI B, The spectral representation of two-point boundary value problems for linear PDEs, *Proc. Royal Soc. Lond. A* **461** (2005) 2965–2984.

[11] PELLONI B, The asymptotic behaviour of the solution of boundary value problems for the sine-Gordon equation on a finite interval, *J. Nonlin. Math. Phys.* **12** (4) (2005).

[12] PELLONI B, Linear and nonlinear generalised Fourier transforms, *Phil. Trans. Royal Soc. London* **364** (2006), 3231–3249.

[13] PINOTSIS D A, The Riemann-Hilbert formalism for certain linear and nonlinear integrable PDEs, *J. Nonlin. Math. Phys.* **14** (3) (2007), 466–485.

[14] SKLIANIN E K, Boundary conditions for integrabel quantum systems, *J. Phys. A* **21** (1988), 2375–2389.