Dependencies between fuzzy conjunctions and implications

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Abstract

This paper deals with some dependencies between fuzzy conjunctions and fuzzy implications. More precisely, a fuzzy implication generated from a fuzzy conjunction and a fuzzy conjunction induced by a fuzzy implication is considered. In the case of a fuzzy conjunction only border conditions and monotonicity are assumed. The results are illustrated by examples of weak fuzzy connectives.

Keywords: Fuzzy conjunction, fuzzy implication, R-implication

1. Introduction

Multivalued logic with truth values in [0,1] was developed after the paper of J. Łukasiewicz [9]. Fuzzy set theory introduced by L.A. Zadeh [11] brought new applications of multivalued logic and new directions in examination of logical connectives. After the contribution of B. Schweizer and A. Sklar [10] the notions of the triangular norm and conorm have played the role of a fuzzy conjunction and disjunction. J. Fodor and M. Roubens [8], M. Baczyński and B. Jayaram [1] examined families of multivalued connectives based on triangular norms and conorms. However, some authors (e.g. I. Batyrshin and O. Kaynak [3], F. Durante et al. [6]) underline that the assumptions made on these multivalued connectives are sometimes too strong and difficult to obtain. Thus, some of the conditions are omitted.

In this contribution a way of generating of fuzzy implication from a fuzzy conjunction by the use of residuation is considered. Implications created in such a way have been considered in the literature in the case when the conjunction is a triangular norm by J. Fodor and M. Roubens [8], M. Baczyński and B. Jayaram [1] and are called R-implications. Also, the case when a conjunction is replaced by an appropriate uninorm was examined by B. De Baets and J. Fodor [4] or M. Baczyński and B. Jayaram [2] (RU-implications). In these considerations only border conditions and monotonicity are required for a fuzzy conjunction.

In the following section the definitions and examples of fuzzy connectives used in the sequel are presented. Next, in Section 3, fuzzy implications generated from fuzzy conjunctions are considered. Finally, Section 4, presents the results concerning fuzzy conjunctions generated from fuzzy implications.

2. Basic definitions

First, we present the notion and examples of a fuzzy conjunction.

Definition 1 ([5])**.** An operation $C : [0,1]^2 \to [0,1]$ is called a fuzzy conjunction if it is increasing with respect to each variable and

$$C(1,1) = 1, \quad C(0,0) = C(0,1) = C(1,0) = 0.$$

Corollary 1. A fuzzy conjunction has an absorbing element 0.

Example 1. The operations C_0 and C_1 are the least and the greatest fuzzy conjunction, respectively, where

$$C_0(x,y) = \begin{cases} 1, & \text{if } x = y = 1 \\ 0, & \text{else} \end{cases},$$

$$C_1(x,y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } \ y = 0 \\ 1, & \text{else} \end{cases}.$$

The following are the other examples of fuzzy conjunctions. The triangular norms are denoted in the traditional way.

$$C_2(x,y) = \begin{cases} y, & \text{if } x = 1 \\ 0, & \text{if } x < 1 \end{cases},$$

$$C_3(x,y) = \begin{cases} x, & \text{if } y = 1 \\ 0, & \text{if } y < 1 \end{cases},$$

$$C_4(x,y) = \begin{cases} 0, & \text{if } x + y \leqslant 1 \\ y, & \text{if } x + y > 1 \end{cases},$$

$$C_5(x,y) = \begin{cases} 0, & \text{if } x + y \leqslant 1 \\ x, & \text{if } x + y > 1 \end{cases},$$

$$T_M(x,y) = \min(x,y),$$

$$T_P(x,y) = xy,$$

$$T_{LK}(x,y) = \max(x+y-1,0),$$

$$T_D(x,y) = \begin{cases} x, & \text{if } y = 1 \\ y, & \text{if } x = 1 \\ 0, & \text{else} \end{cases},$$

$$T_{nM}(x,y) = \begin{cases} 0, & \text{if } x + y \leqslant 1 \\ \min(x,y), & \text{else} \end{cases}.$$

Next, we recall the notion of a fuzzy implication.

Definition 2 ([1], pp. 2,9)**.** A binary operation $I\colon [0,1]^2 \to [0,1]$ is called a fuzzy implication if it is decreasing with respect to the first variable and increasing with respect to the second variable and

$$I(0,0) = I(0,1) = I(1,1) = 1, \quad I(1,0) = 0.$$

We say that a fuzzy implication I fulfils:

- the neutral property (NP) if
$$I(1,y) = y, \quad y \in [0,1], \tag{NP}$$
- the exchange principle (EP) if
$$I(x, I(y,z)) = I(y, I(x,z)), \quad x,y,z \in [0,1], \tag{EP}$$
- the identity principle (IP)
$$I(x,x) = 1, \quad x \in [0,1], \tag{IP}$$
- the ordering property (OP) if
$$I(x,y) = 1 \Leftrightarrow x \leqslant y, \quad x,y \in [0,1]. \tag{OP}$$

Example 2 ([1], pp. 4,5)**.** The operations I_0 and I_1 are the least and the greatest fuzzy implication, respectively, where

$$I_0(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } \ y = 1 \\ 0, & \text{else} \end{cases},$$

$$I_1(x,y) = \begin{cases} 0, & \text{if } x = 1, y = 0 \\ 1, & \text{else} \end{cases}.$$

The following are the other examples of fuzzy implications.

$$I_{\text{ŁK}}(x,y) = \min(1 - x + y, 1),$$

$$I_{\text{GD}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ y, & \text{if } x > y \end{cases},$$

$$I_{\text{RC}}(x,y) = 1 - x + xy,$$

$$I_{\text{DN}}(x,y) = \max(1 - x, y),$$

$$I_{\text{GG}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \frac{y}{x}, & \text{if } x > y \end{cases},$$

$$I_{\text{RS}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ 0, & \text{if } x > y \end{cases},$$

$$I_{\text{YG}}(x,y) = \begin{cases} 1, & \text{if } x, y = 0 \\ y^x, & \text{else} \end{cases},$$

$$I_{\text{FD}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \max(1 - x, y), & \text{if } x > y \end{cases},$$

$$I_{\text{WB}}(x,y) = \begin{cases} 1, & \text{if } x \le 1 \\ y, & \text{if } x = 1 \end{cases}.$$

Corollary 2. A fuzzy implication has the right absorbing element 1 and fulfils the condition

$$I(0,y) = 1, \quad x,y \in [0,1]. \tag{1}$$

3. Implications induced from conjunctions

A fuzzy implication can be generated from a fuzzy conjunction by means of residuation as in the formula (2) below.

Lemma 1. *For an arbitrary $C\colon [0,1]^2 \to [0,1]$ with the right absorbing element 0, the function $I_C\colon [0,1]^2 \to [0,1]$ given by the formula*

$$I_C(x,y) = \sup\{t \in [0,1] : C(x,t) \le y\}, \ \ x,y \in [0,1] \tag{2}$$

is increasing with respect to the second variable. If additionally C is increasing with respect to the first variable then I_C is decreasing with respect to the first variable.

Proof. Firstly, let us observe that the function I_C is defined correctly. Let us fix $x,y \in [0,1]$ and denote

$$R(x,y) := \{t \in [0,1] : C(x,t) \le y\}. \tag{3}$$

From the existence of the right absorbing element 0 it follows that $0 \leqslant C(x,0) = 0$, so $0 \in R(x,y)$. This means that $R(x,y) \neq \emptyset$ and $\sup R(x,y) \in [0,1]$.
Let $x,y,v \in [0,1]$, $y \leqslant v$. We have

$$\{t \in [0,1] : C(x,t) \leqslant y\} \subset \{t \in [0,1] : C(x,t) \leqslant v\},$$

$$\sup\{t \in [0,1] : C(x,t) \leqslant y\} \leqslant \\ \leqslant \sup\{t \in [0,1] : C(x,t) \leqslant v\},$$

$$I_C(x,y) \leqslant I_C(x,v).$$

This means, that the function I_C is increasing with respect to the second variable.
Now, let $x,u,y \in [0,1]$, $x \leqslant u$. From the monotonicity of the operation C with respect to the first variable we have $C(x,t) \leqslant C(u,t)$ for all $t \in [0,1]$. Thus, we obtain as follows

$$\{t \in [0,1] : C(u,t) \leqslant y\} \subset \{t \in [0,1] : C(x,t) \leqslant y\},$$

$$\sup\{t \in [0,1] : C(u,t) \leqslant y\} \leqslant \\ \leqslant \sup\{t \in [0,1] : C(x,t) \leqslant y\},$$

$$I_C(u,y) \leqslant I_C(x,y).$$

Hence, the function I_C is decreasing with respect to the first variable. □

Lemma 2. *If C is a fuzzy conjunction then $I_C(0,0) = I_C(0,1) = I_C(1,1) = 1$, where I_C is defined by (2).*

Proof. From (2) and Corollary 1 we obtain

$$I_C(0,0) = \sup\{t \in [0,1] : C(0,t) \le 0\} =$$

$$= \sup\{t \in [0,1] : 0 \le 0\} = 1,$$

$$I_C(0,1) = \sup\{t \in [0,1] : C(0,t) \le 1\} = 1,$$

$$I_C(1,1) = \sup\{t \in [0,1] : C(1,t) \le 1\} = 1.$$

□

Lemma 3. *Let C be a fuzzy conjunction. Then $I_C(1,0)=0$ if and only if C fulfils the condition*

$$C(1,y)>0, \quad y\in(0,1], \tag{4}$$

Proof. From (2) we have

$$I_C(1,0)=\sup\{t\in[0,1]:C(1,t)=0\}.$$

It is enough to observe that $I_C(1,0)=0$ if and only if the condition (4) is fulfilled. □

From Lemmas 1–3 it follows the next statement.

Theorem 1. *Let C be a fuzzy conjunction. The function $I_C\colon[0,1]^2\to[0,1]$ given by (2) is a fuzzy implication if and only if C fulfils the condition (4).*

Definition 3. Let C be a fuzzy conjunction fulfilling the condition (4). The function I_C given by (2) is called the induced implication.

Example 3. The following table shows fuzzy implications with their generators (cf. Examples 1, 2). The symbol – means that, the function I_C is not a fuzzy implication.

Conjunction C	Implication I_C
C_0	–
C_1	I_0
C_2	I_{WB}
C_3	–
C_4	I_{DN}
C_5	$I_2(x,y)=\begin{cases}1, & \text{if } x\le y\\ 1-x, & \text{if } x>y\end{cases}$
T_M	I_{GD}
T_P	I_{GG}
T_{LK}	I_{LK}
T_D	I_{WB}
T_{nM}	I_{FD}

For example, let us check, that the conjunction C_2 induces the implication I_{WB}. We can observe that

$$\begin{aligned}I_{C_2}(1,y)&=\sup\{t\in[0,1]:C_2(1,t)\le y\}=\\&=\sup\{t\in[0,1]:t\le y\}=y=I_{WB}(1,y).\end{aligned}$$

Moreover, for $x\neq 1$ we obtain

$$\begin{aligned}I_{C_2}(x,y)&=\sup\{t\in[0,1]:C_2(x,t)\le y\}=\\&=\sup\{t\in[0,1]:0\le y\}=1=I_{WB}(x,y).\end{aligned}$$

In the similar way one can prove that the implication I_{WB} is generated by the triangular norm T_D as well.

The condition (4) put on a conjunction guarantees, that the function I_C given by the formula (2) is a fuzzy implication. However, this assumption is not sufficient for the functions C and I_C to fulfil an additional, important condition called residuation principle.

Definition 4. We say, that the functions C and I_C fulfil the residuation principle if

$$C(x,z)\leqslant y \Leftrightarrow I_C(x,y)\geqslant z, \quad x,y,z\in[0,1]. \tag{RP}$$

Theorem 2. *Let C be a fuzzy conjunction fulfilling (4). The following conditions are equivalent:*
(i) C is a left-continuous function with respect to the second variable;
(ii) C and I_C fulfil the residuation principle (RP)*;*
(iii) $I_C(x,y)=\max\{t\in[0,1]:C(x,t)\le y\}$, $x,y\in[0,1]$.

Proof. The proof is based on similar considerations for triangular norms (cf. [1], Proposition 2.5.2).

$(i)\Rightarrow(ii)$ Let us assume that C is left-continuous with respect to the second variable. We can observe that if for some $x,y,z\in[0,1]$ the inequality $C(x,z)\leqslant y$ holds then $z\in R(x,y)$ (cf. (3)), so $z\leqslant I_C(x,y)$.
Now, let us assume that $z\leqslant I_C(x,y)$ for some $x,y,z\in[0,1]$. If $z<I_C(x,y)$ then there exists $t_0>z$ such that $C(x,t_0)\leqslant y$. From the monotonicity of the fuzzy conjunction C we obtain $C(x,z)\leqslant y$. If $z=I_C(x,y)$ then either $z\in R(x,y)$ and also $C(x,z)\leqslant y$ or $z\notin R(x,y)$. In the last case, from the property of supremum, there exists such increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ that for all $n\in\mathbb{N}$ the inequalities $t_n<z$ and $C(x,t_n)\leqslant y$ hold and $\lim_{n\to\infty}t_n=z$. From the left-continuity with respect to the second variable of C it follows

$$C(x,z)=C(x,\lim_{n\to\infty}t_n)=\lim_{n\to\infty}C(x,t_n)\leqslant y,$$

which proves that $C(x,z)\leqslant y$.
$(ii)\Rightarrow(iii)$ Let us assume that C and I_C fulfil the residuation principle (RP). Because for arbitrary $x,y\in[0,1]$ the trivial inequality $I_C(x,y)\geqslant I_C(x,y)$ holds, we have $C(x,I_C(x,y))\leqslant y$. This means that $I_C(x,y)\in R(x,y)$ and $\sup R(x,y)=\max R(x,y)$.
$(iii)\Rightarrow(i)$ Firstly, we will prove that C is infinitely right-sup-distributive, i.e.

$$C(x,\sup_{s\in S}y_s)=\sup_{s\in S}C(x,y_s), \quad x,y_s\in[0,1],\ s\in S.$$

Let us observe that from the monotonicity of C we have the inequality

$$C(x,\sup_{s\in S}y_s)\geqslant\sup_{s\in S}C(x,y_s).$$

Let $y=\sup_{s\in S}C(x,y_s)$. For an arbitrary $s\in S$ we have $C(x,y_s)\leqslant y$. This is why $y_s\in R(x,y)$ for every $s\in S$, and $y_s\leqslant I_C(x,y)$ for every $s\in S$. Hence, $\sup_{s\in S}y_s\leqslant I_C(x,y)$. Again, by the monotonicity of C and from (iii) we obtain

$$C(x,\sup_{s\in S}y_s)\leqslant C(x,I_C(x,y))\leqslant y=\sup_{s\in S}C(x,y_s).$$

From the above inequalities it follows that the fuzzy conjunction C is infinitely right-sup-distributive. Let $x, y_n \in [0,1]$, $y_n \leqslant y_{n+1}$, $n \in \mathbb{N}$. We have

$$C(x, \lim_{n\to\infty} y_n) = C(x, \sup_{n\in\mathbb{N}} y_n) = \\ = \sup_{n\in\mathbb{N}} C(x, y_n) = \lim_{n\to\infty} C(x, y_n).$$

This means that the function C is a left-continuous function with respect to the second variable. □

Now, we are presenting properties of fuzzy implication I_C that together with its generator C fulfil the residuum principle. To this end let us denote by $\mathcal{C}$ the family of all fuzzy conjunctions which are left-continuous with respect to the second variable and fulfilling the condition (4).

Theorem 3. *Let $C \in \mathcal{C}$. The induced implication I_C is right-continuous with respect to the second variable.*

Proof. The proof is based on similar considerations for triangular norms (por. [1], Theorem 2.5.7). Let us assume that the function I_C is not right-continuous with respect to the second variable in a point $(x_0, y_0) \in [0,1] \times [0,1)$. Because the implication I_C is increasing with respect to the second variable, so there exist such $a, b \in [0,1]$ that $a > b$ and

$$I_C(x_0, y) \geqslant a, \quad y > y_0$$

and also $I_C(x_0, y_0) = b$. Thus, by the Theorem 2 we have

$$C(x_0, a) \leqslant y, \quad y > y_0$$

From a property the supremum we obtain $C(x_0, a) \leqslant y_0$. Again from the residuation principle we have $b = I_C(x_0, y_0) \geqslant a$ and this is the contradiction to the assumption that $a > b$. Hence, I_C is right-continuous with respect to the second variable. □

Theorem 4. *Let $C \in \mathcal{C}$ be additionally left-continuous with respect to the first variable. The induced implication I_C is left-continuous with respect to the first variable.*

Proof. The proof is based on similar considerations for triangular norms (por. [1], Theorem 2.5.7).

Let us suppose that the function I_C is not left-continuous with respect to the first variable in a point $(x_0, y_0) \in (0,1] \times [0,1]$. Because the implication I_C is decreasing with respect to the first variable, so there exist such $a, b \in [0,1]$ that $a > b$ and

$$I_C(x, y_0) \geqslant a, \quad x < x_0$$

and also $I_C(x_0, y_0) = b$. Thus, by the Theorem 2 we have

$$C(x, a) \leqslant y_0, \quad x > x_0$$

From the property of supremum we obtain $C(x_0, a) \leqslant y_0$. From the residuation principle we have $b = I_C(x_0, y_0) \geqslant a$ and this is the contradiction to the supposition that $a > b$. Hence, I_C is left-continuous with respect to the first variable. □

The following consideration is based on the contribution [7].

Theorem 5. *Let $C \in \mathcal{C}$. The induced implication I_C*
(i) has left neutral element (NP) *if and only if C has left neutral element* 1*;*
(ii) fulfils exchange principle (EP) *if and only if C fulfills* (EP)*;*
(iii) fulfils identity principle (IP) *if and only if C fulfils*

$$C(x, 1) \leqslant x, \quad x \in [0,1]; \tag{5}$$

(iv) has ordering property (OP) *if and only if C has right neutral element* 1*.*

Proof. (i)(⇒) Let us assume that for all $y \in [0,1]$

$$I_C(1, y) = \max\{t \in [0,1] : C(1,t) \leq y\} = y. \tag{6}$$

For an arbitrary $y \in [0,1]$ we obtain $C(1, y) \leqslant y$. Let us presume that for a $y_0 \in [0,1]$ we have $C(1, y_0) < y_0$. Thus, there exists $z < y_0$ such that $C(1, y_0) \leqslant z$. From the residuation principle one receives $I_C(1, z) \geqslant y_0 > z$ and this is the contradiction to the assumption (6). Hence,

$$C(1, y) = y, \quad y \in [0,1]. \tag{7}$$

(⇐) Let us assume that $C(1, y) = y$, $y \in [0,1]$. We obtain

$$I_C(1, y) = \max\{t \in [0,1] : C(1,t) \leq y\} = \\ = \max\{t \in [0,1] : t \leq y\} = y$$

for $y \in [0,1]$, so I_C fulfils (NP).

(ii) (⇒) Let us assume that the implication fulfils the exchange principle (EP). For the proof of contradiction let us suppose that there exist such $x, y, z \in [0,1]$ that $C(x, C(y,z)) \neq C(y, C(x,z))$. Without loss of generality we can assume that $C(x, C(y,z)) < C(y, C(x,z))$. Applying twice the residuation principle (RP) we obtain as follows

$$I_C(y, C(x, C(y,z))) < C(x,z), \\ I_C(x, I_C(y, C(x, C(y,z))) < z.$$

Due to (EP) we have $I_C(y, I_C(x, C(x, C(y,z))) < z$. Again, applying twice (RP) we obtain

$$I_C(x, C(x, C(y,z))) < C(y,z), \\ C(x, C(y,z)) < C(x, C(y,z)),$$

that is a trivial contradiction. Thus, C fulfils (EP). (⇐) Now, let us assume that C fulfils (EP). From residuation principle we obtain

$$\begin{aligned}&I_C(x, I_C(y,z)) = \\&\quad = \max\{t \in [0,1] : C(x,t) \le I_C(y,z)\} = \\&\quad = \max\{t \in [0,1] : C(y, C(x,t)) \le z\} = \\&\quad = \max\{t \in [0,1] : C(x, C(y,t)) \le z\} = \\&\quad = \max\{t \in [0,1] : C(y,t) \le I_C(x,z)\} = \\&\quad\quad = I_C(y, I_C(x,z)),\end{aligned}$$

which proves (EP).

(iii) It is enough to observe, that for an arbitrary $x \in [0,1]$

$$I_C(x,x) = \max\{t \in [0,1] : C(x,t) \leqslant x\} = 1 \Leftrightarrow \Leftrightarrow C(x,1) \leqslant x.$$

(iv) ($\Rightarrow$) Let us assume that the implication I_C fulfils the ordering property, i.e.

$$I_C(x,y) = 1 \Leftrightarrow x \leqslant y, \quad x,y \in [0,1]$$

and let $x \in [0,1]$. Then

$$I_C(x,x) = \max\{t \in [0,1] : C(x,t) \leqslant x\} = 1.$$

This means that $C(x,1) \leqslant x$. Moreover, let us observe that by the monotonicity of the conjunction C we obtain

$$I_C(x, C(x,1)) = \max\{t \in [0,1] : C(x,t) \leqslant \leqslant C(x,1)\} = 1.$$

Thus we obtain $x \leqslant C(x,1)$. and for an arbitrary $x \in [0,1]$ we have $C(x,1) = x$. This means that C has the right neutral element 1.

($\Leftarrow$) Now, let us assume that C has the right neutral element 1. If for some $x,y \in [0,1]$ there is

$$I_C(x,y) = \max\{t \in [0,1] : C(x,t) \leqslant y\} = 1. \quad (8)$$

then we have $x = C(x,1) \leqslant y$. If, on the other hand for some $x,y \in [0,1]$ we have $x \leqslant y$, then from neutral element C we obtain $C(x,1) = x \leqslant y$. Thus from the residuation principle it follows that $1 \leqslant I_C(x,y) \leqslant 1$ that is $I_C(x,y) = 1$. Hence I_C fulfils (OP). □

From the Theorem 5 it follows the next result.

Corollary 3 (por. [1])**.** Every implication induced by a triangular norm that is left continuous with respect to the second variable fulfils the conditions (NP), (EP), (IP), (OP).

In general a fuzzy conjunction is not commutative, the induced implication can be generated in a different way. This method is presented in Theorem 6, which can be proved in the similar way as Theorem 1.

Theorem 6. *Let C be a fuzzy conjunction. The function $I_C^*\colon [0,1]^2 \to [0,1]$ given by the formula*

$$I_C^*(x,y) = \sup\{t \in [0,1] : C(t,x) \le y\}, \ \ x,y \in [0,1] \quad (9)$$

is a fuzzy implication if and only if C fulfils the condition

$$C(y,1) > 0, \quad y \in (0,1], \quad (10)$$

Similar to the case of the implication I_C there is an important dependency between functions C i I_C^*.

Definition 5. We say, that the functions C and I_C^* fulfil the residuation principle of the second type if

$$C(z,x) \leqslant y \Leftrightarrow I_C^*(x,y) \geqslant z \quad x,y,z \in [0,1]. \quad \text{(RP*)}$$

The next Theorem can be proved similarly to Theorem 2.

Theorem 7. *Let C be a fuzzy conjunction fulfilling* (10)*. The following conditions are equivalent:*
(i) C is a left continuous function with respect to the first variable;
(ii) C and I_C^* fulfil the residuation principle (RP*)*;*
(iii) $I_C^*(x,y) = \max\{t \in [0,1] : C(t,x) \le y\}$, $x,y \in [0,1]$.

Theorem 8. *Let C be a fuzzy conjunction that is a left continuous with respect to each of the variables. Then $I_C = I_C^*$ if and only if C is commutative.*

Proof. ($\Rightarrow$) If C is a commutative operation then for all $x,y \in [0,1]$ we have the equality of the sets

$$\{t \in [0,1] : C(t,x) \le y\} = = \{t \in [0,1] : C(x,t) \le y\},$$

so $I_C = I_C^*$.

($\Leftarrow$) Let us assume that $I_C = I_C^*$. From the assumption on C and from Theorems 2 and 7 we have the condition (RP) and (RP*) fulfilled. Let us suppose that C is not a commutative operation. Without loss of generality we can assume that for some $x,y \in [0,1]$ we have $C(x,y) < C(y,x)$. Then from (RP*) we obtain $I_C^*(x, C(x,y)) < y$. From the equality of implication we have $I_C(x, C(x,y)) < y$. As a result, from (RP), we obtain $C(x,y) < C(x,y)$ and this is an obvious contradiction. Thus C is commutative. □

4. Conjunctions induced from implications

In this section we consider fuzzy conjunctions generated from fuzzy implications. Moreover, we indicate the assumption under which it is possible to regain implications from such conjunctions and also conjunctions from the induced implications.

Lemma 4. *For an arbitrary* $I\colon [0,1]^2 \to [0,1]$ *with the right absorbing element* 1*, the function* $C_I\colon [0,1]^2 \to [0,1]$ *given by the formula*

$$C_I(x,y) = \inf\{t \in [0,1] : I(x,t) \geqslant y\}, \quad x,y \in [0,1] \tag{11}$$

is increasing with respect to the second variable. If additionally I *is increasing with respect to the first variable then* C_I *is increasing with respect to the first variable.*

Proof. Firstly, let us observe, that the function C_I is defined correctly. Let $x, y \in [0,1]$. Let us denote

$$P(x,y) := \{t \in [0,1] : I(x,t) \geqslant y\}. \tag{12}$$

From the existence of the right absorbing element 1 it follows that $1 \geqslant I(x,1) = 1 \geqslant y$, so $1 \in P(x,y)$. It means that $P(x,y) \neq \emptyset$, thus $\inf P(x,y) \in [0,1]$. Let $x, y, v \in [0,1]$, $y \leqslant v$. We have as follows

$$\{t \in [0,1] : I(x,t) \geqslant y\} \supset \{t \in [0,1] : I(x,t) \geqslant v\},$$

$$\inf\{t \in [0,1] : I(x,t) \geqslant y\} \leqslant \leqslant \inf\{t \in [0,1] : I(x,t) \geqslant v\},$$

$$C_I(x,y) \leqslant C_I(x,v),$$

which proves that C_I is increasing with respect to the second variable. Now, let $x, u, y \in [0,1]$, $x \leqslant u$. From monotonicity of the function I with respect to the first variable we have $I(x,t) \geqslant I(u,t)$ for all $t \in [0,1]$. Hence, we obtain as follows

$$\{t \in [0,1] : I(u,t) \geqslant y\} \subset \{t \in [0,1] : I(x,t) \geqslant y\},$$

$$\inf\{t \in [0,1] : I(u,t) \geqslant y\} \geqslant \geqslant \inf\{t \in [0,1] : I(x,t) \geqslant y\},$$

$$C_I(u,y) \geqslant C_I(x,y).$$

Thus, the function C_I is increasing with respect to the first variable. □

Lemma 5. *If* I *is a fuzzy implication then* $C_I(0,0) = C_I(0,1) = C_I(1,0) = 0$.

Proof. From the Corollary 2 we obtain

$$C_I(0,0) = \inf\{t \in [0,1] : I(0,t) \geqslant 0\} = 0,$$

$$C_I(1,0) = \inf\{t \in [0,1] : I(1,t) \geqslant 0\} = 0,$$

$$C_I(0,1) = \inf\{t \in [0,1] : I(0,t) \geqslant 1\} = = \inf\{t \in [0,1] : 1 \geqslant 1\} = 0.$$

□

Lemma 6. *Let* I *be a fuzzy implication. Then* $C_I(1,1) = 1$ *if and only if*

$$I(1,y) < 1 \quad y \in [0,1). \tag{13}$$

Proof. From (18) we have $C_I(1,1) = \inf\{t \in [0,1] : I(1,t) = 1\}$. It is enough to observe that $C_I(1,1) = 1$ if and only if the C fulfils the condition (13). □

From Lemmas 4–6 it follows the next statement.

Theorem 9. *Let* I *be a fuzzy implication. The function* $C_I\colon [0,1]^2 \to [0,1]$ *given by the formula*

$$C_I(x,y) = \inf\{t \in [0,1] : I(x,t) \geqslant y\}, \quad x,y \in [0,1] \tag{14}$$

is a fuzzy conjunction if and only if the C *fulfils the condition* (13).

Definition 6. Let I be a fuzzy implication fulfilling the condition (13). The function C_I given by (18) is called the induced conjunction.

Example 4. The following table shows fuzzy conjunctions with their generators (cf. Examples 1, 2). The symbol – means that, the function C_I is not a fuzzy conjunction.

Implication I	Conjunction C_I
I_0	C_1
I_1	–
I_{LK}	T_{LK}
I_{GD}	T_M
I_{RC}	$C_6(x,y) = \begin{cases} 0, & \text{if } x+y \leq 1 \\ \frac{x+y-1}{x}, & \text{if } x+y > 1 \end{cases}$
I_{DN}	C_4
I_{GG}	T_P
I_{RS}	$C_7(x,y) = \begin{cases} 0, & \text{if } y = 0 \\ x, & \text{if } y > 0 \end{cases}$
I_{YG}	$C_8(x,y) = \begin{cases} 0, & \text{if } x = 0 \\ y^{\frac{1}{x}}, & \text{if } x > 0 \end{cases}$
I_{FD}	T_{nM}
I_{WB}	C_2

For example, we will show that the implication I_{RS} induces the conjunction C_7. Let us observe that

$$C_{I_{RS}}(x,0) = \inf\{t \in [0,1] : I_{RS}(x,t) \geqslant 0\} = 0$$

and for $y > 0$ we have

$$C_{I_{RS}}(x,y) = \inf\{t \in [0,1] : I_{RS}(x,t) = 1\} = = \inf\{t \in [0,1] : x \leqslant t\} = x.$$

Thus we can see that

$$C_{I_{RS}}(x,y) = \begin{cases} 1, & \text{if } y = 0 \\ x, & \text{if } y > 0 \end{cases} = C_7(x,y).$$

The condition (13) put on an implication guarantees, that the function C_I given by (18) is a fuzzy conjunction. However it is not sufficient for the functions I and C_I to fulfil the residuation principle that in this case has a form

$$C_I(x,z) \leqslant y \Leftrightarrow I(x,y) \geqslant z, \quad x,y,z \in [0,1]. \tag{RP**}$$

Theorem 10. *Let I be a fuzzy implication fulfilling* (13)*. The following conditions are equivalent:*
(i) I is a right-continuous function with respect to the second variable;
(ii) I and C_I fulfil the residuation principle (RP**)*;*
(iii) $C_I(x,y) = \min\{t \in [0,1] : I(x,t) \geqslant y\}$, $x,y \in [0,1]$.

Proof. The proof is based on similar considerations for triangular norms (cf. [1], Proposition 2.5.13)

$(i) \Rightarrow (ii)$ Let us assume, that I is right-continuous with respect to the second variable. We can observe, that if for some $x,y,z \in [0,1]$ the inequality $I(x,z) \geqslant y$ holds then $z \in P(x,y)$ (cf. (12)), so $z \geqslant C_I(x,y)$.
Let us assume that $z \geqslant C_I(x,y)$ for some $x,y,z \in [0,1]$. If $z > C_I(x,y)$ then there exists $t_0 < z$ such that $I(x,t_0) \geqslant y$. From monotonicity of the fuzzy implication I with respect to the second variable we obtain $I(x,z) \geqslant y$. If $z = C_I(x,y)$ then either $z \in P(x,y)$ and also $I(x,z) \geqslant y$ or $z \notin P(x,y)$. In the last case, from the property of the infimum, there exists such decreasing sequence $\{t_n\}_{n\in\mathbb{N}}$ that for all $n \in \mathbb{N}$ the inequalities $t_n > z$ and $I(x,t_n) \geqslant y$ hold and $\lim_{n\to\infty} t_n = z$. From the right-continuity with respect to the second variable of I it follows

$$I(x,z) = I(x, \lim_{n\to\infty} t_n) = \lim_{n\to\infty} I(x,t_n) \geqslant y,$$

which proves that $I(x,z) \geqslant y$.
$(ii) \Rightarrow (iii)$ Let us assume that I and C_I fulfil the residuation principle (RP**). Because for arbitrary $x,y \in [0,1]$ the trivial inequality $C_I(x,y) \leqslant C_I(x,y)$ holds, we have $I(x, C_I(x,y)) \geqslant y$. This means that $C_I(x,y) \in P(x,y)$ and $\inf P(x,y) = \min P(x,y)$.
$(iii) \Rightarrow (i)$ Firstly, we will prove that I is infinitely right-inf-distributive, i.e.

$$I(x, \inf_{s\in S} y_s) = \inf_{s\in S} I(x,y_s), \quad x, y_s \in [0,1],\ s \in S.$$

Let us see, that from the monotonicity with respect to the second variable of I we have inequality

$$I(x, \inf_{s\in S} y_s) \leqslant \inf_{s\in S} I(x,y_s).$$

Let $y = \inf_{s\in S} I(x,y_s)$, then for an arbitrary $s \in S$ we have $I(x,y_s) \geqslant y$. This is why $y_s \in P(x,y)$ for every $s \in S$, and $y_s \geqslant C_I(x,y)$ for every $s \in S$. Hence, $\inf_{s\in S} y_s \geqslant C_I(x,y)$. Again, by the monotonicity with respect to the second variable of I and from (iii) we obtain

$$I(x, \inf_{s\in S} y_s) \geqslant I(x, C_I(x,y)) \geqslant y = \inf_{s\in S} I(x,y_s).$$

From the above inequalities it follows that the fuzzy implication I is infinitely right-inf-distributive. Let $x, y_n \in [0,1]$, $y_n \geqslant y_{n+1}$, $n \in \mathbb{N}$, then we have

$$I(x, \lim_{n\to\infty} y_n) = I(x, \inf_{n\in\mathbb{N}} y_n) = \inf_{n\in\mathbb{N}} I(x,y_n) = \lim_{n\to\infty} I(x,y_n).$$

This means that the function I is right-continuous function with respect to the second variable. □

Now, we are presenting properties of fuzzy conjunction C_I that together with its generator I fulfil the residuum principle (RP**). To this end let us denote by $\mathcal{I}$ the family of all fuzzy implications which are right-continuous with respect to the second variable and fulfilling the condition (13).

Theorem 11. *Let $I \in \mathcal{I}$. The induced implication C_I is left-continuous with respect to the second variable.*

Proof. The proof is based on similar considerations for triangular norms (por. [1], Theorem 2.5.7). Let us assume that the function I_C is not left-continuous with respect to the second variable in a point $(x_0,y_0) \in [0,1] \times (0,1]$. Because the conjunction C_I is increasing with respect to the second variable, so there exist such $a,b \in [0,1]$ that $a < b$ and

$$C_I(x_0,y) \leqslant a, \quad y < y_0$$

and also $C_I(x_0,y_0) = b$. Thus, from the Theorem 2 we have

$$I(x_0,a) \geqslant y, \quad y < y_0$$

From a property of supremum we obtain $I(x_0,a) \geqslant \sup[a,y_0) = y_0$. Again from the residuation principle (RP**) we have $b = C_I(x_0,y_0) \leqslant a$ and this is the contradiction to the assumption that $a < b$. Hence, C_I is left-continuous with respect to the second variable. □

Theorem 12. *Let $I \in \mathcal{I}$. Then $I = I_{C_I}$, that is*

$$I(x,y) = \max\{t \in [0,1] : C_I(x,t) \leqslant y\},\ x,y \in [0,1]. \tag{15}$$

Proof. Let $x,y \in [0,1]$. From definition of C_I and from monotonicity with respect to the second variable of implication I we obtain

$$C_I(x, I(x,y)) = \\ = \min\{t \in [0,1] : I(x,t) \geqslant I(x,y)\} \leqslant y.$$

Thus we have $I(x,y) \in \{t \in [0,1] : C_I(x,t) \leqslant y\}$ and

$$I(x,y) \leqslant I_{C_I}(x,y). \tag{16}$$

From $C_I(x, I_{C_I}(x,y)) \leqslant C_I(x, I_{C_I}(x,y))$ by (RP**) we obtain

$$I(x, C_I(x, I_{C_I}(x,y))) \geqslant I_{C_I}(x,y). \tag{17}$$

Because C_I is a conjunction which is left-continuous with respect to the second variable (from Theorem 11), C_I and I_{C_I} fulfil the residuation principle (from the Theorem 10). Thus, by the trivial inequality $I_{C_I}(x,y) \geqslant I_{C_I}(x,y)$ we obtain $C_I(x, I_{C_I}(x,y)) \leqslant y$. Additionally from (17) and the monotonicity with respect to the second variable of the implication it follows $I(x,y) \geqslant I_{C_I}(x,y)$. Hence and by (16) we obtain the equality $I = I_{C_I}$. □

Theorem 13. *Let $C \in \mathcal{C}$. Then $C = C_{I_C}$, that is*

$$C(x,y) = \min\{t \in [0,1] : I_C(x,t) \geqslant y\},\ x,y \in [0,1]. \quad (18)$$

Proof. Let $x, y \in [0,1]$. From definition of I_C and from monotonicity of the conjunction C we obtain

$$I_C(x, C(x,y)) = \\ = \max\{t \in [0,1] : C(x,t) \leqslant C(x,y)\} \geqslant y.$$

Thus we have $C(x,y) \in \{t \in [0,1] : I_C(x,t) \geqslant y\}$ and

$$C(x,y) \geqslant C_{I_C}(x,y). \quad (19)$$

From $I_C(x, C_{I_C}(x,y)) \geqslant I_C(x, C_{I_C}(x,y))$ by (RP) we obtain

$$C(x, I_C(x, C_{I_C}(x,y))) \leqslant C_{I_C}(x,y). \quad (20)$$

Because C_I is an implication which is right-continuous with respect to the second variable (from Theorem 10), I_C and C_{I_C} fulfil the residuation principle (from the Theorem 2). Thus, by the trivial inequality $I_{C_I}(x,y) \geqslant I_{C_I}(x,y)$ we obtain $I_C(x, C_{I_C}(x,y)) \geqslant y$. Additionally from (20) and the monotonicity of the conjunction it follows $C(x,y) \leqslant C_{I_C}(x,y)$. Hence and by (19) we obtain the equality $C = C_{I_C}$. □

5. Conclusion

In this contribution the residuation concept that connects the fuzzy implication together with the fuzzy conjunction is examined. The method of regaining the connectives that play the roles of generators are shown in Theorems 12 and 13. Further examination can concern the characterizations of induced implications fulfilling other properties.

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