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Integrating Factors and $\lambda-$Symmetries

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Abstract

We investigate the relationship between integrating factors and $\lambda-$symmetries for ordinary differential equations of arbitrary order. Some results on the existence of $\lambda-$symmetries are used to prove an independent existence theorem for integrating factors. A new method to calculate integrating factors and the associated first integrals is derived from the method to compute $\lambda-$symmetries and the associated reduction algorithm. Several examples illustrate how the method works in practice and how the computations that appear in other methods may be simplified.

1 Introduction

A classic result of A. Clairaut (1739) ensures that any first-order equation

$$M(x,u)+N(x,u)u_x=0 \tag{1.1}$$

admits an integrating factor, i.e. a function, $\mu(x,u)$, such that the equivalent equation $\mu(M(x,u)+N(x,u)u_x)=0$ is exact. Integrating factors are a powerful tool to integrate some first-order differential equations although finding integrating factors can be more difficult than solving the original equation.

For higher-order equations integrating factors have not been so widely used. The determination of integrating factors for an nth-order ordinary differential equation requires the solution of an nth-order linear partial differential equation in $n+1$ variables.

An alternative approach ([1],[2]) to find integrating factors is based on variational derivatives of the form

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_x\frac{\partial}{\partial u_x}+D_x^2\frac{\partial}{\partial u_{xx}}+\cdots, \tag{1.2}$$

where D_x denotes the total derivative operator with respect to x. Denote by $u^{(k)}=(u,u_1,\cdots,u_k)$, where u_i denotes the derivative of order i of u with respect to the independent variable x. The integrating factors $\mu(x,u^{(n-1)})$ of an n-th order equation

$$M(x,u^{(n-1)})+N(x,u^{(n-1)})u_n=0 \tag{1.3}$$

are determined by the following equation:

$$\frac{\delta}{\delta u}(\mu(M+Nu_n))=0. \tag{1.4}$$

For first-order equations Eq. (1.4) becomes the single linear partial differential equation $(\mu M)_u-(\mu N)_x=0$, which always has an infinite number of solutions. For $n=2$ the determining Eq. (1.4) gives an over-determined system of two second-order linear partial differential equations ([2], Eqs. 6.6.17-6.6.18), with only one dependent variable and the compatibility of which cannot be *a priori* ensured. For equations of arbitrary order the equations derived from (1.4) are rather complicated and they have scarcely been studied.

In this paper we present a new independent approach to find integrating factors for equations of arbitrary order. For first-order scalar ordinary differential equations a well-known result of Sophus Lie [5] states that a Lie point symmetry can be used to construct an integrating factor and conversely any integrating factor determines a Lie point symmetry of the original equation. However, there exist higher-order equations without Lie symmetries that admit integrating factors or that are exact ([11], pag. 182). In these cases Lie point symmetries are of no help to find integrating factors neither to characterize exact equations. Therefore we cannot expect a one-to-one correspondence between Lie point symmetries and integrating factors for equations of order higher than one. As it is remarked in [4] generalised and nonlocal symmetries must also be considered in the context of the existence of integrating factors.

Although there are exact equations without Lie symmetries, any exact equation always admit a $\lambda-$symmetry (also called $\mathscr{C}^{\infty}-$symmetry) and the associated algorithm of reduction gives the corresponding conserved form ([7], Theorem 3.2). As a consequence of this result in this paper we firstly prove that there exists a $\lambda-$symmetry associated to any integrating factor (Theorem 1). Next we investigate the converse problem and we establish the connection between $\lambda-$symmetries and integrating factors for equations of any arbitrary order. The main result (Theorem 2) states that any $\lambda-$symmetry yields an integrating factor. This generalizes the result of Sophus Lie for ordinary differential equations of arbitrary order.

For ordinary differential equations with analytic terms, we prove (Theorem 3) an existence theorem of $\lambda-$symmetries. This result leads to an indirect proof of the existence of integrating factors.

These theoretical results have interesting practical consequences: once a $\lambda-$symmetry has been calculated, the associated integrating factor and the corresponding first integral can be derived from the reduction algorithm without additional computation.

Several examples have been selected to exhibit different aspects of the method: to show how the method works in practice, to compare our method with those based on variational derivatives and to illustrate the induction method used in the proof of Theorem 2.

For simplicity all the results have been presented for scalar equations. With some minor changes the results and methods can be extended to systems of ordinary differential equations.

2 $\lambda-$Symmetries and Integrating Factors

2.1 $\lambda-$Symmetries associated to integrating factors

Consider a nth-order ordinary differential equation

$$\widetilde{\Delta}(x,u^{(n)})=0, \tag{2.1}$$

where variables (x,u) are in some open set $M \subset X \times U \simeq \mathbb{R}^2$. For $k \in \mathbb{N}$, $M^{(k)} \subset X \times U^{(k)}$ denotes the corresponding k-jet space and the elements of $M^{(k)}$ are denoted by $(x,u^{(k)}) = (x,u,u_1,\cdots,u_k)$.

Suppose that Eq. (2.1) admits an integrating factor, $\mu(x,u^{(k)})$, for some k such that $0 \le k \le n-1$. The multiplication by μ converts the left-hand side of (2.1) into the total derivative of some function $\Delta(x,u^{(n-1)})$:

$$\mu(x,u^{(k)}) \cdot \widetilde{\Delta}(x,u^{(n)}) = D_x(\Delta(x,u^{(n-1)})). \tag{2.2}$$

The exact equation $D_x(\Delta(x,u^{(n-1)})) = 0$ admits a $\lambda-$symmetry and the trivial reduction of order $\Delta(x,u^{(n-1)}) = C, \quad C \in \mathbb{R}$, appears as consequence of the reduction algorithm associated to that $\lambda-$symmetry ([7], Theorem 3.2). If $\lambda \in C^\infty(M^{(k)}), 0 \le k \le n-1$, is any solution of the partial differential equation:

$$\sum_{i=0}^{n-1} (D_x+\lambda)^i(1)\frac{\partial \Delta}{\partial u_i} = 0 \quad \text{when} \quad D_x(\Delta(x,u^{(n-1)})) = 0, \tag{2.3}$$

then the vector field $v = \partial_u$ is a $\lambda-$symmetry of Eq. (2.1). By (2.2), we have that

$$v^{[\lambda,(n)]}(D_x(\Delta(x,u^{(n-1)}))) = \mu \cdot v^{[\lambda,(n)]}(\widetilde{\Delta}) + \widetilde{\Delta} \cdot v^{[\lambda,(n)]}(\mu). \tag{2.4}$$

It is clear that any solution of (2.1) is also a solution of $D_x(\Delta(x,u^{(n-1)})) = 0$. Hence, if $v = \partial_u$ is a $\lambda-$symmetry of $D_x(\Delta(x,u^{(n-1)})) = 0$, with λ satisfying (2.3), the left-hand side of (2.4) is equal to zero when it is evaluated on solutions of Eq. (2.1). Similarly the second term in the right member of (2.4) becomes equal to zero when it is evaluated on $\widetilde{\Delta} = 0$. Therefore

$$\mu|_{\widetilde{\Delta}=0} \cdot v^{[\lambda,(n)]}(\widetilde{\Delta})|_{\widetilde{\Delta}=0} = 0. \tag{2.5}$$

From (2.5) we deduce the following theorem:

Theorem 1. *Assume that (2.1) is a nth-order ordinary differential equation that admits an integrating factor μ such that $\mu|_{\widetilde{\Delta}=0} \neq 0$. If λ is any particular solution of (2.3), then the vector field $v = \partial_u$ is a $\lambda-$symmetry of (2.1).*

Example. The second-order equation

$$u_{xx} + \frac{u_x^2}{u} + 3\frac{u_x}{x} = 0 \tag{2.6}$$

admits two integrating factors $\mu_1 = xu$ and $\mu_2 = x^3u$:

$$\mu_1(x,u) \cdot (u_{xx} + \frac{u_x^2}{u} + 3\frac{u_x}{x}) = D_x(xuu_x + u^2), \tag{2.7}$$

$$\mu_2(x,u) \cdot (u_{xx} + \frac{u_x^2}{u} + 3\frac{u_x}{x}) = D_x(x^3uu_x). \tag{2.8}$$

Denote $\Delta_1(x,u^{(1)}) = xuu_x + u^2$. For Eq. (2.6) and μ_1 it can be checked that Eq. (2.3) can be written as

$$xu_x + 2u + \lambda_1 xu = 0. \tag{2.9}$$

Since $\lambda_1 = -\left(\frac{u_x}{u} + \frac{2}{x}\right)$ is a solution of (2.9), Theorem 1 ensures that the vector field $v = \partial_u$ is a λ_1-symmetry.

Similarly for $\lambda_2 = -\frac{u_x}{u}$ it can be checked that $v = \partial_u$ is a λ_2-symmetry associated to the integrating factor μ_2.

2.2 Integrating factors associated to $\lambda-$symmetries

In this section we show how to obtain an integrating factor of a given ordinary differential equation with a known $\lambda-$symmetry.

Theorem 2. *Let*

$$\widetilde{\Delta}(x,u^{(n)})=0 \tag{2.10}$$

be a nth-order ordinary differential equation that admits a vector field v as a $\lambda-$symmetry for some function $\lambda\in\mathscr{C}(M^{(k)})$, $0\leq k<n$. The $\lambda-$symmetry yields an integrating factor $\mu(x,u^{(n-1)})$ of the equation.

Proof. By introducing canonical coordinates for v, if necessary, it can be assumed that $v=\partial_u$. We proceed by induction on the order of the equation. Since any first-order equation does always admit an integrating factor, we prove the theorem for $n=2$. The corresponding Eq. (2.10) can be written in explicit form as

$$u_{xx}=F(x,u,u_x). \tag{2.11}$$

Let $w(x,u,u_x)$ be a first-order invariant of $v^{[\lambda,(1)]}$ i.e., any particular solution of the equation:

$$w_u+w_{u_x}\lambda(x,u,u_x)=0. \tag{2.12}$$

The reduction process associated to the $\lambda-$symmetry v gives a first-order reduced equation of the form $\Delta_R(x,w,w_x)=0$, the general solution of which is implicitly given by an equation of the form $G(x,w)=C,C\in\mathbb{R}$. It is clear that $D_x(G(x,w(x,u,u_x)))=0$ is an equivalent form of Eq. (2.11). Therefore

$$\mu(x,u,u_x)=G_w(x,u,w(x,u,u_x))\cdot w_{u_x}(x,u,u_x) \tag{2.13}$$

is an integrating factor of (2.11).

Assume that the theorem is true for any ordinary differential equation of order $m<n$. Let $w=w(x,u^{(k)})$ be an invariant of $v^{[\lambda,(n)]}$. The set $\{x,w^{(n-k)}\}$ is a set of functionally independent invariants of $v^{[\lambda,(n)]}$. Since the subvariety

$$\mathscr{L}_\Delta=\{(x,u^{(n)}):\widetilde{\Delta}(x,u^{(n)})=0\} \tag{2.14}$$

is invariant for $v^{[\lambda,(n)]}$, by Proposition 2.18 in [10] there exists an invariant function $\Delta_R(x,w^{(n-k)})$ the solution set of which coincides, locally, with $\mathscr{L}_\Delta$. By Proposition 2.10 in [10] there exists a function $H(x,u,w^{(n-k)})$ such that

$$\widetilde{\Delta}(x,u^{(n)})=H(x,u,w^{(n-k)})\cdot\Delta_R(x,w^{(n-k)}). \tag{2.15}$$

The equation $\Delta_R(x,w^{(n-k)})=0$ is the reduced equation associated to the $\lambda-$symmetry v. By our inductive assumption there exists an integrating factor $\mu(x,w^{(n-k-1)})$ of the reduced equation, namely

$$\mu(x,w^{(n-k-1)})\cdot\Delta_R(x,w^{(n-k)})=D_x(\phi(x,w^{(n-k-1)})). \tag{2.16}$$

By writing $w^{(n-k)}$ in terms of variables $(x,u^{(n)})$, (2.16) becomes

$$\widetilde{\mu}(x,u^{(n-1)})\cdot\frac{\widetilde{\Delta}(x,u^{(n)})}{\widetilde{H}(x,u^{(n)})}=D_x(\widetilde{\phi}(x,u^{(n-1)})), \tag{2.17}$$

where $\widetilde{\mu},\widetilde{\phi}$ and $\widetilde{H}$ denote the functions μ, ϕ and H, respectively, written in terms of variables $(x,u^{(n)})$. Therefore

$$\frac{\widetilde{\mu}(x,u^{(n-1)})}{\widetilde{H}(x,u^{(n)})} \tag{2.18}$$

is an integrating factor of Eq. (2.10).

■

3 Existence of $\lambda-$Symmetries and Integrating Factors

In the previous section it was proved the equivalence between integrating factors and $\lambda-$symmetries of ordinary differential equations of arbitrary order. We now present an existence theorem for $\lambda-$symmetries, which provides an indirect proof of an existence theorem for integrating factors.

Theorem 3. *Let $u_n=F(x,u^{(n-1)})$ be a nth-order ordinary differential equation, where F is an analytic function of its arguments. There exists a function $\lambda(x,u^{(k)})$, for some $k<n$ such that the vector field $v=\partial_u$ is a $\lambda-$symmetry of the equation.*

Proof. We try to find some function $\lambda(x,u^{(k)})$, $k\le n-1$, such that

$$v^{[\lambda,(n)]}(u_n-F(x,u^{(n-1)}))=0 \quad \text{when} \quad u_n=F(x,u^{(n-1)}). \tag{3.1}$$

In terms of the characteristic $Q\equiv 1$ of v we have

$$v^{[\lambda,(n)]}=\sum_{i=0}^{n}(D_x+\lambda)^i(1)\frac{\partial}{\partial u_i}. \tag{3.2}$$

Hence Eq. (3.1) can be written as

$$(D_x+\lambda)^n(1)=\sum_{i=0}^{n-1}(D_x+\lambda)^i(1)\frac{\partial F}{\partial u_i} \quad \text{when} \quad u_n=F(x,u^{(n-1)}). \tag{3.3}$$

Since the set of analytic functions is closed under differentiation, it is clear that, if λ is an analytic function in $(x,u^{(k)})$, then for $i=1,\cdots,n-1$ the expression $(D_x+\lambda)^i(1)$ defines an analytical function in variables $(x,u^{(k+i-1)})$, which depends on the partial derivatives of λ up to order $i-1$. To evaluate $(D_x+\lambda)^i(1)$ when $u_n=F(x,u^{(n-1)})$ we replace u_{n+h}, $h\ge 0$, by the corresponding derivative of F. The resulting expression depends in an analytic way upon the variables $(x,u^{(n-1)})$ and on the partial derivatives of λ with respect to all its arguments up to order $i-1$. It is easy to see that in (3.3) the term

$$\frac{\partial^{n-1)}\lambda}{\partial x^{n-1)}} \tag{3.4}$$

only appears in the first member and its coefficient is 1. Therefore condition (3.3) can be written as a single partial differential equation for λ of the form:

$$\frac{\partial^{n-1)}\lambda}{\partial x^{n-1}} = G(x, u^{(n-1)}, \lambda^{(n-1)}), \tag{3.5}$$

where $\lambda^{(n-1)}$ denotes the partial derivatives of λ, of order $\leq n-1$, with respect to all its arguments and function G is an analytic function which does not depend on $\frac{\partial^{n-1)}\lambda}{\partial x^{n-1)}}$. With these conditions the Cauchy-Kowalevski Theorem ([3]) ensures the existence of an analytic solution of Eq. (3.5). Any of these solutions $\lambda = \lambda(x, u^{(k)})$, $k < n$, satisfies condition (3.3) and the vector field v is a $\lambda-$symmetry of the equation.

■

As a corollary of Theorem 2 and Theorem 3 we have proved the following result:

Corollary 1. *Let $u_n = F(x, u^{(n-1)})$ be an n-th order ordinary differential equation, where F is a an analytic function in some open subset $M^{(n-1)}$. There exists an integrating factor $\mu(x, u^{(k)})$ for some $k < n$.*

4 Examples

4.1 Example 1

By using a method based on variational derivatives, N Ibragimov ([2], Example 6.6.5) has proved that the second-order equation

$$u_{xx} - \frac{u_x^2}{u} - \frac{\left(x^2+x\right)u_x}{u} + 2x + 1 = 0 \tag{4.1}$$

admits an integrating factor.

For this example we use the method based on $\lambda-$symmetries to calculate an integrating factor for Eq. (4.1). Although this equation does not admit Lie point symmetries, by Theorem 3 we know that Eq. (4.1) admits the vector field $v = \partial_u$ as a $\lambda-$symmetry, where λ is any particular solution to the following equation:

$$\begin{aligned}&(-2xu^2 - u^2 + uu_x^2 + x^2uu_x + xuu_x)\lambda_{u_x} + u^2u_x\lambda_u + u^2\lambda_x\\&+\lambda^2u^2 - (x^2u + xu + 2uu_x)\lambda + x^2u_x + xu_x = 0,\end{aligned} \tag{4.2}$$

that corresponds to Eq. (3.3).

For the sake of simplicity we try to find a solution λ of (4.2) of the form $\lambda(x,u) = \lambda_1(x,u)u_x + \lambda_2(x,u)$. This ansatz leads to the following system

$$\lambda_1^2u^2 + \lambda_{1u}u^2 - \lambda_1u + 1 = 0, \tag{4.3a}$$

$$2\lambda_1\lambda_2u^2 + \lambda_{2u}u^2 + \lambda_{1x}u^2 - 2\lambda_2u + x^2 + x = 0, \tag{4.3b}$$

$$\lambda_2^2u^2 - 2x\lambda_1u^2 - \lambda_1u^2 + \lambda_{2x}u^2 - x^2\lambda_2u - x\lambda_2u = 0. \tag{4.3c}$$

A particular solution of the first equation is given by $\lambda_1(x,u) = \frac{1}{u}$. The two remaining equations become:

$$\lambda_{2u}u^2 + x^2 + x = 0, \tag{4.4a}$$
$$\lambda_2^2 u^2 + \lambda_{2x}u^2 - 2xu - x^2\lambda_2 u - x\lambda_2 u - u = 0. \tag{4.4b}$$

The general solution of the first equation is given by $\lambda_2(x,u) = \frac{x^2}{u} + \frac{x}{u} + \lambda_{21}(x)$. Since the last equation becomes $\left(\lambda_{21}(x)^2 + \lambda_{21}'(x)\right)u^2 + \left(\lambda_{21}(x)x^2 + \lambda_{21}(x)x\right)u = 0$, we can choose $\lambda_{21}(x) = 0$. In consequence $v = \partial_u$ is a $\lambda-$symmetry for $\lambda(x,u) = \frac{u_x+x^2+x}{u}$.

In order to find an integrating factor associated to λ we must find a first-order invariant $w(x,u,u_x)$ of $v^{[\lambda,(1)]}$. The equation that corresponds to (2.11) is

$$w_u + \frac{u_x + x^2 + x}{u} w_{u_x} = 0. \tag{4.5}$$

It is clear that $w(x,u,u_x) = \frac{u_x+x^2+x}{u}$ is a solution of (4.5). In terms of $\{x,w,w_1\}$ Eq. (4.1) becomes $uw_1 = 0$, the general solution of which is given by $w = C, C \in \mathbb{R}$. According to (2.13) an integrating factor is given by:

$$\mu(x,u,u_x) = w_{u_x}(x,u,u_x) = \frac{1}{u}. \tag{4.6}$$

We observe that the method we have followed not only provides the integrating factor but also gives the conserved form of the equation without additional computation. In this example the conserved form of the resulting equation is given by $D_x\left(\frac{u_x+x^2+x}{u}\right) = 0$.

4.2 Example 2

The equation

$$u_{xx} + \frac{x^2}{4u^3} + u + \frac{1}{2u} = 0 \tag{4.7}$$

has been considered in [6] as an example of equation without Lie symmetries that can be integrated, because it has a λ-symmetry.

According with the method proposed in [2] the integrating factors $\mu(x,u,u_x)$ of (4.7) are determined by the equations (see Eqs. 6.6.17-18 in [2]):

$$(-4u^4 - 2u^2 - x^2)\mu_{u_1u_1} + 4u_1\mu_{uu_1}u^3 + 4\mu_{xu_1}u^3 + 8\mu_u u^3 = 0, \tag{4.8a}$$
$$\begin{aligned}&(-4u^5 - 2u^3 - x^2u)\mu_{xu_1} + (-4u_1u^5 - 2u_1u^3 - u_1x^2u)\mu_{uu_1}\\ &\quad + 4u^4u_1^2\mu_{uu} + 8u_1u^4\mu_{xu} + 4\mu_{xx}u^4\\ &\quad + (-4u_1u^4 + 2u_1u^2 - 2xu + 3u_1x^2)\mu_{u_1} + (4u^5 + 2u^3 + x^2u)\mu_u\\ &\quad + (4u^4 - 2u^2 - 3x^2)\mu = 0.\end{aligned} \tag{4.8b}$$

Although this system is linear in μ, the compatibility of the equations cannot *a priori* be determined. The method based on $\lambda-$symmetries may help to simplify the computations to find integrating factors and conserved forms: it does only require a particular solution of the first-order partial differential equation corresponding to (3.3):

$$(-4u^5-x^2u-2u^3)\lambda_{u_x}+4u_1u^4\lambda_u+4u^4\lambda_x=-4\lambda^2u^4-4u^4+2u^2+3x^2. \tag{4.9}$$

This equation is satisfied, for instance, by $\lambda(x,u,u_x)=\frac{u_x}{u}+\frac{x}{u^2}$. A first-order invariant of $v^{[\lambda,(1)]}$ is given by $w(x,u,u_x)=\frac{u_x}{u}+\frac{x}{2u^2}$ and the general solution of the reduced equation $w_x+w^2+1=0$ is given in implicit form by $\arctan(w)+x=C, C\in\mathbb{R}$. Therefore an integrating factor that corresponds to (2.13) is given by

$$\mu(x,u,u_x)=\frac{1}{1+(\frac{u_x}{u}+\frac{x}{2u^2})^2}+\frac{1}{u} \tag{4.10}$$

and $D_x\left(\arctan(\frac{u_x}{u}+\frac{x}{2u^2})+x\right)=0$ is a conserved form of the equation. It does not seem obvious how to find μ directly from (4.8).

4.3 Example 3

The method we have used in the proof of Theorem 2 allows the construction of integrating factors of an equation by using integrating factors of the reduced equation via a λ-symmetry of the original equation. In this example we show how the method works in practice and how the original equation can be written in a conserved form.

When an $(n-1)$th-order equation of the form

$$\Delta(y,w^{(n-1)})=0 \tag{4.11}$$

is transformed by

$$\left.\begin{array}{l} y=y(x,u),\\ w=w(x,u,u_x), \end{array}\right\} \tag{4.12}$$

we obtain a new nth-order equation

$$\widetilde{\Delta}(x,u^{(n)})=0. \tag{4.13}$$

In [7] (Th. 3.1) it is proved that any of these equations (4.13) admits a $\lambda-$symmetry and that (4.11) is the corresponding reduced equation. In this way the method we have used in the proof of Theorem 2 permits the construction of integrating factors of any equation of the family (4.13) by using a known integrating factor of (4.11).

In this example we show how the method works in practice. We consider again the equation

$$w_{xx}+\frac{x^2}{4w^3}+w+\frac{1}{2w}=0. \tag{4.14}$$

By means of the transformation $y=x$, $w=\frac{u_x-1}{u}$ Eq. (4.14) is transformed into the third-order equation

$$u_{xxx}=u\left(-\frac{x^2u^3}{4(u_x-1)^3}-\frac{u}{2(u_x-1)}-\frac{u_x}{u}+\frac{1}{u}-\frac{(1-3u_x)u_{xx}}{u^2}-\frac{2(u_x-1)u_x^2}{u^3}\right). \tag{4.15}$$

This equation admits the vector field $v = \partial_u$ as a $\lambda-$symmetry [7] for

$$\lambda = -\frac{w_u}{w_{u_x}}, \tag{4.16}$$

i.e. $\lambda = \frac{u_x-1}{u}$. In terms of variables $\{x, u, w^{(2)}\}$ Eq. (4.15) becomes

$$u(w_{xx} + \frac{x^2}{4w^3} + w + \frac{1}{2w}) = 0. \tag{4.17}$$

As we proved in Example 2 $\mu(x,w,w_x) = \frac{1}{1+(\frac{w_x}{w}+\frac{x}{2w^2})^2} + \frac{1}{w}$ is an integrating factor of the reduced equation (4.14) and

$$\mu(x,w,w_x)\left(w_{xx} + \frac{x^2}{4w^3} + w + \frac{1}{2w}\right) = D_x\left(\arctan(\frac{w_x}{w} + \frac{x}{2w^2}) + x\right). \tag{4.18}$$

In consequence by (2.18)

$$\frac{\mu(x,w,w_1)}{u} = \frac{4u^2(u_x-1)^3}{x^2u^6+4(u_x-1)(-u_x^2+u_x+uu_{xx})xu^3+4(u_x-1)^4u^2+4(u_x-1)^2(-u_x^2+u_x+uu_{xx})^2} \tag{4.19}$$

is an integrating factor of Eq. (4.15) and

$$D_x\left(x + \arctan\left(\frac{xu^2}{2(u_x-1)^2} + \frac{-u_x^2+u_x+uu_{xx}}{(u_x-1)u}\right)\right) = 0 \tag{4.20}$$

is the associated conserved form of Eq. (4.15).

5 Conclusions

In this paper we prove the equivalence between the existence of integrating factors and the existence of $\lambda-$symmetries for ordinary differential equations of any arbitrary order n. This provides an alternative approach to the problem of determining integrating factors.

The method may simplify the computations derived by other methods: from the $\lambda-$symmetry and the associated algorithm of reduction, the integrating factor and the associated first integrals can be determined without additional computations.

We also provide an existence theorem of $\lambda-$symmetries for ordinary differential equations of arbitrary order that gives a new independent proof of the existence of integrating factors.

Several examples show how the method works in practice, including several equations that have no Lie point symmetries.

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