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On a graded q-differential algebra

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Abstract

Given an associative unital $\mathbb{Z}_N$-graded algebra over the complex numbers we construct the graded q-differential algebra by means of a graded q-commutator, where q is a primitive N-th root of unity. The N-differential d of the graded q-differential algebra is a homogeneous endomorphism of degree 1 satisfying the graded q-Leibniz rule and $d^N = 0$. We apply this construction to a reduced quantum plane and study the exterior calculus on a reduced quantum plane induced by the N-differential of the graded q-differential algebra. Making use of the higher order differentials $d^k x$ induced by the N-differential d we construct an analogue of an algebra of differential forms with exterior differential satisfying $d^N = 0$.

1 Introduction

The basic property of the differential d of a cochain complex is $d^2 = 0$, and the study of the cohomologies of a cochain complex can be viewed as the study of the nontrivial solutions of the equation $d^2 = 0$. This basic property of the differential of a cochain complex is based on the operation of alternation which twice successively applied gives zero because of the identity $1 + (-1) = 0$. It should be mentioned that an intensive and fruitful development of the q-deformed structures within the framework of quantum groups, non-commutative geometry and field theories on quantum spaces stimulated the interest towards a possible q-generalization of some classical structures in mathematics. In our case the above mentioned identity $1 + (-1) = 0$ can be easily generalized because it is a particular case of the well known identity $1 + q + q^2 + \ldots + q^{N-1} = 0$, where q is a primitive N-th root of unity. This leads to a natural idea to generalize the notion of a cochain complex replacing the classical property $d^2 = 0$ by the more general one $d^N = 0, N \geq 2$. This generalization of a cochain complex called N-complex was described in [10]. An immediate consequence of the basic property $d^N = 0$ of the differential of a N-complex is a new type of cohomologies $\mathrm{Ker} d^p/\mathrm{Im} d^{N-p}$, where $p = 1, 2, \ldots, N$. The theory of N-complexes was elaborated in the series of papers [6, 7, 8], where one can find several examples of N-complexes with detailed analysis of their generalized cohomologies and their applications to constrained systems (BRS-method).

A generalization of a graded differential algebra called a graded q-differential algebra [7] arises naturally within the framework of the theory of N-complexes. The differential of a graded q-differential algebra satisfies the graded q-Leibniz rule and $d^N = 0$. It is well known that one of the most important examples of graded differential algebra is the algebra of differential forms on a smooth manifold (de Rham complex) and q-generalization of a graded differential algebra raises an intriguing question: is it possible to construct a realization of a graded q-differential algebra by analogues of differential forms on a geometric space? The first attempt in this direction was made in [10], where the author constructed the analogues of differential forms on $\mathbb{R}^n$ with exterior differential d satisfying $d^N = 0$. Unfortunately it can be shown that the algebraic structure of the analogues of differential forms proposed in [10] is not consistent (this was explained to the author of the present paper by M. Dubois-Violette in the private communication). The main purpose of the present paper is to construct an example of a graded q-differential algebra with the help of analogues of differential forms on a geometric space. Given an associative unital graded $\mathbb{C}$-algebra we prove that if this algebra contains an element v satisfying $v^N = \alpha\, e$, where e is the unit element of an algebra and $\alpha \in \mathbb{C}$, then this algebra becomes a graded q-differential algebra if we endow it with a differential d constructed with the help of a graded q-commutator. Then we apply this construction to a reduced quantum plane [4] which is the algebra of polynomials generated by two variables x, y satisfying $xy = q\, yx, x^N = y^N = 1$, where q is a primitive N-th root of unity. It is obvious that this algebra contains an element satisfying $v^N = 1$ and according to the previously proved theorem we have a structure of a graded q-differential algebra. It should be mentioned that the algebra of polynomials on a reduced quantum plane is a particular case of a generalized Clifford algebra [9]. Making use of the generators x, y which can be interpreted as non-commutative coordinates of a reduced quantum plane we express the differential d and the elements of the graded q-differential algebra in terms of differentials $d^k x, k = 1, 2, \ldots, N-1$ interpreting the elements of the graded q-differential algebra as differentials forms and d as exterior differential. We would like to point out that the peculiar property of the exterior calculus on a reduced quantum plane with exterior differential satisfying $d^N = 0$ is the appearance of the higher order differentials $d^k x, k > 1$ [11, 12, 2, 5, 13].

2 Graded q-differential algebra

In this section given a $\mathbb{Z}_N$-graded algebra we construct the graded q-differential algebra. Let us remind the definition of a graded q-differential algebra [7]. A unital associative algebra is said to be a graded q-differential algebra ($q \in \mathbb{C}, q \neq 1$) if it is a $\mathbb{Z}_N$-graded (or $\mathbb{Z}$-graded) algebra endowed with the linear mapping d of degree 1 satisfying the graded q-Leibniz rule and $d^N = 0$ in the case when q is a primitive N-th root of unity. The linear mapping d is called an N-differential of a graded q-differential algebra.

Let $\mathcal{A}$ be an associative unital $\mathbb{Z}$ (or $\mathbb{Z}_N$)-graded algebra over the complex numbers $\mathbb{C}$ and $\mathcal{A}^k \subset \mathcal{A}$ be the subspace of homogeneous elements of a grading k. The grading of a homogeneous element w will be denoted by $|w|$, which means that if $w \in \mathcal{A}^k$ then $|w| = k$. Let q be a complex number such that $q \neq 1$. The q-commutator of two homogeneous elements $w, w' \in \mathcal{A}$ is defined by the formula

$$[w, w']_q = ww' - q^{|w||w'|} w'w. \tag{2.1}$$

Using the associativity of an algebra $\mathcal{A}$ and the property $|ww'| = |w| + |w'|$ of its graded structure it is easy to show that for any homogeneous elements $w, w', w'' \in \mathcal{A}$ the q-commutator has the property

$$[w, w'w'']_q = [w, w']_q w'' + q^{|w||w'|} w'[w, w'']_q. \tag{2.2}$$

Given an element $v \in \mathcal{A}^1$ one can define the mapping $d_v : \mathcal{A}^k \to \mathcal{A}^{k+1}$ by the following formula

$$d_v w = [v, w]_q, \qquad w \in \mathcal{A}^k.$$

It follows from the property of q-commutator (2.2) that d_v is a graded q-differential on an algebra $\mathcal{A}$, i.e. it is a homogeneous linear mapping of degree 1 satisfying the graded q-Leibniz rule

$$d_v(ww') = d_v(w)w' + q^{|w|} w d_v(w'), \tag{2.3}$$

where w, w' are the homogeneous elements of $\mathcal{A}$.

Lemma 1. *For any integer $k \geq 2$ the k-th power of the q-differential d_v can be written as follows*

$$d_v^k w = \sum_{i=0}^{k} p_i^{(k)} v^{k-i} w v^i, \tag{2.4}$$

where w is a homogeneous element of $\mathcal{A}$ and

$$p_i^{(k)} = (-1)^i q^{|w|_i} \frac{[k]_q!}{[i]_q![k-i]_q!} = (-1)^i q^{|w|_i} \left[\begin{array}{c} k \\ i \end{array} \right]_q, \tag{2.5}$$

$$|w|_i = i|w| + \frac{i(i-1)}{2}. \tag{2.6}$$

Proof. The coefficients $p_i^{(k)}$ of an expansion (2.4) satisfy the following recurrence relations

$$p_0^{(k)} = p_0^{(k+1)} = 1, \;\; p_{k+1}^{(k+1)} = -q^{|w|+k} p_k^{(k)}, \;\; p_i^{(k+1)} = p_i^{(k)} - q^{|w|+k} p_{i-1}^{(k)}, \tag{2.7}$$

where $1 \leq i \leq k$. Now we prove the formula (2.4) by means of a mathematical induction (with respect to k) and the recurrence relations (2.7). ■

Theorem 1. *If N is an integer such that $N \geq 2$, $\mathcal{A}$ is an associative unital $\mathbb{Z}_N$-graded $\mathbb{C}$-algebra containing an element v of grading one such that $v^N = \alpha e$, where $\alpha \in \mathbb{C}$ and e is the unity element of an algebra $\mathcal{A}$, q is a primitive N-th root of unity, then $d_v w = [v, w]_q$ satisfies $d_v^N w = 0$ for any $w \in \mathcal{A}$.*

Proof. It follows from the Lemma 1 that if q is a primitive N-th root of unity then for any integer $l = 1, 2, \ldots, N-1$ the coefficient $p_l^{(N)}$ contains the factor $[N]_q$ which is equal to zero in the case of q being a primitive N-th root of unity and this implies $p_l^{(N)} = 0$. Thus $d_v^N(w) = v^N w + (-1)^N q^{|w|_N} w v^N$. Taking into account that $v^N = \alpha e$ we obtain $d_v^N(w) = (1 + (-1)^N q^{|w|_N}) \alpha w$. The first factor in the right-hand side of the above formula equals to zero. Indeed if N is an odd number then $1 - (q^N)^{\frac{N-1}{2}} = 0$. In the case of an even integer N we have $1 + (q^{\frac{N}{2}})^{N-1} = 1 + (-1)^{N-1} = 0$, and this ends the proof. ■

Let $\mathcal{A}$ be an associative unital $\mathbb{Z}_N$-graded algebra over the complex numbers $\mathbb{C}$ with unit element denoted by e. Then from the property (2.3) and the Theorem 1 it follows

Corollary. *If there exists an element $v \in \mathcal{A}$ of grading 1 such that $v^N = \alpha e, \alpha \in \mathbb{C}$ then an algebra $\mathcal{A}$ endowed with the homogeneous linear mapping $d_v : \mathcal{A}^k \to \mathcal{A}^{k+1}$ of degree 1 defined by $d_v w = [v, w]_q$, where $w \in \mathcal{A}$, and q is a primitive N-th root of unity, is a $\mathbb{Z}_N$-graded q-differential algebra and d_v is its N-differential.*

Let us remind that a first order differential calculus over an associative unital algebra $\mathcal{B}$ is a pair $(\mathcal{M}, d)$, where $\mathcal{M}$ is a $(\mathcal{B}, \mathcal{B})$-bimodule and d is a linear mapping $d : \mathcal{B} \to \mathcal{M}$ which satisfies the Leibniz rule $d(ww') = d(w)w' + wd(w')$, where $w, w' \in \mathcal{B}$. The subspace $\mathcal{A}^0$ of elements of grading zero of a $\mathbb{Z}_N$-graded algebra $\mathcal{A}$ is a subalgebra, and N-differential d_v restricted to this subalgebra induces a first order differential calculus $(\mathcal{A}^1, d_v)$ where the space $\mathcal{A}^1$ of elements of grading 1 has a $(\mathcal{A}^0, \mathcal{A}^0)$-bimodule structure. Indeed it follows from the associativity of the algebra $\mathcal{A}$ and its $\mathbb{Z}_N$-graded structure that for each k the mappings $\mathcal{A}^0 \times \mathcal{A}^k \to \mathcal{A}^k$ and $\mathcal{A}^k \times \mathcal{A}^0 \to \mathcal{A}^k$ determined by the algebra multiplication $(r, w) \to rw,\ (w, s) \to ws$, where $w \in \mathcal{A}^k$ and $r, s \in \mathcal{A}^0$, induce a $(\mathcal{A}^0, \mathcal{A}^0)$-bimodule structure on $\mathcal{A}^k$. In the next section we consider a reduced quantum plane from a point of view of graded q-differential algebra and study the exterior calculus induced by the N-differential.

3 Reduced quantum plane as a q-differential algebra

An exterior calculus with exterior differential d satisfying $d^N = 0$ has been studied in [1, 2, 12]. In this section we study the exterior calculus on a reduced quantum plane constructed with the help of a graded q-differential algebra described in the previous section. Let us remind that the unital associative algebra $\mathbb{C}_{rq}$ generated, over the complex numbers $\mathbb{C}$, by the two variables x and y satisfying the relations $x\,y = q\ y\,x,\ x^N = y^N = 1$, where q is a primitive N-th root of unity, can be considered as an algebra of polynomials over a reduced quantum plane. Let us mention that this algebra has a representation by $N \times N$ complex matrices. It should be also mentioned that the algebra of polynomials over a reduced quantum plane is a particular case of a generalized Clifford algebra which is an algebra generated by variables $x_1, x_2, \ldots, x_p$ obeying the relations $x_i x_j = q^{\mathrm{sg}(j-i)} x_j x_i,\ x_i^N = 1$, where sg is the sign function.

The set of monomials $B = \{\mathbf{1}, y, x, x^2, yx, y^2, \ldots, y^k x^l, \ldots, y^{N-1}x^{N-1}\}$ can be taken as the basis of the vector space of the algebra $\mathbb{C}_{rq}$. Having chosen the basis B we can endow this vector space with a $\mathbb{Z}_N$-graded structure as follows: if a polynomial $w \in \mathbb{C}_{rq}$ expressed in terms of the monomials of B has the form

$$w = \sum_{l=0}^{N-1} \beta_l y^k x^l, \quad \beta_l \in \mathbb{C}, \tag{3.1}$$

then we shall refer to it as the homogeneous polynomial of grading k, where $k \in \mathbb{Z}_N$. Let us denote the grading of a homogeneous polynomial w by $|w|$ and the subspace of the homogeneous polynomials of grading k by $\mathbb{C}^k_{rq}$. It is obvious that

$$\mathbb{C}_{rq} = \mathbb{C}^0_{rq} \oplus \mathbb{C}^1_{rq} \oplus \ldots \oplus \mathbb{C}^{N-1}_{rq}. \tag{3.2}$$

In particular a polynomial r of grading zero has the form

$$r = \sum_{l=0}^{N-1} \beta_l x^l, \qquad \beta_l \in \mathbb{C}, r \in \mathbb{C}^0_{rq}. \tag{3.3}$$

It is easy to show that the $\mathbb{Z}_N$-graded structure defined on a vector space $\mathbb{C}_{rq}$ by (3.2) is consistent with the algebra structure of $\mathbb{C}_{rq}$, i.e. for any two homogeneous polynomials we have $|ww'| = |w| + |w'|$. Consequently $\mathbb{C}_{rq}$ is a $\mathbb{Z}_N$-graded algebra with respect to (3.2), and there exists an element v of grading one of this algebra satisfying $v^N = \alpha$, where $\alpha \in \mathbb{C}$. Indeed one can take for instance $v = y$ which satisfies all mentioned above conditions. According to the Theorem 1 proved in the previous section we can endow a reduced quantum plane with the structure of a graded q-differential algebra defining the N-differential by the formula $d_v w = [v, w]_q$, where q is a primitive N-th root of unity and $w \in \mathbb{C}_{rq}$.

In order to give a differential-geometric interpretation to the graded q-differential algebra structure of $\mathbb{C}_{rq}$ induced by the N-differential d_v we interpret the commutative subalgebra $\mathbb{C}^0_{rq}$ of the x-polynomials (3.3) of $\mathbb{C}_{rq}$ as an algebra of polynomial functions on a one dimensional space with coordinate x. Since $\mathbb{C}^k_{rq}$ for $k > 0$ is a $\mathbb{C}^0_{rq}$-bimodule we interpret this $\mathbb{C}^0_{rq}$-bimodule of the elements of grading k as a bimodule of differential forms of degree k and we shall call an element of this bimodule a differential k-form on a one dimensional space with coordinate x. The N-differential d_v can be interpreted within the framework of this approach as an exterior differential.

It is easy to show that in one dimensional case we have a simple situation when every bimodule $\mathbb{C}^k_{rq}, k > 0$ of the differential k-forms is a free right module over the commutative algebra of functions $\mathbb{C}^0_{rq}$. Indeed if we write a differential k-form w as follows

$$w = y^k \sum_{l=0}^{N-1} \beta_l x^l = y^k\, r, \quad r = \sum_{l=0}^{N-1} \beta_l x^l \in \mathbb{C}^0_{rq}, \tag{3.4}$$

and take into account that the polynomial $r = (y^k)^{-1} w = y^{N-k} w$ is uniquely determined then we can conclude that $\mathbb{C}^k_{rq}$ is a free right module over $\mathbb{C}^0_{rq}$ generated by y^k.

It is well known that a bimodule structure on a free right module over an algebra $\mathcal{B}$ generated freely by p generators is uniquely determined by the homomorphism from an algebra $\mathcal{B}$ to the algebra of $(p \times p)$-matrices over $\mathcal{B}$. Since in the case of a reduced quantum plane every right module $\mathbb{C}^k_{rq}$ is freely generated by one generator (for instant we can take y^k as a generator of this module) its bimodule structure induces the homomorphism of the algebra of functions $\mathbb{C}^0_{rq}$ and denoting this homomorphism in the case of the generator y^k by $A_k : \mathbb{C}^0_{rq} \to \mathbb{C}^0_{rq}$ we have

$$r\, y^k = y^k\, A_k(r), \tag{3.5}$$

for any function $r \in \mathbb{C}^0_{rq}$. Making use of the relations (3) of the algebra of polynomials on a reduced quantum plane we easily find that $A_k(x) = q^k\, x$. Since the algebra of functions $\mathbb{C}^0_{rq}$ may be viewed as a bimodule over the same algebra we can consider the functions as degree zero differentials forms and the corresponding homomorphism is the identity mapping of $\mathbb{C}_{rq}$, i.e. $A_0 = I$, where $I : \mathbb{C}^0_{rq} \to \mathbb{C}^0_{rq}$ is the identity mapping. Thus the bimodule structures of the free right modules $\mathbb{C}^0_{rq}, \mathbb{C}^0_{rq}, \ldots, \mathbb{C}^{N-1}_{rq}$ of differential forms induce the associated homomorphisms $A_0, A_1, \ldots, A_{N-1}$ of the algebra $\mathbb{C}^0_{rq}$. It is easy to see that for any k it holds $A_k = A_1^k$.

Let us start with the first order differential calculus on the algebra of functions $\mathbb{C}^0_{rq}$ induced by the N-differential d_v which is the pair $(\mathbb{C}^1_{rq}, d_v)$, where $d_v : \mathbb{C}^0_{rq} \to \mathbb{C}^1_{rq}$ and $\mathbb{C}^1_{rq}$ is the bimodule over $\mathbb{C}^0_{rq}$. Since $\mathbb{C}^1_{rq}$ is a free right module over $\mathbb{C}^0_{rq}$ generated by y there exists an invertible element $u \in \mathbb{C}^0_{rq}$ such that $v = y \cdot u$. We take v as a generator of the free right module $\mathbb{C}^1_{rq}$. Using the relation (3.5) and the commutativity of the algebra $\mathbb{C}^0_{rq}$ we find that the relation determining the bimodule structure in terms of the new generator v has the same form (3.5) as in the case of the generator y. Let us express the differential $d_v w$ of a function $w \in \mathbb{C}^0_{rq}$ in terms of the new generator v. We get

$$d_v w = vw - wv = vw - vA_1(w) = v(w - A_1(w)) = v\,\Delta_q(w), \tag{3.6}$$

where $\Delta_q = I - A_1 : \mathbb{C}^0_{rq} \to \mathbb{C}^0_{rq}$. It is easy to verify that for any functions $w, w' \in \mathbb{C}^0_{rq}$ the mapping Δ_q has the following properties

$$\Delta_q(ww') = \Delta_q(w)w' + A_1(w)\Delta_q(w'), \tag{3.7}$$

$$\Delta_q(x^k) = (1-q)[k]_q\, x^k. \tag{3.8}$$

Particularly $d_v x = y\Delta_q(x)$, and this formula shows that $d_v x$ can be taken as a generator of the free right module $\mathbb{C}^1_{rq}$.

Since the bimodule $\mathbb{C}^1_{rq}$ of the first order differential calculus $(\mathbb{C}^1_{rq}, d_v)$ is a free right module we deal with a free coordinate calculus over the algebra $\mathbb{C}^0_{rq}$ [3], and in the case of a calculus of this type the differential induces the derivative $\partial : \mathbb{C}^0_{rq} \to \mathbb{C}^0_{rq}$ which is defined by the formula $d_v w = d_v x\ \partial w,\ \forall w \in \mathbb{C}^0_{rq}$. Using this definition we find that for any function w it holds

$$\partial w = (1-q)^{-1} x^{N-1} \Delta_0(w). \tag{3.9}$$

From this formula and (3.7),(3.8) it follows that this derivative satisfies the generalized Leibniz rule

$$\partial(ww') = \partial(w) \cdot w' + A_1(w) \cdot \partial(w'), \tag{3.10}$$

and

$$\partial x^k = [k]_q\, x^{k-1}. \tag{3.11}$$

Let us study the structure of the higher order exterior calculus on a reduced quantum plane or, by other words, the structure of the bimodule $\mathbb{C}^k_{rq}$ of differential k-forms, when $k > 1$. In this case we have a choice for the generator of the free right module. Indeed since the k-th power of the exterior differential d_v is not equal to zero when $k < N$, i.e. $d_v^k \neq 0$ for $k < N$, a differential k-form w may be expressed either by means of $(d_v x)^k$ or by means of $d_v^k x$. Straightforward calculation shows that we have the following relation between these generators

$$d_v^k x = \frac{[k]_q}{q^{\frac{k(k-1)}{2}}} (d_v x)^k\, x^{1-k}. \tag{3.12}$$

We will use the generator $(d_v x)^k$ of the free right module $\mathbb{C}^k_{rq}$ as a basis in our calculations with differential k-forms. For any differential k-form $w \in \mathbb{C}^k_{rq}$ we have $d_v w \in \mathbb{C}^{k+1}_{rq}$. Let us express these two differential forms in terms of the generators of the modules $\mathbb{C}^k_{rq}$ and $\mathbb{C}^{k+1}_{rq}$. We have $w = (d_v x)^k\, r$, $d_v w = (d_v x)^{k+1}\, \tilde{r}$, where $r, \tilde{r} \in \mathbb{C}^0_{rq}$ are the functions. Making use of the definition of the exterior differential d_v we calculate the relation between the functions $r, \tilde{r}$ which is

$$\tilde{r} = (\Delta_q x)^{-1}(q^{-k} r - q^k A_1(r)), \tag{3.13}$$

where A_1 is the homomorphism of the algebra of functions $\mathbb{C}^0_{rq}$. This relation shows that the exterior differential d_v considered in the case of the differential k-forms induces the mapping $\Delta^{(k)}_q : \mathbb{C}^0_{rq} \to \mathbb{C}^0_{rq}$ on the algebra of the function which is defined by the formula

$$d_v w = (d_v x)^{k+1} \Delta^{(k)}_q (r), \tag{3.14}$$

where

$$w = (d_v x)^k\, r. \tag{3.15}$$

It is obvious that

$$\Delta^{(k)}_q (r) = (\Delta_q x)^{-1}(q^{-k} r - q^k A_1(r)). \tag{3.16}$$

It is obvious that for $k = 0$ the mapping $\Delta^{(0)}_q$ coincides with the derivative induced by the differential d_v in the first order calculus, i.e.

$$\Delta^{(0)}_q (r) = \partial r = (\Delta_q x)^{-1}(r - A_1(r)). \tag{3.17}$$

The higher order mappings $\Delta^{(k)}_q$, which we do not have in the case of a classical exterior calculus on a one dimensional space, have the derivation like property

$$\Delta^{(k)}_q (r\, r') = \Delta^{(k)}_q (r)\, r' + q^k\, A_1(r)\, \Delta^{(0)}_x (r'), \tag{3.18}$$

where $k = 0, 1, 2, \ldots, N-1$. A higher order mapping $\Delta^{(k)}_q$ can be expressed in terms of the derivative ∂ as a differential operator on the algebra of functions as follows

$$\Delta^{(k)}_q = q^k\, \partial\; + \frac{q^{-k} - q^k}{1 - q}\, x^{-1}. \tag{3.19}$$

Thus we see that exterior calculus on a one dimensional space with coordinate x satisfying $x^N = 1$ generated by the exterior differential d_v satisfying $d_v^N = 0$ has the differential forms of higher order which are not presented in the case of a classical exterior calculus with $d^2 = 0$. The formula for the exterior differential of these kind of differential forms uses not an ordinary derivative but a deformed derivative (3.19). The exterior calculus with exterior differentials d_v satisfying $d_v^N = 0$ can be considered in a more general case of a generalized Clifford algebra with even number $2p$ of generators. Then we will have an exterior calculus with $d_v^N = 0$ on a p-dimensional space and the structure of this exterior calculus and associated structures such as connections and curvatures will be a main object of a forthcoming paper.

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