Journal of Nonlinear Mathematical Physics Volume 15, Supplement 3 (2008), 251–263 ARTICLE

Fedosov Quantization in White Noise Analysis

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Abstract

We define the deformation quantization in the Fedosov sense for a limit model of Taubes in white noise analysis.

1 Introduction

Let us recall the program of deformation quantization ([3], [4]).

Let A be a Frechet unital commutative complex algebra endowed with a Poisson structure. It is a continuous antisymmetric bilinear map $\{,\}$ from $A \times A$ into A satisfying the Jacobi equation and which vanishes on 1 which is a derivation in each variable. Deformation quantization is the following:

We consider the set of formal series $A[[h]]$ in A. It is a $C[[h]]$-algebra, where $C[[h]]$ denotes the set of formal series with values in the complex numbers . We want to define a $C[[h]]$ linear product $*$ on $A[[h]]$ such that:

i)$A[[h]]$ endowed with $*$ is a non commutative algebra.

ii)If F and G are in A

$$F * G = \sum h^k P_k[F,G] \tag{1.1}$$

where P_k is continuous from $A \times A$ into A and $P_k[1,F] = 0$ if $k > 0$.

iii)$P_0[F,G] = FG$ for F and G in A and

$$P_1[F,G] - P_1[G,F] = -ih\{F,G\} \tag{1.2}$$

If we consider the case where the algebra A is the space of smooth functions on R^d endowed with a non degenerated antisymmetric form, the algebra A inherits of a Poisson structure and we can consider the Moyal product on it.

Let M be a compact symplectic manifold. $A = C^\infty(M)$ inherits of a Poisson structure and Fedosov ([9]) constructed a $*$-product on it by glueing together all the Moyal product on its tangent spaces through a suitable connection on the Weyl bundle ([24]) on M.

We are motivated by an extension of Fedosov construction in the infinite dimensional setting. Dito ([6]) has defined the Moyal product on a Hilbert space. If we consider as algebra the algebra of functionals of Malliavin type on a Wiener space, Dito-Léandre ([7]) has extended the construction of [6]. Léandre ([17]) has considered another constant symplectic form on the underlying Hilbert space of the Wiener space, which leads to a non-bounded Poisson structure on the space of test functionals in the Malliavin sense on the Wiener space. This leads Léandre ([17]) to consider a Hida test Bosonic Fock space instead of the algebra of functionals smooth in the Malliavin sense on the Wiener space.

We are motivated by an extension of the works [6], [7], [17] to the free loop space of a manifold. Analysis on the free loop space of a manifold is very important for mathematical physics: see for instance the seminal work of Witten ([29]). Taubes ([27]) has considered a limit model for the free loop space: it is constituted as the family of flat loop spaces in each tangent space of the manifold. On each loop space on each tangent space $T_x(M)$ of M, Taubes considers a supersymmetric Fock space, carries Gaussian analysis on it and studies how this analysis depends on the finite dimensional parameter x. Taubes limit model has its counterpart in stochastic analysis in the works of Jones-Léandre ([12]), Léandre-Roan ([18]) and Léandre ([13], [14], [15], [16]) motivated by the Index theory on the free loop space.

In this paper, we consider a Taubes limit model, but instead of considering a family of ordinary Fock spaces as Taubes did, we consider a family of weighted Hida Fock spaces. We can repeat in this infinite dimensional situation the considerations of Fedosov.

For people interested by deformation quantization, we refer to the survey of Dito-Sternheimer ([8]), Maeda ([20]) and Weinstein ([28]).

We thank G. Dito and I. Mitoma for helpfull discutions.

2 Construction of the model

Let M be a compact Riemannian manifold endowed with a symplectic form ω. Let $T(M)$ be the tangent bundle of M endowed with the natural projection p on M. We consider the set T_∞ of finite energy paths $s \to \gamma(s)$ from $[0,1]$ into $T(M)$ such that for s, s' $p(\gamma(s)) = p(\gamma(s'))$. We consider on the set on paths starting from 0 in the tangent space $T_x(M)$ at x of M the Hilbert norm:

$$\int_0^1 |d/ds\gamma(s)|^2 ds \tag{2.1}$$

If $e_i(x)$ is a local orthonormal basis of $T_x(M)$, we get an orthonormal basis of this Hilbert space H_x by putting

$$X_n(e_i(x))(s) = \frac{\cos[2\pi ns] - 1}{2\pi n} e_i(x) \tag{2.2}$$

if $n > 0$ and if $n < 0$

$$X_n(e_i(x))(s) = \frac{\sin[2\pi ns]}{2\pi n} e_i(x) \tag{2.3}$$

and $X_0(e_i(x)) = e_i(x)$ We consider this Hilbert space H_x of paths of the type (3) such that H_x has an orthonormal basis given by the $X_n(e_i(x))$ $n \neq 0$

Let $A = (n_1, i_1), .., (n_{i_{|A|}}, i_{i_{|A|}})$ with $n_i \neq 0$. We consider the normalized symmetric tensor product

$$X_A(e)(x) = X_{n_1}(e_{i_1}(x)) \hat{\otimes} ... \hat{\otimes} X_{n_{|A|}}(e_{i_{|A|}}(x)) \tag{2.4}$$

We introduce the Hida weight

$$\|A\| = \prod_{(n_l,i_l)\in A} (|n_l|^2+1) \tag{2.5}$$

Let $C > 0$ and $p > 0$. We consider the weighted Hida Fock space $S_{C,p}(x)$ of sums

$$\sum \lambda_A(x) X_A(e)(x) = \xi(x) \tag{2.6}$$

such that

$$\sum |\lambda_A(x)|^2 C^{|A|} \|A\|^p = \|\xi(x)\|^2_{C,p,x} < \infty \tag{2.7}$$

$S_{C,p}$ realizes a bundle on M in complex Hilbert spaces because the components $\lambda_A(x)$ are chosen complex. The Levi-Civita connection lifts on it into a Hermitian connection.

Definition 1. $W.N_{\infty-}$ is the space of sections ξ belonging to all $S_{C,p}$ such that for all C, p, k, r

$$\sum_{k'\leq k} \int_M \|\nabla^{k'} \xi(x)\|^r_{C,p,x} dx = \|\xi\|^r_{C,p,k,r} < \infty \tag{2.8}$$

for all $C > 0, p > 0, k \in N$.

If we consider two normalized elementary tensor products, we take as product the normalized tensor product got by concatenation of the indices of each. We get therefore a product which can be extended to $W.N_{\infty-}$.

Theorem 1. *$W.N_{\infty-}$ is a topological algebra.*

Proof. Let be

$$\xi^1(x) = \sum \lambda^1_A(x) X_A(e)(x) \tag{2.9}$$

$$\xi^2(x) = \sum \lambda^2_A(x) X_A(e)(x) \tag{2.10}$$

be two elements of $W.N_{\infty-}$. We have clearly

$$(\xi^1.\xi^2)(x) = \sum \mu_A(x) X_A(e)(x) \tag{2.11}$$

where

$$\mu_A(x) = \sum \lambda^1_{B^1}(x) \lambda^2_{B^2}(x) \tag{2.12}$$

where we take the sum on multiindices B^1 and B^2 whose concatenation is A. By Jensen inequality and chain rules for derivatives, we get that

$$\sum_{k'\leq k} \|\nabla^{k'} \mu_A(x)\|^2 \leq K C^{|A|} \sum (\sum_{k'\leq k} \|\nabla^{k'} \lambda^1_{B^1}(x)\|^2)(\sum_{k'\leq k} \|\nabla^{k'} \lambda^1_{B^1}(x)\|^2) \tag{2.13}$$

where we take the sum on B^1 and B^2 whose concatenation is equal to A. But in such a case

$$\|A\| = \|B^1\| \|B^2\| \tag{2.14}$$

By Hoelder inequality we deduce that:

$$\|(\xi^1.\xi^2)\|_{C,p,k,r} \leq K \|\xi^1\|_{C',p,k,2r} \|\xi^2\|_{C',p,k,2r} \tag{2.15}$$

for some $C' > C$. ■

We can introduce the following symplectic form on the limit model T_∞:

$$\Omega(\gamma^1,\gamma^2) = \int_0^1 \omega(d/ds\gamma^1(s), d/ds\gamma^2(s))ds + \omega(\gamma^1(0),\gamma^2(0)) \tag{2.16}$$

Let $I=(n,i)$ and $J=(m,j)$.We put

$$\Omega^{I,J}(x) = \delta_{n,m}\omega^{i,j}(x) \tag{2.17}$$

where $\omega^{i,j}(x)$ are the elements of the inverse matrix of the matrix of ω in a normal local coordinates system centered in x. $\delta_{n,m}$ is the Kronecker symbol.

We put

$$\nabla_I = a_{n,i} \tag{2.18}$$

which is the normalized annihilation operator on the Bosonic Fock space if $n \neq 0$ ([17]). We put $\nabla_I = \frac{\partial}{\partial x_i}$ if $I=(0,i)$.

We define if ξ^1 and ξ^2 belong to $W.N_{\infty-}$:

$$\{\xi^1,\xi^2\}(x) = \sum \Omega^{I,J}(x)\nabla_I\xi^1(x)\nabla_J\xi^2(x) \tag{2.19}$$

Theorem 2. *$\{,\}$ defines a continuous Poisson Bracket on $W.N_{\infty-}$.*

Proof. The fact that $\{,\}$ satisfies the same algebraic properties than a standard Poisson bracket holds by the same way than in finite dimension.

Let us show the continuity of $\{,\}$. Only in the series the sum where in $I=(n,j)$ $n\neq 0$ put some difficulties.

We have, modulo this restriction,

$$\{\xi^1,\xi^2\}(x) = \sum \mu_A(x)X_A(e)(x) \tag{2.20}$$

where

$$\mu_A(x) = \sum \lambda^1_{B^1\cup I}(x)\lambda^2_{B^2\cup J}(x)\Omega^{I,J}(x) \tag{2.21}$$

where we sum on B^1, I, B^2, J whose concatenation is equal to A. There are at most $C^{|A|}$ terms in the previous sum. By Jensen inequality

$$|\mu_A(x)|^2 \leq C^{|A|}\sum |\lambda^1_{B^1\cup I}|^2|\lambda^2_{B^2\cup J}|^2|\Omega^{I,J}(x)|^2 \tag{2.22}$$

where we sum on the same set.

On the other hand for any p there exists an enough big p_1 such that:

$$\|A\|^p|\mu_A(x)|^2 \leq KC^{|A|}\sum |\lambda^1_{B^1\cup I}(x)|^2\|B^1\cup I\|^{p_1}|\lambda^2_{B^2\cup J}(x)|^2\|B^2\cup J\|^{p_1} \tag{2.23}$$

where we take the sum on the same set. We can patch together this last inequality and the inequalities of the type (2.14) where we take the derivatives of $\mu_A(x)$ in x in order to show that;

$$\|\{\xi^1,\xi^2\}\|_{C,p,k,r} \leq K\|\xi^1\|_{C_1,p_1,k,2r}\|\xi^2\|_{C_1,p_1,k,2r} \tag{2.24}$$

for enough big C_1 and p_1. ■

The goal of this paper is then to define a $*$-product on $W.N_{\infty-}$ endowed with this Poisson bracket by using the apparatus of Fedosov.

We can define following the lines of [17] a Moyal-Weyl product on the Hida Fock space associated to $T_x(M)\oplus H_x$. The algebra is the same as in [17], the only difference which leads here to a simplification is that $\Omega^{I,J}$ is bounded.

3 The Weyl bundle in Hida sense

We consider $H_0 = T_x(M)$ and the real Hilbert space H_x of finite energy paths in $T_x(M)$ starting from 0. We put

$$H_x^t = H_0 \oplus H_x \tag{3.1}$$

The previous sum is orthogonal. If $A^s = (I_1, ..., I_{|A^s|})$ we consider the normalized symmetric tensor product

$$X_{A^s}^t(e) = X_{I_1}(e)\hat{\otimes}...\hat{\otimes}X_{I_{|A^s|}}(e) \tag{3.2}$$

and if $A^a = (I_1, .., I_{|A^a|})$ we consider the normalized exterior product:

$$X_{A^a}^t(e) = X_{I_1}(e) \wedge ... \wedge X_{I_{|A^a|}}(e) \tag{3.3}$$

We order all the multiindices in lexicographic order in order to avoid some redundances.

If A_1^s and A_2^s are two multindices, we denote by $A_1^s \cup A_2^s$ the concatenation of each multiindices after reordering them. If A_1^a and A_2^a are two indices we do the same operations denoted by the same symbol if their intersection is empty and we get the empty set if their intersection is non empty.

We consider the Hida weights $\|A^s\|$ and $\|A^a\|$ of the first part. We consider the Hida supersymmetric test Fock space $(S^t \otimes \Lambda^t)_{\infty-}$ of element of the supersymmetric Fock space

$$\Xi^t = \sum \lambda_{A^s,A^a} X_{A^s}^t(e) \otimes X_{A^a}^t(e) \tag{3.4}$$

such that for all $p > 0$ and all $C > 0$

$$\|\Xi^t\|_{C,p}^2 = \sum |\lambda_{A^s,A^a}|^2 \|A^a\|^p \|A^s\|^p C^{|A^a|+|A^s|} < \infty \tag{3.5}$$

These system of norms are invariant under an orthonormal transformation depending only on x on the basis $e_i(x)$.

The Weyl bundle in the Hida sense is the bundle $p_\infty^*(S^t \otimes \Lambda^t)_{\infty-})$ on the limit model T_∞ where p_∞ is the natural projection from the limit model on M.

We consider the moyal product on $(S^t \otimes \Lambda^t)_{\infty-}$. If

$$\Xi^{i,t} = \sum \lambda^i_{A^s,A^a} X_{A^s}^t(e) \otimes X_{A^a}^t(e) \tag{3.6}$$

we have:

$$\Xi^{1,t} \circ \Xi^{2,t} = \sum \lambda^1_{A_1^s,A_1^a} \lambda^2_{A_2^s,A_2^a} \varepsilon (X_{A_1^s}^t(e) \circ X_{A_2^s}^t(e)) \otimes X_{A_1^a \cup A_2^a}^t(e) \tag{3.7}$$

ε denotes a sign. We define the Moyal product $X_{A_1^s}^t(e) \circ X_{A_2^s}^t(e)$ as in the first part or in [17]: The only difference is that if $I = (0,i)$ we consider a standard annihilation operator on the Fock space.

Theorem 3. *The Moyal product* $\circ$ *is continuous from* $(S^t \otimes \Lambda^t)_{\infty-} \times (S^t \otimes \Lambda^t)_{\infty-}$ *into* $(S^t \otimes \Lambda^t)_{\infty-}[[h]]$.

Proof. We write

$$\Xi^{i,t} = \sum \xi_{A^a}^{i,t} \otimes X_{A^a}^t(e) \tag{3.8}$$

such that

$$\Xi^{1,t} \circ \Xi^{2,t} = \sum_{A_1^a \cup A_2^a = A^a} \varepsilon(\xi^{1,t}_{A_1^a} \circ \xi^{2,t}_{A_2^a}) X^t_{A^a}(e) \tag{3.9}$$

We consider the component of degree n in the formal series in h. We get that

$$(\Xi^{1,t} \circ \Xi^{2,t})_n = \sum_{A_1^a \cup A_2^a = A^a} \varepsilon(\xi^{1,t}_{A_1^a} \circ \xi^{2,t}_{A_2^a})_n X^t_{A^a}(e) \tag{3.10}$$

We get that

$$\|(\Xi^{1,t} \circ \Xi^{2,t})_n\|^2_{C,p} = \sum_{A^a} \| \sum_{A_1^a \cup A_2^a = A^a} (\xi^{1,t}_{A_1^a} \circ \xi^{2,t}_{A_2^a})_n \|^2_{C,p} C^{|A^a|} \|A^a\|^p \tag{3.11}$$

By Jensen inequality,

$$\| \sum_{A_1^a \cup A_2^a = A^a} (\xi^{1,t}_{A_1^a} \circ \xi^{2,t}_{A_2^a})_n \|^2_{C,p} \le C^{|A^a|} \sum_{A_1^a \cup A_2^a = A^a} \|(\xi^{1,t}_{A_1^a} \circ \xi^{2,t}_{A_2^a})_n\|^2_{C,p} \tag{3.12}$$

By using the same considerations as in [17], but in a simpler way, because $\Omega^{I,J}$ is bounded,

$$\|(\xi^{1,t}_{A_1^a} \circ \xi^{2,t}_{A_2^a})_n\|^2_{C,p} \le C_n \|\xi^{1,t}_{A_1^a}\|^2_{C_1,p_1} \|\xi^{2,t}_{A_2^a}\|^2_{C_1,p_1} \tag{3.13}$$

We remark that $C^{|A^a|} = C^{|A_1^a|} C^{|A_2^a|}$ and that $\|A^a\| = \|A_1^a\| \|A_2^a\|$ if $A_1^a \cup A_2^a = A^a$. We deduce that

$$\|(\Xi^{1,t} \circ \Xi^{2,t})\|^2_{C,p} \le K_n \|\Xi^{1,t}\|^2_{C_1,p_1} \|\Xi^{2,t}\|^2_{C_1,p_1} \tag{3.14}$$

Therefore the result. ■

Let be the ordinary Fock space. We consider the Shigekawa complex

$$\delta = \sum a_I^s a_I^{a*} \tag{3.15}$$

where a_I^s is the family of annihilation operators on the Bosonic Fock space, a_I^{s*} the family of creation operators on the Bosonic Fock space, a_I^a the family of annihilation operators on the Fermionic Fock space and a_I^{a*} the family of creation operators on the Fermionic Fock space. We have:

$$\delta^* = \sum a_I^a a_I^{s*} \tag{3.16}$$

Let us recall ([25])

$$\delta\delta^* + \delta^*\delta = N_B + N_F \tag{3.17}$$

where N_B denotes the Bosonic number operator on the Bosonic Fock space and N_F denotes the Fermionic number operator on the Fermionic Fock space.

Theorem 4. *δ and δ^* are continuous on $(S^t \otimes \Lambda^t)_{\infty-}$.*

Proof. Let be

$$\Xi^t = \sum \lambda_{A^s,A^a} X^t_{A^s}(e) \otimes X^t_{A^a}(e) \tag{3.18}$$

We have

$$\delta \Xi^t = \sum \mu_{A^s,A^a} X^t_{A^s}(e) \otimes X^t_{A^a}(e) \tag{3.19}$$

where

$$|\mu_{A^s,A^a}|^2 \leq (\sum_{I \in A^a} |\lambda_{A^s \cup I, A^a - I}|)^2 \leq C^{|A^a|} \sum_{I \in A^a} |\lambda_{A^s \cup I, A^a - I}|^2 \tag{3.20}$$

such that

$$\|\delta \Xi^t\|^2_{C,p} \leq \sum_{A^s,A^a} \sum_{I \in A^a} |\lambda_{A^s \cup I, A^a - I}|^2 \|A^s \cup I\|^{p_1} \|A^a - I\|^{p_1} C_1^{|A^s \cup I| + |A^a - I|} \tag{3.21}$$

There are at most $|A^s|$ way to write $\lambda_{A^s,A^a} = \lambda_{A_1^s \cup I, A_1^s - I}$ Therefore

$$\|\delta \Xi^t\|^2_{C,p} \leq K \|\Xi^t\|^2_{C_1,p_1} \tag{3.22}$$

for K, C_1, p_1 independent of x in M.

Let us consider δ^*:

$$\delta^* \Xi^t = \sum \mu_{A^s,A^a} X^t_{A^s}(e) \otimes X^t_{A^a}(e) \tag{3.23}$$

where

$$|\mu_{A^s,A^a}| \leq \sum_{I \in A^s} |\lambda_{A^s - I, A^a \cup I}| \tag{3.24}$$

We do as before by interchanging the role of A^s and A^a in order to show that:

$$\|\delta^* \Xi^t\|^2_{C,p} \leq K \|\Xi^t\|^2_{C_1,p_1} \tag{3.25}$$

for K, C_1, p_1 independent from x in the compact manifold M. ■

Remark: We refer to the works of Arai-Mitoma ([1], [2]) for this statement.

Let us recall classically ([24]) that

$$\delta^2 = 0;\ \delta^{*2} = 0;\ \delta\delta^* + \delta^*\delta = N_B + N_F \tag{3.26}$$

We can speak of sections in Hida sense of the Weyl bundle on the Taubes limit model of $p^*_\infty(S^t \otimes \Lambda^t)_{\infty-}$. Let us recall for that the definition of $S_{C,p}(x)$ in the first part. We consider the Levi-Civita connection on M. This Levi-Civita connection lifts to the bundle on M in Hilbert space $S_{C,p} \otimes (S^t \otimes \Lambda^t)_{C_1,p_1}$.

Definition 2. The space of sections in the Taubes-Hida sense of the bundle on the limit model $p^*_\infty(S^t \otimes \Lambda^t)_{\infty-}$ is the space of sections ψ of the bundle $S_{C,p} \otimes (S^t \otimes \Lambda^t)_{C_1,p_1}$ such that for all C, p, C_1, p_1, k, r positive numbers (k is an integer)

$$\sum_{k' \leq k} \int_M \|\nabla^k \psi(x)\|^r_{C,p,C_1,p_1} dx = \|\psi\|^r_{C,p,C_1,p_1} < \infty \tag{3.27}$$

We call this space $W.N_{\infty-}(p^*_\infty(S^t \otimes \Lambda^t)_{\infty-})$.

Remark: This space does not depend on the choice of the unitary connection on $T(M)$, because we consider Hida test functional spaces.

We have the following properties:

1) δ and δ^* are continuous on $W.N_{\infty-}(p_\infty^*(S^t\otimes\Lambda^t)_{\infty-})$.

2) $\circ$ is continuous from $W.N_{\infty-}(p_\infty^*(S^t\otimes\Lambda^t)_{\infty-})\times W.N_{\infty-}(p_\infty^*(S^t\otimes\Lambda^t)_{\infty-})$ into the Hida space $W.N_{\infty-}(p_\infty^*(S^t\otimes\Lambda^t)_{\infty-})$.

3) If ψ_i belong to $W.N_{\infty-}(p_\infty^*(S^t\otimes\Lambda^t)_{\infty-})$,

$$\delta(\psi_1\circ\psi_2)=(\delta\psi_1)\circ\psi_2+\varepsilon\psi_1\circ\delta\psi_2 \tag{3.28}$$

where $\varepsilon=(-1)^{r_1}$ if $N_F\psi_1=r_1\psi_1$.

We consider a symplectic connection Γ^s on $T(M)$: it is a connection without torsion which preserves the symplectic form ω. The tangent bundle of the Taubes limit model is $p_\infty^*H^t$. The symplectic connection lifts to a symplectic connection $\Gamma^{s,\infty}$ for the symplectic structure given by Ω on the limit model. This lifts to a symplectic connection denoted by $\partial_{\Gamma^{s,\infty}}$ on $p_\infty^*(S^t\otimes\Lambda^t)_{\infty-}$.

We have the main theorem of this part:

Theorem 5. *$\partial_{\Gamma^{s,\infty}}$ is a continuous application on $W.N_{\infty-}(p_\infty^*(S^t\otimes\Lambda^t)_{\infty-})$.*

Proof. We can work in local coordinates in x. Let Γ^s be the connection 1-form of the symplectic connection form on M. The connection 1-form on $(p_\infty^*(S^t\otimes\Lambda^t)_{\infty-})$ acts as a second quantized operator of Γ^s and is therefore continuous. This means that $X^t_{A^s}(e)$ is transformed into

$$\sum_i X^t_{I_1}(e)\hat{\otimes}..\hat{\otimes}X^t_{I_{i-1}}(e)\hat{\otimes}\Gamma^s(X^t_{I_i}(e))\hat{\otimes}X^t_{I_{i+1}}(e)\hat{\otimes}..\hat{\otimes}X^t_{I_{i_{|A|}}}(e)=\Gamma^{s,\infty}X^t_{A^s}(e) \tag{3.29}$$

So we have only to show that d_∞ is in local coordinates in $x\in M$ continuous on $W.N_{\infty-}(p_\infty^*(S^t\otimes\Lambda^t)_{\infty-})$. Let ψ an element of this space:

$$\psi=\sum_{A,A^{t,s},A^{t,a}}\lambda_{A,A^{t,s},A^{t,a}}X_A(e)\otimes X^t_{A^{t,s}}(e)\otimes X^a_{A^{t,a}}(e) \tag{3.30}$$

$X_A(e)$ is considered as an element of $S_{C,p}(x)$. Only the series which appears when we take derivatives in the direction of H_x put any problem. We consider the Bosonic annihilation a_I^s on $S_{C,p}(x)$ and associated to it the creation Fermionic operator $a_I^{*,a,t}$ on this time on $\Lambda^t(x)$. The possible divergent series is

$$d^1_\infty\psi=\sum_{A,A^{t,s},A^{t,a}}\lambda_{A,A^{t,s},A^{t,a}}\sum_{I\in A}a_I^sX_A(e)\otimes X^t_{A^{t,s}}(e)\otimes a_I^{*,a,t}X^a_{A^{t,a}}(e) \tag{3.31}$$

such that

$$d^1_\infty\psi=\sum_{A,A^{t,s},A^{t,a}}\mu_{A,A^{t,s},A^{t,a}}X_A(e)\otimes X^t_{A^{t,s}}(e)\otimes X^a_{A^{t,a}}(e) \tag{3.32}$$

where

$$\mu_{A,A^{t,s},A^{t,a}}=\sum_{I\in A^{t,a}}\lambda_{A\cup I,A^{t,s},A^{t,a}-I} \tag{3.33}$$

By Jensen inequality, we have that:

$$|\mu_{A,A^{t,s},A^{t,a}}|^2 \leq C^{|A^{t,a}|} \sum_{I \in A^{t,a}} |\lambda_{A\cup I,A^{t,s},A^{t,a}-I}|^2 \tag{3.34}$$

We remark that in the previous sum

$$\|A\cup I\|^p \|A^{t,s}\|^{p_1} \|A^{t,a}-I\|^{p_1} \leq (\|A\| \|A^{t,s}\| \|A^{a,s}\|)^{p_2} \tag{3.35}$$

We conclude as in the in the proof of Theorem 5. ■

As in [9], we have that:

$$\partial_{\Gamma^{s,\infty}} \delta + \delta \partial_{\Gamma^{s,\infty}} = 0 \tag{3.36}$$

and if ψ_i are elements of $W.N_{\infty-}(p_\infty^*(S^t \otimes \Lambda^t)_{\infty-})$ with $N_F \psi_1 = r_1 \psi_1$ we have that:

$$\partial_{\Gamma^{s,\infty}}(\psi_1 \circ \psi_2) = (\partial_{\Gamma^{s,\infty}}(\psi_1)) \circ \psi_2 + (-1)^{r_1} \psi_1 \circ \partial_{\Gamma^{s,\infty}}(\psi_2) \tag{3.37}$$

4 Abelian connection and quantization

We consider the formal series in $(S^t \otimes \Lambda^t)_{\infty-}$ called $(S^t \otimes \Lambda^t)_{\infty-}[[h]]$. On this space, we have a grading by counting twice the power of h and 1 the length of the Boson. We get a space $(S^t \otimes \Lambda^t)^l_{\infty-}$ and we consider formal series $\sum_{l\geq 0}(S^t \otimes \Lambda^t)^l_{\infty-}$. The Moyal product applies continuously $(S^t \otimes \Lambda^t)^{l_1}_{\infty-} \times (S^t \otimes \Lambda^t)^{l_2}_{\infty-}$ into $(S^t \otimes \Lambda^t)^{l_1+l_2}_{\infty-}$.

We consider a continuous operator R^l from $(S^t)^l_{\infty-}$ into $\prod_{L'\geq l}(S^t)^{l'}_{\infty-} \otimes \Lambda_1(T_x(M))$ such that $R = \sum R^l$ is a continuous operator from $\prod_{l\geq 0}(S^t)^l_{\infty-}$ into $\prod_{l\geq 0}(S^t)^l_{\infty-} \otimes \Lambda_1(T_x(M))$.

This definition arises from the work of Fedosov ([9]):

Definition 3. We consider the operator $\partial = \partial_{\Gamma^{s,\infty}} - \delta + R$. It is called an Abelian Hida-Taubes connection if:

-)∂ maps continuously $W.N_{\infty-}(p_\infty^* \prod (S^t \otimes \Lambda^t)_{\infty-})$.

-)$\partial(\psi_1 \circ \psi_2) = \partial \psi_1 \circ \psi_2 + \psi_1 \circ \partial \psi_2$ for any elements ψ_i of $W.N_{\infty-}(p_\infty^* \prod (S^t)^l_{\infty-})$.

-)$\partial^2 = 0$.

We consider $(S^{f,t})^l$ the set of finite combinations of $X_A^t(e)$ of total grading l.

The fiberwise Moyal product can be defined on $(S^{f,t})^{l_1} \times (S^{f,t})^{l_2}$ and we can consider formal series in $(S^{f,t})^l$: this space is denoted by $\prod (S^{f,t})^l$. We can define $R^{f,l}$ in this situation as before and $R^f = \sum R^{f,l}$ and the notion of combinatorial connection ∂^f in this context. In order to do that, we replace in the Hida Fock space $S_{C,p}(x)$ the series by finite combinations and we get a combinatorial Fock space S^f. So we get a combinatorial limit model $W^f(p_\infty^* \prod_l (S^{f,t})^l)$ and a combinatorial Abelian connection

$$\partial^f = \partial_{\Gamma^{s,\infty}} - \delta + R^f \tag{4.1}$$

We remark that ∂^f acts on $W^f(p_\infty^*(\prod_{l\geq 0}(S^{f,t})^l)$ such that $(\partial^f)^2 = 0$.

Lemma 1. *There exists a combinatorial Abelian connection.*

Proof. We consider the space $T(M)$. Ω restricts to $T(M)$, a subspace of $T_\infty(M)$, due to (2.17). We consider the Weyl bundle on $T(M)$, the connection $\partial_{\Gamma^{s,\infty}}$ restricts naturally to it as well as δ and $\circ$. By using the considerations of Fedosov ([9]), we can a suitable Abelian connection:

$$\partial^1 = \partial^1_{\Gamma^{s,\infty}} - \delta^1 + h^{-1}[r^1,.] \tag{4.2}$$

where $r^1 \in \prod_{l\geq 3} S^l[T_x(M) \oplus (T_x(M)] \otimes \Lambda^1(T_x(M))$ and which depends only from $p(x)$ in $T(M)$. The first $T_x(M)$ holds for the x part in the Weyl algebra and the second holds for the Fourier mode considered (here the first one).

We recall that $H^f_x = \otimes_{n\neq 0} T^n_x(M)$ each part corresponding to the associated Fourier mode. We consider the n part in the Fourier modes given by (2.2) and (2.3). We get Bosons given by $X_{n,i}(e)$. Each bundle $p^*_\infty T_x(M)^n \sim p^*_\infty T_x(M)$ is parallel for the connection $\partial_{\Gamma^{s\infty}}$ and this decomposition is orthogonal for Ω. Let us consider $X^t_{A^s}(e)$ where:

$$X^t_{A^s}(e) = (X_{n_1,l_{1_{n_1}}} \hat{\otimes} ... \hat{\otimes} X_{n_1,l_{r_{n_1}}}) \hat{\otimes} .. \hat{\otimes} (X_{n_m,l_{1_{nm}}} \hat{\otimes} ... \hat{\otimes} X_{n_m,l_{r_{nm}}}) \tag{4.3}$$

for $n_1,..,n_m$.

We have that:

$$R^{f,l} X^t_{A^s}(e) = \sum (X_{n_1,l_{1_{n_1}}} \circ ... \circ R^{f,l} X_{n_1,r_{l_{n_1}-1}} \circ X_{n_1,r_{l_{n_1}}}) \circ .. \circ (X_{n_m,l_{1_{nm}}} \circ ... \circ X_{n_m,l_{r_{nm}}}) \tag{4.4}$$

such that $\partial^f X^t_{A^s}(e)$ is determined by (4.4). Namely on the symmetric Fock space determined by x and fixed Fourier modes, the combinatorial Abelian connection is determined by the previous considerations. ■

Therefore we have determined the existence of a combinatorial Abelian connection ∂^f. We will show that ∂^f can be extended in an Abelian Hida-Taubes connection.

Theorem 6. *∂^f can be extended in a Hida-Taubes Abelian connection.*

Proof. $\partial^f_{\Gamma^{s,\infty}} - \delta^f$ can be extended continuously on the Taubes-Hida test functional space: it is the object of the previous part. It is enough to show that R^f can be extended continuously on the Hida-Taubes test sections in the sense of the beginning of this part.

For that, we put in (4.3) that

$$X^t_{A^s} = X^t_{A^s_1} \circ ... \circ X^t_{A^s_m} \tag{4.5}$$

We consider an element of the Hida space $(S^t_{\infty-})^l$. It is written

$$\Xi^l = \sum_{m,|A^s_1|+..+|A^s_m|+2r=l;|A^s_i|\neq 0} h^r \lambda_{A^s_1,...,A^s_m} X^t_{A^s_1} \circ ... \circ X^t_{A^s_m} \tag{4.6}$$

such that

$$R^l \Xi^l = \sum_{m,|A^s_1|+..+|A^s_m|+2r=l;|A^s_i|\neq 0} h^r \lambda_{A^s_1,...,A^s_m} \sum X^t_{A^s_1} \circ ... \circ R^l X^t_{A^s_i} \circ X^t_{A^s_m} \tag{4.7}$$

where

$$||\Xi^l||^2_{C,p} = \sum |\lambda_{A^s_1,..,A^s_m}|^2 C^{|A^s_1|+...|A^s_m|} \prod ||A_i||^p < \infty \tag{4.8}$$

By the considerations of [9], [24] and the previous considerations:

$$R^{i,l}X^t_{A_i^s} = h^{-1}[\sum r^j, X^t_{A_i^s}] \tag{4.9}$$

where r^i belongs to $\prod_{l\geq 3} S^l[T_x(M)\oplus T_x(M)]\otimes\Lambda^1(T_x(M))$ and corresponds to the Fedosov counterterm for the i^{th} Fourier mode. It is the same formal expression for all i, which depends smoothly of x. The sum in (4.9) is reduced to one element.

If we consider the part $R^{l,l'}$ of R^l which goes from $(S^t)^l_{\infty-}$ to $(S^t)^{l+l'}_{\infty-}\otimes\Lambda_1(T_x(M))$ and if we compute the various Hida weights which appear in $R^{l,l'}X^t_{A_i^s}$, they are bounded by $\|A_i^s\|^{p+k(l,l')}$ where $k(l,l')$ is bounded. So the Hida weights which appear in

$$\sum X^t_{A_1^s}\circ\ldots\circ R^{l,l'}X^t_{A_i^s}\circ X^t_{A_m^s} \tag{4.10}$$

have a bound in $\prod\|A_i^s\|^{p+k(l,l')}$. Moreover, in

$$R^{l,l'}X^t_{A_i^s} = 1/h[\sum r^{j,l,l'}, X^t_{A_i^s}] \tag{4.11}$$

where the previous sum is reduced only to one element, there are at most $C^{|A_i^s|}$ terms.

So we can write

$$\begin{aligned}\|R^{l,l'}\Xi^l\|_{C,p} &\leq \sum|\lambda_{A_1^s,\ldots,A_m^s}|\|\sum X^t_{A_1^s}\circ\ldots\circ R^l X^t_{A_i^s}\circ X^t_{A_m^s}\|_{C,p}\\ &\leq \sum|\lambda_{A_1^s,..,A_m^s}|C_1^{|A_1^s|+..|A_m^s|}\prod\|A_i^s\|^{p+k(l,l')}\end{aligned} \tag{4.12}$$

Let us recall that if C_2 is small enough and p_2 is big enough

$$K = \sum C_2^{|A|}\|A\|^{-p_2} < \infty \tag{4.13}$$

because it is equal to

$$\prod_n(\sum C_2\sqrt{n^2+1}^{-p_2 l}+1) = \prod(1-\frac{C_2}{\sqrt{n^2+1}^{p_2}})^{-1} < \infty \tag{4.14}$$

We use Cauchy-Schwartz inequality and the previous inequality in order to show that

$$\sum|\lambda_{A_1,..,A_m}|C_1^{|A_1|+..|A_m|}\prod\|A_i\|^{p+k(l,l')} \leq K\|\Xi^l\|_{C_3,p_3} \tag{4.15}$$

So $R^{l,l'}$ is continuous from $(S^t)^l_{\infty-}(x)$ into $(S^t)^{l+l'}_{\infty-}(x)$. We can prove in an exact similar way that $R^{l,l'}$ is continuous from $W.N_{\infty-}(p^*_\infty(S^t)^l_{\infty-})$ into $W.N_{\infty-}(p^*_\infty(S^t)^{l+l'}_{\infty-})$. ■

Theorem 7. *Let F,G belonging to $W.N_{\infty-}$ and let ∂ be the Abelian Hida-Taubes connection of the previous part. There exists a unique $\psi\in W.N_{\infty-}(\prod_{l\geq 0}(S^t)^l_{\infty-})$ such that $\partial\psi = 0$ and such that the component in $(S^t)^0_{\infty-}$ of ψ equals F. We call that $\psi = QF$. The map $F\to QF$ is continuous.*

Proof. On the combinatorial model $R = \sum i/h[r^i,.]$. and there exists a flat section of the combinatorial domain such its component on $(S^{f,t})^0$ is F^f. ψ^f satisfies to

$$\delta\psi^f = \partial^f_{\Gamma^{s,\infty}}\psi^f + i/h[\sum r^i,\psi^l] \tag{4.16}$$

and this equation can be solved step by step as in theorem (3.3) of [9] by applying δ^{-1} iteratively as in [9]. But since the combinatorial model is densely continuously imbedded in the Hida-Taubes model, and since R is continuous on the Hida-Taubes model, we can solve by continuity the equation

$$\delta\psi = \partial_{\Gamma^{s,\infty}}\psi + R\psi \tag{4.17}$$

because δ and δ^{-1} are continuous (see [9] for the definition of δ^{-1}). ■

Theorem 8. *There exists a $*$ product on $W.N_{\infty-}$.*

Proof. As in [9], we put

$$\psi_1 * \psi_2 = Q^{-1}[Q(\psi_1)\circ Q(\psi_2)] \tag{4.18}$$

where Q is the isomorphism between $W.N_{\infty-}$ and the space of flat sections in the Hida-Taubes sense of $\prod_{l\geq 0}(S^t)^l_{\infty-}$. ■

Acknowledgments. This research was supported by the J.S.P.S. grant S-06029 during the visit of the author at the department of mathematics at Kyoto University.

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