Intuitionistic fuzzy preference relations

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Abstract

We consider properties of intuitionistic fuzzy preference relations. We study preservation of a preference relation by lattice operations, composition and some Atanassov's operators like $F_{\alpha,\beta}$, $P_{\alpha,\beta}$, $Q_{\alpha,\beta}$, where $\alpha, \beta \in [0, 1]$. We also define semi-properties of intuitionistic fuzzy relations, namely reflexivity, irreflexivity, connectedness, asymmetry, transitivity. Moreover, we study under which assumptions intuitionistic fuzzy preference relations fulfil these properties. In all these cases, if possible, we try to give characterizations of adequate properties.

Keywords: intuitionistic fuzzy preference relations, properties of intuitionistic fuzzy relations.

1. Introduction

We deal with Atanassov's intuitionistic fuzzy relations (for short, intuitionistic fuzzy relations) which were introduced by Atanassov [1] as a generalization of the concept of a fuzzy relation defined by Zadeh [18]. Fuzzy sets and relations have applications in diverse types of areas, for example in data bases, pattern recognition, neural networks, fuzzy modelling, economy, medicine, multicriteria decision making. Similarly, intuitionistic fuzzy sets are widely applied, for example in multiattribute decision making [10]. If it comes to the composition of intuitionistic fuzzy relations the effective approach to deal with decision making in medical diagnosis was proposed [5]. We take into account intuitionistic fuzzy preference relations which are applied in group decision making problems where a solution from the individual preferences over some set of options should be derived. The concept of a preference relation was considered by many authors, in the crisp case for example in [13] and in the fuzzy environment in [4]. The first authors who generalized the concept of preference from the fuzzy case to the intuitionistic fuzzy one, were Szmidt and Kacprzyk [14]. Next, other papers were devoted to this topic, for example [16], [15], [17].

This work is a continuation of the results presented during IWIFSGN 2010 conference. Firstly, we recall some concepts and results useful in our further considerations (section 2). Next, we put results connected with the preservation of a preference relation by lattice operations, composition and Atanassov's operators (section 3). Finally, we define some new properties of intuitionistic fuzzy relations and we check when such properties are fulfilled by intuitionistic fuzzy preference relations (section 4).

2. Basic definitions

Now we recall some definitions which will be helpful in our investigations.

Definition 1 ([1])**.** *Let $X, Y \neq \emptyset$, R, $R^d : X \times Y \to [0, 1]$ be fuzzy relations fulfilling the condition*

$$R(x, y) + R^d(x, y) \leq 1, \quad (x, y) \in (X \times Y). \quad (1)$$

A pair $\rho = (R, R^d)$ is called an Atanassov's intuitionistic fuzzy relation. The family of all Atanassov's intuitionistic fuzzy relations described in the given sets X, Y is denoted by $AIFR(X \times Y)$. In the case $X = Y$ we will use the notation $AIFR(X)$.

The boundary elements in $AIFR(X \times Y)$ are $\mathbf{1} = (1, 0)$ and $\mathbf{0} = (0, 1)$, where $0, 1$ are the constant fuzzy relations. Basic operations for $\rho = (R, R^d)$, $\sigma = (S, S^d) \in AIFR(X \times Y)$ are the union and the intersection, respectively

$$\rho \vee \sigma = (R \vee S, R^d \wedge S^d), \ \rho \wedge \sigma = (R \wedge S, R^d \vee S^d). \quad (2)$$

Similarly, for arbitrary set $T \neq \emptyset$

$$(\bigvee_{t\in T} \rho_t)(x, y) = (\bigvee_{t\in T} R_t(x, y), \bigwedge_{t\in T} R^d_t(x, y)),$$

$$(\bigwedge_{t\in T} \rho_t)(x, y) = (\bigwedge_{t\in T} R_t(x, y), \bigvee_{t\in T} R^d_t(x, y)).$$

Moreover, the order is defined by

$$\rho \leq \sigma \ \Leftrightarrow \ (R \leq S, \ S^d \leq R^d). \quad (3)$$

The pair $(AIFR(X \times Y), \leq)$ is a partially ordered set. Operations $\vee, \wedge$ are the binary supremum and infimum in the family $AIFR(X \times Y)$, respectively. The family $(AIFR(X \times Y), \vee, \wedge)$ is a complete, distributive lattice. Now, let us recall the notion of the composition in its standard form

Definition 2 (cf. [9],[3])**.** *Let $\sigma = (S, S^d) \in AIFR(X \times Y)$, $\rho = (R, R^d) \in AIFR(Y \times Z)$. By the composition of relations σ and ρ we call the relation $\sigma \circ \rho \in AIFR(X \times Z)$,*

$$(\sigma \circ \rho)(x, z) = ((S \circ R)(x, z), (S^d \circ' R^d)(x, z)),$$

where

$$(S \circ R)(x, z) = \bigvee_{y\in Y} (S(x, y) \wedge R(y, z)), \quad (4)$$

$$(S^d \circ' R^d)(x,z) = \bigwedge_{y \in Y} (S^d(x,y) \vee R^d(y,z)). \quad (5)$$

The fuzzy relation $\pi_\rho\colon X \times Y \to [0,1]$ is associated with each Atanassov's intuitionistic fuzzy relation $\rho = (R, R^d)$, where

$$\pi_\rho(x,y) = 1 - R(x,y) - R^d(x,y), \qquad x \in X, y \in Y. \quad (6)$$

The number $\pi_\rho(x,y)$ is called an index of an element (x,y) in an Atanassov's intuitionistic fuzzy relation ρ. It is also described as an index (a degree) of hesitation whether x and y are in the relation ρ or not. This value is also regarded as a measure of non-determinacy or uncertainty (see [11]) and is useful in applications. Intuitionistic fuzzy indices allow to calculate the best final result and the worst one that may be expected in a process leading to a final optimal decision (see [11]).

If we consider decision making problems in the intuitionistic fuzzy environment we deal with the finite set of alternatives $X = \{x_1, \ldots, x_n\}$ and an expert who needs to provide his/her preference information over alternatives. In the sequel, we will consider a preference relation on a finite set $X = \{x_1, \ldots, x_n\}$. In this situation intuitionistic fuzzy relations may be represented by matrices.

Definition 3 ([16], cf. [14]). *Let $\overline{\overline{X}} = n$. An intuitionistic fuzzy preference relation ρ on the set X is represented by a matrix $\rho = (\rho_{ij})_{n\times n}$ with $\rho_{ij} = (R(i,j), R^d(i,j))$, for all $i,j = 1,...,n$, where ρ_{ij} is an intuitionistic fuzzy value, composed by the degree $R(i,j)$ to which x_i is preferred to x_j, the degree $R^d(i,j)$ to which x_i is non-preferred to x_j, and the uncertainty degree $\pi(i,j)$ to which x_i is preferred to x_j. Furthermore, $R(i,j), R^d(i,j)$ satisfy the following characteristics for all $i,j = 1,...,n$:*

$$0 \le R(i,j) + R^d(i,j) \le 1,$$

$$R(i,j) = R^d(j,i), \;\; R(j,i) = R^d(i,j),$$
$$R(i,i) = R^d(i,i) = 0.5.$$

Directly from this definition it follows that $\pi(i,j) = \pi(j,i)$ for all $i,j = 1,...,n$.

3. Operations on preference relations

Lattice operations and the composition in the family $AIFR(X)$ do not preserve a preference relation, i.e. if ρ and σ are intuitionistic fuzzy preference relations, then their sum, intersection and composition need not have this property.

Example 1. *Let card $X = 2$ and $\rho = (R, R^d), \sigma = (S, S^d) \in AIFR(X)$ be preference relations represented by the matrices:*

$$\rho = \begin{bmatrix} (0.5, 0.5) & (0.3, 0.6) \\ (0.6, 0.3) & (0.5, 0.5) \end{bmatrix},$$

$$\sigma = \begin{bmatrix} (0.5, 0.5) & (1, 0) \\ (0, 1) & (0.5, 0.5) \end{bmatrix}.$$

Then according to (2), (3), (4), (5), *we obtain*

$$\rho \vee \sigma = \begin{bmatrix} (0.5, 0.5) & (1, 0) \\ (0.6, 0.3) & (0.5, 0.5) \end{bmatrix},$$

$$\rho \wedge \sigma = \begin{bmatrix} (0.5, 0.5) & (0.3, 0.6) \\ (0, 1) & (0.5, 0.5) \end{bmatrix},$$

$$\rho \circ \sigma = \begin{bmatrix} (0.5, 0.5) & (0.5, 0.5) \\ (0.5, 0.5) & (0.6, 0.3) \end{bmatrix},$$

$$\rho \circ \rho = \begin{bmatrix} (0.5, 0.5) & (0.3, 0.6) \\ (0.5, 0.5) & (0.5, 0.5) \end{bmatrix}.$$

We see that none of the relations $\rho \vee \sigma$, $\rho \wedge \sigma$, $\rho \circ \sigma$, $\rho \circ \rho$ is a preference relation.

Now we put definitions of some Atanassov's operators

Definition 4 ([2]). *Let $\rho \in AIFR(X \times Y)$, $\rho = (R, R^d)$, α, $\beta \in [0,1]$, $\alpha + \beta \le 1$. The operators $F_{\alpha,\beta}, P_{\alpha,\beta}, Q_{\alpha,\beta} : AIFR(X \times Y) \to AIFR(X \times Y)$ are defined as follows*

$$F_{\alpha,\beta}(\rho(x,y)) =$$

$$(R(x,y) + \alpha\pi_\rho(x,y), R^d(x,y) + \beta\pi_\rho(x,y)),$$

$$P_{\alpha,\beta}(\rho(x,y)) = (\max(\alpha, R(x,y)), \min(\beta, R^d(x,y))),$$

$$Q_{\alpha,\beta}(\rho(x,y)) = (\min(\alpha, R(x,y)), \max(\beta, R^d(x,y))).$$

We examine whether Atanassov's operators preserve intuitionistic fuzzy preference relations.

Proposition 1. *Let $\rho \in AIFR(X)$, $\overline{\overline{X}} = n$, α, $\beta \in [0,1]$, $\alpha + \beta \le 1$ and $\rho = (R, R^d)$ be an intuitionistic fuzzy preference relation.*
- *$F_{\alpha,\beta}(\rho)$ is an intuitionistic fuzzy preference relation if and only if $\alpha = \beta$;*
- *$P_{\alpha,\beta}(\rho)$ is an intuitionistic fuzzy preference relation if and only if $\alpha \le R(i,j) \le \beta$ for all $i,j = 1,...,n$;*
- *$Q_{\alpha,\beta}(\rho)$ is an intuitionistic fuzzy preference relation if and only if $\beta \le R(i,j) \le \alpha$ for all $i,j = 1,...,n$.*

Proof. First we consider operation $F_{\alpha,\beta}(\rho)$ and we observe for $1 \le i,j \le n$ that

$$F_{\alpha,\beta}(\rho_{ii}) = (R(i,i) + \alpha\pi_\rho(i,i), R^d(i,i) + \beta\pi_\rho(i,i)) =$$

$$(R(i,i), R^d(i,i)) = (0.5, 0.5).$$

Moreover

$$R(i,j) + \alpha\pi_\rho(i,j) = R^d(j,i) + \beta\pi_\rho(j,i),$$

because $R(i,j) = R^d(j,i)$ and $\pi_\rho(i,j) = \pi_\rho(j,i)$. Thus $F_{\alpha,\beta}(\rho)$ preserves the preference property if and only if $\alpha = \beta$.

Now we will examine operator $P_{\alpha,\beta}$.
For $\alpha \le R(i,j) \le \beta$ we have

$$\max(\alpha, R(i,j)) = R(i,j) = R^d(j,i) = \min(\beta, R^d(j,i)).$$

This proves that $P_{\alpha,\beta}(\rho)$ preserves the preference property.
If $P_{\alpha,\beta}(\rho)$ and ρ are intuitionistic fuzzy preference relations, then
$P_{\alpha,\beta}(\rho_{ii}) = (\max(\alpha, R(i,i)), \min(\beta, R^d(i,i))) = (\max(\alpha, 0.5), \min(\beta, 0.5)) = (0.5, 0.5)$. As a result $\alpha \le 0.5 = R(i,i)$ and $\beta \ge 0.5 = R^d(i,i)$.
For $i \ne j$ we obtain

$$\max(\alpha, R(i,j)) = \min(\beta, R^d(j,i)) = \min(\beta, R(i,j)).$$

This condition is true only for $\alpha \le R(i,j) \le \beta$, so these inequalities are also true.
The case of $Q_{\alpha,\beta}(\rho)$ can be proven in a similar way. $\square$

4. Properties of intuitionistic fuzzy preference relations

In this section we consider some properties of intuitionistic fuzzy relations and intuitionistic fuzzy preference relations. First, we recall the concept of a partially included relation in which the $sgn : \mathbb{R} \to \mathbb{R}$ function occurs, where

$$sgn(t) = \begin{cases} 1, & \text{for } t > 0 \\ 0, & \text{for } t = 0 \\ -1, & \text{for } t < 0 \end{cases}.$$

Definition 5 (cf. [3])**.** *An intuitionistic fuzzy relation $\rho = (R, R^d) \in AIFR(X)$ is partially included, if for all $x, y, z \in X$*

$$sgn(R(x,y) - R(y,z)) = sgn(R^d(y,z) - R^d(x,y)). \tag{7}$$

Definition 6. *An intuitionistic fuzzy relation $\rho = (R, R^d) \in AIFR(X)$ is transitive, if $\rho \circ \rho \le \rho$ ($\rho^2 \le \rho$).*

Thus we have

Lemma 1 (cf. [12])**.** *Let $\rho \in AIFR(X)$, $\alpha, \beta \in [0,1]$, $\alpha + \beta \le 1$. If ρ is partially included and transitive, then $F_{\alpha,\beta}(\rho)$ is transitive.*

Proof. Let $\rho^2 \le \rho$ and ρ be partially included, $x, y \in X$. From (7) we obtain
$((1-\alpha)R(x,z) + \alpha(1 - R^d(x,z)))\wedge$
$((1-\alpha)R(z,y) + \alpha(1 - R^d(z,y))) =$
$(1-\alpha)(R(x,z) \wedge R(z,y))+$
$\alpha((1 - R^d(x,z)) \wedge (1 - R^d(z,y)))$ and
$((1-\beta)R^d(x,z) + \beta(1 - R(x,z)))\vee$
$((1-\beta)R^d(z,y) + \beta(1 - R(z,y))) =$
$(1-\beta)(R^d(x,z) \vee R^d(z,y))+$
$\beta((1 - R(x,z)) \vee (1 - R(z,y)))$.
Then $F^2_{\alpha,\beta}(\rho)(x,y) =$
$((R(x,y) + \alpha\pi_\rho(x,y))^2, (R^d(x,y) + \beta\pi_\rho(x,y))^2) =$
$(((1-\alpha)R(x,y)+\alpha(1-R^d(x,y)))^2, ((1-\beta)R^d(x,y)+$
$\beta(1 - R(x,y)))^2) =$
$(\bigvee_{z\in X}((1-\alpha)R(x,z) + \alpha(1 - R^d(x,z)))\wedge$
$((1-\alpha)R(z,y) + \alpha(1 - R^d(z,y)))$,
$\bigwedge_{z\in X}((1-\beta)R^d(x,z) + \beta(1 - R(x,z)))\vee$
$((1-\beta)R^d(z,y) + \beta(1 - R(z,y))))$.
From the above considerations we have
$(\bigvee_{z\in X}((1-\alpha)(R(x,z) \wedge R(z,y)) + \alpha(1 - R^d(x,z)) \wedge$
$(1 - R^d(z,y)))$,
$\bigwedge_{z\in X}((1-\beta)(R^d(x,z) \vee R^d(z,y)) + \beta((1 - R(x,z)) \vee$
$(1 - R(z,y))))) \le$
$(\bigvee_{z\in X}(1-\alpha)(R(x,z) \wedge R(z,y))+$
$\bigvee_{z\in X}\alpha((1 - R^d(x,z)) \wedge (1 - R^d(z,y)))$,
$\bigwedge_{z\in X}(1-\beta)(R^d(x,z) \vee R^d(z,y))+$
$\bigwedge_{z\in X}\beta((1 - R(x,z)) \vee (1 - R(z,y)))) =$
$F_{\alpha,\beta}(\rho^2)(x,y)$, so by the isotonicity of $F_{\alpha,\beta}$ we obtain $F^2_{\alpha,\beta}(\rho)(x,y) \le F_{\alpha,\beta}(\rho)(x,y)$. $\square$

By Lemma 1 and by condition: $\rho_{ij} + \rho_{ji} = (1,1)$, which means that $R(i,j) + R(j,i) = 1$ and $R^d(i,j) + R^d(j,i) = 1$, we obtain the following

Proposition 2. *Let $\rho \in AIFR(X)$, $\overline{\overline{X}} = n$ and $\alpha, \beta \in [0,1]$. If $\rho = (R, R^d)$ is an intuitionistic fuzzy preference relation fulfilling the property $\rho_{ij} + \rho_{ji} = (1,1)$ for all $i, j = 1, ..., n$ and the transitivity property, then $F_{\alpha,\beta}(\rho)$ ($F_{\alpha,\alpha}(\rho)$) is also an intuitionistic fuzzy transitive relation (intuitionistic fuzzy transitive preference relation).*

Proof. If $\rho_{ij} + \rho_{ji} = (1,1)$, then for an intuitionistic fuzzy preference relation $(R(i,j) + R(j,i) = 1) \Leftrightarrow (R^d(i,j) + R^d(j,i) = 1)$ and ρ is partially included, i.e.

$$sgn(R(i,j) - R(j,k)) = sgn(1 - R(j,i) - (1 - R(k,j))) = sgn(R(k,j) - R(j,i)) = sgn(R^d(j,k) - R^d(i,j)).$$

By Lemma 1 we see that $F_{\alpha,\beta}(\rho)$ is transitive, moreover by Proposition 1, $F_{\alpha,\beta}(\rho)$ for $\alpha = \beta$ is an intuitionistic fuzzy transitive preference relation. $\square$

We also obtain

Lemma 2. *Let $\rho \in AIFR(X)$, $\alpha, \beta \in [0,1]$ and $\alpha + \beta \le 1$. If ρ is partially included and $F_{\alpha,\beta}(\rho)$ is transitive, then ρ is also transitive.*

Proof. We must prove that $F^2_{\alpha,\beta}(\rho) \le F_{\alpha,\beta}(\rho) \Rightarrow \rho^2 \le \rho$. Thus we assume

$$(R^2 + \alpha\pi_{\rho^2}, (R^d)^2 + \beta\pi_{\rho^2}) \le (R + \alpha\pi_\rho, R^d + \beta\pi_\rho).$$

We consider the following cases:
1. If $\pi_{\rho^2} = \pi_\rho$, then by (3) we obtain $R^2 \le R$ and $(R^d)^2 \ge R^d$.
2. If $\pi_{\rho^2} > \pi_\rho$, then $R + \alpha\pi_{\rho^2} > R + \alpha\pi_\rho \ge R^2 + \alpha\pi_{\rho^2}$ so $R^2 \le R$ and $R - R^2 \ge \alpha(\pi_{\rho^2} - \pi_\rho)$. Moreover, by $R - R^2 \ge 0$ and

$$\alpha(\pi_{\rho^2} - \pi_\rho) \ge 0 \Leftrightarrow R - R^2 + R^d - (R^d)^2 \ge 0,$$

we have $R^d - (R^d)^2 \leq 0$, i.e. $R^d \leq (R^d)^2$. This means $\rho^2 \leq \rho$.
3. If $\pi_{\rho^2} < \pi_\rho$, then $R^d + \beta\pi_{\rho^2} < R^d + \beta\pi_\rho \leq (R^d)^2 + \beta\pi_{\rho^2}$, so $R^d \leq (R^d)^2$, i.e. $(R^d)^2 - R^d \geq 0$. Moreover,

$$\beta(\pi_\rho - \pi_{\rho^2}) > 0 \Leftrightarrow R^2 - R + (R^d)^2 - R^d > 0$$

and

$$\beta(\pi_\rho - \pi_{\rho^2}) \leq (R^d)^2 - R^d.$$

We have $R^2 - R \leq 0$, i.e. $R^2 \leq R$. This finishes the proof. □

From the above lemma we obtain, similarly to Proposition 2, the following theorem

Corollary 1. *Let $\rho \in AIFR(X)$, $\overline{\overline{X}} = n$ and $\alpha, \beta \in [0,1]$. If $\rho = (R, R^d)$ is an intuitionistic fuzzy preference relation and $F_{\alpha,\beta}(\rho)$ ($F_{\alpha,\alpha}(\rho)$) is an intuitionistic fuzzy transitive relation (intuitionistic fuzzy transitive preference relation), then ρ is also transitive.*

Now we recall the notion of equivalent fuzzy relations.

Definition 7 (cf. [7]). *Fuzzy relations R, S are equivalent ($R \sim S$) if*

$$\underset{x,y,u,v \in X}{\forall} \ R(x,y) \leqslant R(u,v) \ \Leftrightarrow \ S(x,y) \leqslant S(u,v). \tag{8}$$

The analogical property can be defined for intuitionistic fuzzy relations.

Definition 8 ([8]). *Let $\rho = (R, R^d), \sigma = (S, S^d) \in AIFR(X)$. We say that relations ρ and σ are equivalent ($\rho \sim \sigma$), if for all $x, y, u, v \in X$*

$$R(x,y) \leqslant R(u,v) \ \Leftrightarrow \ S(x,y) \leqslant S(u,v)$$

and

$$R^d(x,y) \leqslant R^d(u,v) \ \Leftrightarrow \ S^d(x,y) \leqslant S^d(u,v).$$

Relation "$\sim$" is an equivalence relation in the family $AIFR(X)$. This fact enables to classify intuitionistic fuzzy information and find some subordinations between this information.

Corollary 2 ([8]). *Let $\rho = (R, R^d), \sigma = (S, S^d) \in AIFR(X)$. Then*

$$\rho \sim \sigma \ \Leftrightarrow \ (R \sim S \text{ and } R^d \sim S^d).$$

Now, let us turn to considerations involving the operations supremum and infimum. These results may be applied in verifying the equivalence between given intuitionistic fuzzy relations.

Theorem 1 ([8]). *Let $\rho = (R, R^d), \sigma = (S, S^d) \in AIFR(X)$. If $\rho \sim \sigma$, then for every non-empty subset P of $X \times X$ and each $x, y, z, t \in P$ the following conditions are fulfilled*

$$\begin{cases} R(x,y) = \bigvee\limits_{(u,v)\in P} R(u,v) \ \Leftrightarrow \\ S(x,y) = \bigvee\limits_{(u,v)\in P} S(u,v) \qquad \text{and} \\ R^d(z,t) = \bigvee\limits_{(u,v)\in P} R^d(u,v) \ \Leftrightarrow \\ S^d(z,t) = \bigvee\limits_{(u,v)\in P} S^d(u,v) \end{cases}, \tag{9}$$

$$\begin{cases} R(x,y) = \bigwedge\limits_{(u,v)\in P} R(u,v) \ \Leftrightarrow \\ S(x,y) = \bigwedge\limits_{(u,v)\in P} S(u,v) \qquad \text{and} \\ R^d(z,t) = \bigwedge\limits_{(u,v)\in P} R^d(u,v) \ \Leftrightarrow \\ S^d(z,t) = \bigwedge\limits_{(u,v)\in P} S^d(u,v) \end{cases}, \tag{10}$$

$$\begin{cases} R(x,y) = \bigvee\limits_{(u,v)\in P} R(u,v) \ \Leftrightarrow \\ S(x,y) = \bigvee\limits_{(u,v)\in P} S(u,v) \qquad \text{and} \\ R^d(z,t) = \bigwedge\limits_{(u,v)\in P} R^d(u,v) \ \Leftrightarrow \\ S^d(z,t) = \bigwedge\limits_{(u,v)\in P} S^d(u,v) \end{cases}, \tag{11}$$

$$\begin{cases} R(x,y) = \bigwedge\limits_{(u,v)\in P} R(u,v) \ \Leftrightarrow \\ S(x,y) = \bigwedge\limits_{(u,v)\in P} S(u,v) \qquad \text{and} \\ R^d(z,t) = \bigvee\limits_{(u,v)\in P} R^d(u,v) \ \Leftrightarrow \\ S^d(z,t) = \bigvee\limits_{(u,v)\in P} S^d(u,v) \end{cases}. \tag{12}$$

Let us notice that the converse statement to Theorem 1 is true and it is enough to assume that only one of the conditions (9) - (12) is fulfilled.

Theorem 2 ([8]). *Let $\rho = (R, R^d), \sigma = (S, S^d) \in AIFR(X)$. If for every finite, non-empty subset P of $X \times X$ and each $x, y, z, t \in P$ one of the conditions* (9) - (12) *holds, then $\rho \sim \sigma$.*

Equivalent relations have connection with transitivity property.

Theorem 3 ([8]). *Let $\rho = (R, R^d), \sigma = (S, S^d) \in AIFR(X)$. If $\rho \sim \sigma$, then ρ is transitive if and only if σ is transitive.*

For intuitionistic fuzzy preference relations we can weaken assumptions from the above theorem.

Proposition 3. *Let $\rho, \sigma \in AIFR(X)$, $\overline{\overline{X}} = n$. If $\rho = (R, R^d), \sigma = (S, S^d)$ are intuitionistic fuzzy preference relations and for arbitrary non-empty set $P \subset X \times X$ and $(i,j) \in P$ holds:*

$$R(i,j) = \bigvee_{(v,w)\in P} R(v,w) \Leftrightarrow S(i,j) = \bigvee_{(v,w)\in P} S(v,w) \tag{13}$$

or

$$R(i,j) = \bigwedge_{(v,w)\in P} R(v,w) \Leftrightarrow S(i,j) = \bigwedge_{(v,w)\in P} S(v,w), \tag{14}$$

then ρ is transitive if and only if σ is transitive.

Proof. For an intuitionistic fuzzy preference relation and conditions (13) and (14) we obtain dual conditions for relations R^d, S^d. Moreover, from definition of an intuitionistic fuzzy preference relation and equivalence relation we observe, that if $\rho = (R, R^d), \sigma = (S, S^d)$ are intuitionistic fuzzy preference relations and $R \sim S$, then $R^d \sim S^d$. As a result, if $\rho = (R, R^d), \sigma = (S, S^d)$ are intuitionistic fuzzy preference relations and $R \sim S$, then $\rho \sim \sigma$. Now by assumptions (13), (14) and Theorems 1- 3 we have transitivity property both for ρ and σ. □

Now we examine weak transitivity property.

Definition 9 ([16])**.** *Let $\overline{\overline{X}} = n$. An intuitionistic fuzzy relation $\rho = (R, R^d) \in AIFR(X)$ is weakly transitive, if for all $1 \le i, j, k \le n$*

$$\rho(i,k) \ge (0.5, 0.5), \rho(k,j) \ge (0.5, 0.5) \Rightarrow \rho(i,j) \ge (0.5, 0.5). \tag{15}$$

In the sequel, we will use the following property of intuitionistic fuzzy relations in a finite set X.

Definition 10. *Let $\overline{\overline{X}} = n$. An intuitionistic fuzzy relation $\rho = (R, R^d) \in AIFR(X)$ is said to be a relation with strictly dominating upper (lower) triangle, if*

$$\forall_{1\le i,j\le n, i<j(i>j)} \rho(i,j) > 0.5. \tag{16}$$

Proposition 4. *Let $\overline{\overline{X}} = n$. If $\rho = (R, R^d) \in AIFR(X)$ is an intuitionistic fuzzy preference relation with strictly dominating lower (upper) triangle, then it is weakly transitive.*

Proof. Let $\rho = (R, R^d)$ be an intuitionistic fuzzy preference relation with strictly dominating upper triangle.
If $i = j$, then $\rho(i,j) = (0.5, 0.5)$. Thus implication (15) is true.
If $i \ne j$, then we consider the following cases:
1. For $i > j$ we have by (16) $\rho(i,j) < (0.5, 0.5)$ and we examine:
- if $i \ge k > j$, then $\rho(k,j) < (0.5, 0.5)$;
- if $k > i > j$, then $\rho(k,j) < (0.5, 0.5)$;
- if $i > j \ge k$, then $\rho(i,k) < (0.5, 0.5)$.

In all these cases we obtained false antecedent and consequence, so implication (15) is true.
2. For $i < j$ we have $\rho(i,j) > (0.5, 0.5)$ so implication (15) is true. The proof for strictly dominating lower triangle property is similar and the intuitionistic fuzzy preference relation $\rho = (R, R^d)$ is weakly transitive. □

The converse property is not true.

Example 2. *Let $\overline{\overline{X}} = 3$. The following intuitionistic fuzzy preference relation $\rho \in AIFR(X)$ is weakly transitive but it is not a relation with strictly dominating lower (upper) triangle:*

$$\rho = \begin{bmatrix} (0.5, 0.5) & (0.5, 0.5) & (0.3, 0.7) \\ (0.5, 0.5) & (0.5, 0.5) & (0.3, 0.5) \\ (0.7, 0.3) & (0.5, 0.3) & (0.5, 0.5) \end{bmatrix}.$$

Now, we define parameterized versions of intuitionistic fuzzy relation properties. We follow the concept of such properties given by Drewniak [6] for fuzzy relations but we restrict ourselves only to parameter $\alpha = 0.5$. This is why we will call these properties *semi-properties.*

Definition 11. *An intuitionistic fuzzy relation $\rho = (R, R^d) \in AIFR(X)$ is called:*
- *semi-reflexive if*

$$\forall_{x\in X} \rho(x,x) \geqslant (0.5, 0.5), \tag{17}$$

- *semi-irreflexive if*

$$\forall_{x\in X} \rho(x,x) \leqslant (0.5, 0.5), \tag{18}$$

- *semi-symmetric if*

$$\forall_{x,y\in X} \rho(x,y) \geqslant (0.5, 0.5) \Rightarrow \rho(y,x) = \rho(x,y), \tag{19}$$

- *semi-asymmetric if*

$$\forall_{x,y\in X} \rho(x,y) \wedge \rho(y,x) \leqslant (0.5, 0.5), \tag{20}$$

- *semi-antisymmetric if*

$$\forall_{x,y\in X, x\ne y} \rho(x,y) \wedge \rho(y,x) \leqslant (0.5, 0.5), \tag{21}$$

- *totally semi-connected if*

$$\forall_{x,y\in X} \rho(x,y) \vee \rho(y,x) \geqslant (0.5, 0.5), \tag{22}$$

- *semi-connected if*

$$\forall_{x,y\in X, x\ne y} \rho(x,y) \vee \rho(y,x) \geqslant (0.5, 0.5), \tag{23}$$

- *semi-transitive if*

$$\forall_{x,y,z\in X} \rho(x,y) \wedge \rho(y,z) \geqslant (0.5, 0.5) \Rightarrow \rho(x,z) \geqslant \rho(x,y) \wedge \rho(y,z). \tag{24}$$

From definition of semi-transitivity and definition of the composition of intuitionistic fuzzy relations it follows

Lemma 3. *Let $\rho = (R, R^d) \in AIFR(X)$ be an intuitionistic fuzzy relation. Relation ρ is semi-transitive if and only if*

$$\forall_{x,z\in X} \rho^2(x,z) \geqslant (0.5, 0.5) \Rightarrow \rho(x,z) \geqslant \rho^2(x,z). \tag{25}$$

Proof. If $\rho = (R, R^d)$ is semi-transitive, then by (24), definition of the order (3) and by applying the tautologies for quantifiers we obtain

$$\underset{x,y,z\in X}{\forall} R(x,y) \wedge R(y,z) \geqslant 0.5 \Rightarrow$$
$$R(x,z) \geqslant R(x,y) \wedge R(y,z)$$

and

$$\underset{x,y,z\in X}{\forall} R^d(x,y) \vee R^d(y,z) \leqslant 0.5 \Rightarrow$$
$$R^d(x,z) \leqslant R^d(x,y) \vee R^d(y,z).$$

As a result

$$\underset{x,z\in X}{\forall} (\underset{y\in X}{\forall} R(x,y) \wedge R(y,z) \geqslant 0.5 \Rightarrow$$
$$\underset{y\in X}{\forall} R(x,z) \geqslant R(x,y) \wedge R(y,z))$$

and

$$\underset{x,z\in X}{\forall} (\underset{y\in X}{\forall} R^d(x,y) \vee R^d(y,z) \leqslant 0.5 \Rightarrow$$
$$\underset{y\in X}{\forall} R^d(x,z) \leqslant R^d(x,y) \vee R^d(y,z)).$$

This implies

$$\underset{x,z\in X}{\forall} \sup_{y\in X}(R(x,y) \wedge R(y,z)) \geqslant 0.5 \Rightarrow$$
$$R(x,z) \geqslant \sup_{y\in X}(R(x,y) \wedge R(y,z)) \tag{26}$$

and

$$\underset{x,z\in X}{\forall} \inf_{y\in X}(R^d(x,y) \vee R^d(y,z)) \leqslant 0.5 \Rightarrow$$
$$R^d(x,z) \leqslant \inf_{y\in X}(R^d(x,y) \vee R^d(y,z)), \tag{27}$$

so by the definition of composition we get (25).
Let us assume that condition (25) is fulfilled which is equivalent to conditions (26) and (27). We will show that ρ is semi-transitive. Let $x, y, z \in X$ and the antecedent in (24) be fulfilled. As a result we have $R(x,y) \wedge R(y,z) \geqslant 0.5$ and $R^d(x,y) \vee R^d(y,z) \leqslant 0.5$. By definition of supremum and infimum we obtain

$$\sup_{y\in X}(R(x,y) \wedge R(y,z)) \geqslant R(x,y) \wedge R(y,z) \geqslant 0.5$$

and

$$\inf_{y\in X}(R^d(x,y) \vee R^d(y,z)) \leqslant R^d(x,y) \vee R^d(y,z) \leqslant 0.5.$$

From (26), (27) and definition of supremum and infimum we have

$$R(x,z) \geqslant \sup_{y\in X}(R(x,y) \wedge R(y,z)) \geqslant R(x,y) \wedge R(y,z)$$

and

$$R^d(x,z) \leqslant \inf_{y\in X}(R^d(x,y) \vee R^d(y,z)) \leqslant$$
$$R^d(x,y) \vee R^d(y,z).$$

This by definition of an intuitionistic fuzzy relation and the order (3) finishes the proof. □

Now, we will check under which assumptions an intuitionistic fuzzy preference relation has each of the semi-property. Directly by the definition of an intuitionistic fuzzy preference relation we obtain

Corollary 3. *Each intuitionistic fuzzy preference relation is semi-reflexive and semi-irreflexive.*

Theorem 4. *Let $\overline{\overline{X}} = n$, $\rho = (R, R^d) \in AIFR(X)$ be an intuitionistic fuzzy preference relation. If*

$$\underset{i,j\in\{1,\dots,n\}, i\neq j}{\forall} \max(R(i,j), R^d(i,j)) \geqslant 0.5, \tag{28}$$

then ρ is totally semi-connected, semi-connected, semi-asymmetric, semi-antisymmetric.

Proof. Let $i, j \in \{1, ..., n\}$. Firstly, we will prove total semi-connectedness of ρ (then semi-connectedness will be obvious). If $i = j$, then condition (22) is fulfilled by definition of a preference relation. Let $i \neq j$. Since ρ is a preference relation $R^d(i,j) = R(j,i)$, so we have

$$\max(R(i,j), R(j,i)) \geqslant 0.5. \tag{29}$$

Relation ρ is the intuitionistic fuzzy one, so by (28) it follows that $\min(R(i,j), R^d(i,j)) \leqslant 0.5$. Moreover, ρ is a preference relation, so we obtain $R(i,j) = R^d(j,i)$. As a result

$$\min(R^d(j,i), R^d(i,j)) \leqslant 0.5. \tag{30}$$

Finally, by (29), (30) and the definition of order for intuitionistic fuzzy relations we get the following inequality $\rho(i,j) \vee \rho(j,i) \geqslant (0.5, 0.5)$. It proves that ρ is totally semi-connected (semi-connected). We will show that ρ is semi-asymmetric (then semi-antisymmetry will be obvious). By assumptions and because of (1) we also have

$$\min(R(i,j), R(j,i)) \leqslant 0.5. \tag{31}$$

and similarly

$$\max(R^d(j,i), R^d(i,j)) \geqslant 0.5. \tag{32}$$

Finally, by (31), (32) and the definition of order for intuitionistic fuzzy relations $\rho(i,j) \wedge \rho(j,i) \leqslant (0.5, 0.5)$, so relation ρ is semi-asymmetric (semi-antisymmetric). □

Similarly, we may give necessary condition for an intuitionistic fuzzy preference relation which is semi-asymmetric, semi-antisymmetric, semi-connected and totally semi-connected.

Theorem 5. *Let $\overline{\overline{X}} = n$, $\rho = (R, R^d) \in AIFR(X)$ be an intuitionistic fuzzy preference relation. If ρ is totally semi-connected (semi-connected, semi-asymmetric, semi-antisymmetric), then*

$$\underset{i,j\in\{1,\dots,n\}}{\forall} \max(R(i,j), R^d(i,j)) \geqslant 0.5. \tag{33}$$

Proof. Let $i, j \in \{1, ..., n\}$, ρ be semi-connected (totally semi-connected). If $i = j$, then by definition of a preference $R(i, i) = R^d(i, i) = 0.5$, so (33) is fulfilled. For $i \neq j$ by semi-connectedness of relation ρ we obtain $\max(R(i, j), R(j, i)) \geqslant 0.5$. Since ρ is a preference we have $R(j, i) = R^d(i, j)$, which gives (33). Let ρ be semi-antisymmetric (semi-asymmetric). According to the first part of proof it is enough to consider $i \neq j$. By semi-antisymmetry of ρ we have $\max(R^d(i, j), R^d(j, i)) \geqslant 0.5$ and by assumptions about preference $R^d(j, i) = R(i, j)$ we obtain (33). This finishes the proof. □

Now, it is time to consider semi-symmetry.

Theorem 6. *Let $\overline{\overline{X}} = n$, $\rho = (R, R^d) \in AIFR(X)$ be an intuitionistic fuzzy preference relation. If for all $i, j \in \{1, ..., n\}, i \neq j$*

$$\rho(i, j) = (0.5, 0.5) \ \textit{or} \ \max(R(i, j), R^d(i, j)) < 0.5, \tag{34}$$

then ρ is semi-symmetric.

Proof. Let $i, j \in \{1, ..., n\}$. If $i = j$, then condition (19) is fulfilled by definition of a preference relation. Let $i \neq j$. If $\rho(i, j) = (0.5, 0.5)$, then since ρ is a preference $R(j, i) = R^d(i, j)$ and $R^d(j, i) = R(i, j)$. As a result $\rho(j, i) = (0.5, 0.5)$ and $\rho(i, j) = \rho(j, i)$. If $\max(R(i, j), R^d(i, j)) < 0.5$, then we have two cases:
1^0) $\max(R(i, j), R^d(i, j)) = R(i, j) < 0.5$. In this case the antecedent of the implication in (19) is false, so the implication is true.
2^0) $\max(R(i, j), R^d(i, j)) = R^d(i, j) < 0.5$. By assumption $R^d(i, j) = R(j, i)$, so $R(j, i) < 0.5$. In this case the antecedent of the implication for the pair (j, i) in (19) is false, so the implication is true. □

Conversely

Theorem 7. *Let $\overline{\overline{X}} = n$, $\rho = (R, R^d) \in AIFR(X)$ be an intuitionistic fuzzy preference relation. If ρ is semi-symmetric, then for all $i, j \in \{1, ..., n\}$*

$$\rho(i, j) = (0.5, 0.5) \ \textit{or} \ \max(R(i, j), R^d(i, j)) < 0.5. \tag{35}$$

Proof. Let $i, j \in \{1, ..., n\}$. If $i = j$, then by definition of a preference $\rho(i, i) = (0.5, 0.5)$. Let us suppose that there exist $i \neq j$ such that $\rho(i, j) \neq (0.5, 0.5)$ and $\max(R(i, j), R^d(i, j)) \geqslant 0.5$. As a result $R(i, j) \neq 0.5$ or $R^d(i, j) \neq 0.5$. We consider the following cases:
1^0) $\max(R(i, j), R^d(i, j)) = R(i, j) \geqslant 0.5$. Thus, if $R(i, j) > 0.5$, then by the fact that ρ is an intuitionistic fuzzy relation $R^d(i, j) < 0.5$. This implies $\rho(i, j) \geqslant (0.5, 0.5)$ and by semi-symmetry of ρ we get $\rho(i, j) = \rho(j, i)$ which means that $R(i, j) = R(j, i)$ and $R^d(i, j) = R^d(j, i)$. By definition of a preference and by the previous assumptions we obtain $0.5 < R(i, j) = R(j, i) = R^d(i, j) < 0.5$, which is a contradiction. If $R(i, j) = 0.5$, then by assumptions and by the fact that ρ is an intuitionistic fuzzy relation it follows $R^d(i, j) < 0.5$. As a result $\rho(i, j) \geqslant (0.5, 0.5)$, so $\rho(i, j) = \rho(j, i)$ and both equalities are fulfilled $R(i, j) = R(j, i)$, $R^d(i, j) = R^d(j, i)$. Finally, $0.5 = R(i, j) = R(j, i) = R^d(i, j) < 0.5$, which finishes indirect proof. In the case 2^0) $\max(R(i, j), R^d(i, j)) = R^d(i, j) \geqslant 0.5$ the proof is similar. □

Now, we turn to considerations connected with semi-transitivity which is a stronger property than weak transitivity discussed before.

Corollary 4. *Let $\overline{\overline{X}} = n$, $\rho = (R, R^d) \in AIFR(X)$ be an intuitionistic fuzzy relation. If ρ is semi-transitive, then it is weakly transitive.*

Proof. Let $i, j, k \in \{1, ..., n\}$. Let $\rho(i, k) \geqslant (0.5, 0.5)$ and $\rho(k, j) \geqslant (0.5, 0.5)$. By semi-transitivity of ρ we obtain $\rho(i, j) \geqslant \rho(i, k) \wedge \rho(k, j)$. This implies $R(i, j) \geqslant \min(R(i, k), R(k, j))$ and $R^d(i, j) \leqslant \max(R^d(i, k), R^d(k, j))$. By assumptions we also obtain $R(i, k) \geqslant 0.5$, $R^d(i, k) \leqslant 0.5$ and $R(k, j) \geqslant 0.5$, $R^d(k, j) \leqslant 0.5$. As a result $R(i, j) \geqslant 0.5$ and $R^d(i, j) \leqslant 0.5$. Finally, $\rho(i, j) \geqslant (0.5, 0.5)$ and by (15) this proves weak transitivity of ρ. □

By Lemma 3 determination of the relation ρ^2 is helpful in checking whether ρ is semi-transitive.

Theorem 8. *Let $\overline{\overline{X}} = n$, $\rho = (R, R^d) \in AIFR(X)$ be an intuitionistic fuzzy preference relation. If*

$$\underset{i,j \in \{1,...,n\}}{\forall} (\rho^2(i, j) < (0.5, 0.5) \ \textit{or} \ \rho(i, j) \geqslant \rho^2(i, j)), \tag{36}$$

then ρ is semi-transitive.

Proof. Let $i, j \in \{1, ..., n\}$. If $\rho^2(i, j) < (0.5, 0.5)$, then the antecedent of the implication is false in (25), so the implication is true. If $\rho(i, j) \geqslant \rho^2(i, j)$, then then the consequence of the implication is true in (25) and this implication is true. By Lemma 3 this finishes the proof. □

5. Conclusion

In this paper we considered properties of intuitionistic fuzzy preference relations in the context of preservation of this property by lattice operations, the composition and by Atanassov's operators. We also introduced semi-properties of intuitionistic fuzzy relations and we investigated fulfilment of these properties by preference relations. In our further considerations we want to study other transitivity properties of intuitionistic fuzzy preference relations introduced in [16].

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