On τ-Function of Conjugate Nets

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Abstract

We study a potential introduced by Darboux to describe conjugate nets, which within the modern theory of integrable systems can be interpreted as a τ-function. We investigate the potential using the nonlocal $\bar{\partial}$-dressing method of Manakov and Zakharov and we show that it can be interpreted as the Fredholm determinant of an integral equation which naturally appears within that approach. Finally we give some arguments extending that interpretation to multicomponent Kadomtsev–Petviashvili hierarchy.

1   Introduction

Conjugate nets are certain parametrized submanifolds theory of which was the object of investigations of XIX-th century differential geometry [5, 2, 9]. The basic system of equations of the theory

$$\partial_i \partial_j h_k(u) = (\partial_j \log h_i(u)) \partial_i h_k(u) + (\partial_i \log h_j(u)) \partial_j h_k(u), \quad i, j, k \quad \text{distinct}, \qquad (1.1)$$

constitutes one-half of the system of the Lamé equations [12] describing orthogonal coordinate systems. Here the functions h_i, $i = 1, \ldots, N$, $N > 2$, called the Lamé coefficients, depend on parameters $u = (u_1, \ldots, u_N)$ of conjugate nets, and ∂_i, $i = 1, \ldots, N$, denote partial derivatives. The Darboux system (1.1) written in terms of the rotation coefficients, β_{ij}, defined by the compatible system

$$\partial_i h_j(u) = h_i(u)\beta_{ij}(u), \quad i \neq j, \qquad (1.2)$$

takes the form

$$\partial_j \beta_{ik}(u) = \beta_{ij}(u)\beta_{jk}(u), \quad j \neq i, k. \qquad (1.3)$$

This system is known in present-day terminology as the N-wave equation [11], which is the simplest equation of the N-component ($N > 2$) Kadomtsev–Petviashvili (KP) hierarchy. The parameters, u_i, of the nets can be identified with the first times of the hierarchy, while

the other times of the KP hierarchy describe integrable isoconjugate deformations of the nets [8].

Equations (1.3) imply that one can introduce the potentials β_{ii}, $i = 1, \dots, N$, such that

$$\partial_j \beta_{ii}(u) = \beta_{ij}(u)\beta_{ji}(u), \qquad i \neq j. \tag{1.4}$$

The symmetry $i \leftrightarrow j$ of the system (1.4) allows, in turn, for the existence of yet another potential field (this fact was known already to Darboux, see [5] p. 363)

$$\partial_i \log \tau(u) = -\beta_{ii}(u), \tag{1.5}$$

which was identified [8] with the τ–function of the multicomponent KP hierarchy.

The τ-functions play the central role [16, 6, 17, 10, 14] in establishing the connections between integrable systems and quantum field theory, statistical mechanics or the theory of random matrices. They are often represented in a determinantal form or can be identified with the Fredholm determinant of the integral Gel'fand–Levitan–Marchenko equation used to solve the model under consideration. In particular the determinantal formula for the τ-function of the KP hierarchy [6] follows from the free fermions (or $\mathfrak{gl}_\infty$) realization of the hierarchy. Within the context of the Zakharov and Shabat dressing method [20] the τ-function of the KP hierarchy was interpreted as the Fredholm determinant in [15].

Manakov and Zakharov introduced in [19] the $\bar{\partial}$-dressing method, which was based on the nonlocal $\bar{\partial}$-problem (see also [1]), as a generalized version of the inverse scattering method [21]. They rediscovered the Darboux system (1.1), in the generalized matrix form, as the basic set of equations solvable by the $\bar{\partial}$-dressing method. For example the KP equation [6] was shown in [4] to be a limiting case of the Darboux system.

The multicomponent KP hierarchy and conjugate nets were studied within the $\bar{\partial}$-dressing method in a number of papers [3, 7, 13]. However, in that approach the determinantal interpretation of the τ-function of conjugate nets was somehow missing. In this paper we show that indeed the τ-function of conjugate nets can be identified with the Fredholm determinant of the integral equation which inverts the nonlocal $\bar{\partial}$-problem.

The paper is constructed as follows. In Section 2 we collect the basic elements of the $\bar{\partial}$-dressing method and we present the way to solve the Darboux equations within this method. Section 3 is devoted to the presentation of the Fredholm determinant interpretation of the τ-function of conjugate nets. Finally in Section 4 we briefly discuss generalization of the above result to the full multicomponent KP hierarchy. In the Appendix we recall some standard facts [18] concerning the Fredholm integral equations.

2 The $\bar{\partial}$-dressing method and the Darboux equations

In this Section we recall ([19], [4]) the method of solution of the Darboux equations, (1.1) or (1.3), within the $\bar{\partial}$-dressing approach.

The basis of the $\bar{\partial}$-dressing method is the following integrodifferential equation in the complex plane $\mathbb{C}$

$$\bar{\partial}(\chi(\lambda) - \eta(\lambda)) + (\hat{S}\chi)(\lambda) = 0. \tag{2.1}$$

Here $\hat{S}$ is an integral operator

$$(\hat{S}\phi)(\lambda) = \int_{\mathbb{C}} S(\lambda, \lambda')\phi(\lambda') \, d\lambda' \wedge d\bar{\lambda}'$$

and the given rational function $\eta(\lambda)$ is called the normalization; it is assumed that

$$\chi(\lambda) - \eta(\lambda) \to 0 \qquad \text{for} \quad |\lambda| \to \infty.$$

Due to the generalized Cauchy formula the solution of the nonlocal $\bar{\partial}$-problem (2.1) can be expressed in terms of the solution of the equation

$$\chi(\lambda) + \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{(\hat{S}\chi)(\lambda')}{\lambda' - \lambda} \, d\lambda' \wedge d\bar{\lambda}' = \eta(\lambda), \tag{2.2}$$

which can be put in the form of the Fredholm integral equation of the second kind

$$\chi(\lambda) = \eta(\lambda) - \int_{\mathbb{C}} K(\lambda, \lambda'')\chi(\lambda'')d\lambda'' \wedge d\bar{\lambda}'', \tag{2.3}$$

with the kernel

$$K(\lambda, \lambda'') = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{S(\lambda', \lambda'')}{\lambda' - \lambda} d\lambda' \wedge d\bar{\lambda}'. \tag{2.4}$$

Remark. In this paper we always assume that the kernel S in the nonlocal $\bar{\partial}$-problem is such that the Fredholm equation (2.2) is uniquely solvable. Then by the Fredholm alternative the homogenous equation with $\eta = 0$ has only the trivial solution.

We assume that the kernel S in the nonlocal $\bar{\partial}$-problem depends on additional parameters. To get the $\bar{\partial}$-dressing method of construction of solutions fo the Darboux equations we introduce the following simple dependence of the kernel S on the variables u_i, $i = 1, \ldots, N$, via

$$S(\lambda', \lambda'', u) = g(\lambda', u)^{-1} S_0(\lambda', \lambda'')g(\lambda'', u), \tag{2.5}$$

where

$$g(\lambda', u) = \exp \sum_{i=1}^{N} u_i A_i(\lambda'), \qquad A_i(\lambda') = \frac{c_i}{\lambda' - \lambda_i} \tag{2.6}$$

the c_i are nonzero constants and $\lambda_i \in \mathbb{C}$ are points of the complex plane. Moreover we assume that $\lambda_i \neq \lambda_j$ for different i and j.

Directly one can verify the following result.

Lemma 1. *The evolution (2.5) of the kernel S implies that the kernel K of the integral equation (2.3) is subject to the equation*

$$\partial_i K(\lambda, \lambda'', u) = A_i(\lambda)K(\lambda_i, \lambda'', u) + A_i(\lambda'')K(\lambda, \lambda'', u) - A_i(\lambda)K(\lambda, \lambda'', u). \tag{2.7}$$

In consequence of the above formula (2.7) we obtain the following useful and crucial for the $\bar{\partial}$-dressing method result.

Lemma 2. *When $\chi(\lambda, u)$ is the unique, by assumption, solution of the $\bar{\partial}$-problem (2.1) with the kernel S evolving according to (2.5) and with normalization given by $\eta(\lambda, u)$, then the function*

$$\partial_i \chi(\lambda, u) + A_i(\lambda)\chi(\lambda, u)$$

is the solution of the same $\bar{\partial}$-problem but with the new normalization

$$\partial_i \eta(\lambda, u) + A_i(\lambda)\eta(\lambda, u) + A_i(\lambda)(\chi(\lambda_i, u) - \eta(\lambda_i, u)).$$

The above Lemma leads to the following Theorem, which gives the system of linear problems for the Darboux equation.

Theorem 1. *Given solution $\chi(\lambda, u)$ of the the $\bar{\partial}$-problem (2.1) and (2.5) with the normalization $\eta(\lambda, u) = 1$, then the function*

$$\psi(\lambda, u) = \chi(\lambda, u)g(\lambda, u)$$

satisfies with respect to variables u_i the following system of Laplace equations

$$\partial_i \partial_j \psi(\lambda, u) = a_{ij}(u)\partial_i \psi(\lambda, u) + a_{ji}(u)\partial_j \psi(\lambda, u), \quad i \neq j \tag{2.8}$$

with the coefficients

$$a_{ij}(u) = \frac{\partial_j \chi(\lambda_i, u)}{\chi(\lambda_i, u)} + A_j(\lambda_i). \tag{2.9}$$

Proof. The idea of the proof is standard within the $\bar{\partial}$-dressing method approach. One collects solutions of the $\bar{\partial}$-problem (2.1) (or, equivalently, the integral equation (2.3)) to obtain a new solution with the vanishing normalization. Then by the Fredholm alternative such a solution must be identically zero. ∎

Define the Lamé coefficients, $h_i(u)$, by

$$h_i(u) = \chi(\lambda_i, u)g_i(\lambda_i, u),$$

where

$$g_i(\lambda, u) = \exp \sum_{j=1, j \neq i}^{N} u_j A_j(\lambda);$$

equivalently they are the nonsingular parts of the function $\psi(\lambda, u)$ in the points λ_i,

$$h_i(u) = \lim_{\lambda \to \lambda_i} \left[\psi(\lambda, u) \exp(-u_i A_i(\lambda)) \right]. \tag{2.10}$$

Then the coefficients $a_{ij}(u)$ of the Laplace equations (2.8) are

$$a_{ij}(u) = \partial_j \log h_i(u) \qquad i \neq j,$$

and the compatibility condition of (2.8) is the Darboux system (1.1). Alternatively the Darboux system can be obtained evaluating equation (2.8) in the points λ_k, $k \neq i, j$ (multiplying it first by $\exp(-u_k A_k(\lambda))$ to remove the singularity, like in equation (2.10)).

Remark. To obtain real solutions one needs special symmetry properties of the kernel S with respect to the complex conjugation,

$$S(\bar{\lambda}, \lambda, \bar{\lambda}', \lambda') = \overline{S(\lambda, \bar{\lambda}, \lambda', \bar{\lambda}')},$$

and reality of the points, $\lambda_i \in \mathbb{R}$, and the parameters, $c_i \in \mathbb{R}$, $i = 1, \ldots, N$, which imply that

$$\psi(\bar{\lambda}, \lambda, u) = \overline{\psi(\lambda, \bar{\lambda}, u)} \qquad \text{and} \quad h_i(u) \in \mathbb{R}.$$

The following result allows one to give the $\bar{\partial}$-method of construction of solutions of the Darboux system (1.3) written in terms of the rotation coefficients. Its proof is analogous to the proof of Theorem 1.

Theorem 2. *Let $\chi_i(\lambda, u)$, $i = 1, \ldots, N$, be solutions of the $\bar{\partial}$-problem (2.1) and (2.5) with the normalizations*

$$\eta_i(\lambda, u) = A_i(\lambda)g_i(\lambda_i, u)^{-1}, \tag{2.11}$$

then the functions

$$\psi_i(\lambda, u) = \chi_i(\lambda, u)g(\lambda, u)$$

satisfy equations

$$\partial_i \psi(\lambda, u) = h_i(u)\psi_i(\lambda, u). \tag{2.12}$$

Moreover the functions $\psi_i(\lambda, u)$ satisfy the linear system

$$\partial_j \psi_i(\lambda, u) = \beta_{ij}(u)\psi_j(\lambda, u), \quad j \neq i, \tag{2.13}$$

with the rotation coefficients

$$\beta_{ij}(u) = \chi_i(\lambda_j, u)g_j(\lambda_j, u) = \lim_{\lambda \to \lambda_j} \left[\psi_i(\lambda, u) \, \exp(-u_j A_j(\lambda)) \right]. \tag{2.14}$$

Evaluating equations (2.12) in the points λ_j, $j \neq i$, (removing first the singularity) we obtain equations (1.2). The compatibility of the linear system (2.13) gives the Darboux equations (1.3) in terms of the rotation coefficients. Alternatively equations (1.3) can be obtained evaluating equations (2.14) in the points λ_k, $k \neq i, j$.

3   The first potentials and the τ−function

To give the meaning of the τ-function within the $\bar{\partial}$-dressing method we firstly present the meaning of the potentials β_{ii} defined by equations (1.4).

Proposition 1. *Within the $\bar{\partial}$-dressing method the potential $\beta_{ii}(u)$ can be identified with the nonsingular part of the function ψ_i at the point λ_i*

$$\beta_{ii}(u) = \lim_{\lambda \to \lambda_i} \left(\psi_i(\lambda, u) \exp(-u_i A_i(\lambda)) - A_i(\lambda) \right). \tag{3.1}$$

Proof. Multiplication of both sides of equation (2.13) by $g_i(\lambda, u)^{-1}$ and evaluation of the result in the limit $\lambda \to \lambda_i$ give

$$\lim_{\lambda \to \lambda_i} \partial_j \left(\psi_i(\lambda, u) \exp(-u_i A_i(\lambda)) \right) = \beta_{ij}(u)\beta_{ji}(u).$$

The expression in brackets in the LHS of the above equation is singular at λ_i. Up to a constant term, which vanishes during differentiation, it agrees with

$$\psi_i(\lambda, u) \exp(-u_i A_i(\lambda)) - A_i(\lambda),$$

which is finite at λ_i due to the normalization condition (2.11). After this replacement we can exchange differentiation with taking the limit. ∎

Finally we give the meaning to the next potential, the τ-function, which is related with potentials $\beta_{ii}(u)$ by equations (1.5).

Theorem 3. *Within the $\bar{\partial}$-dressing method the τ-function of conjugate nets can be identified with the Fredholm determinant of the integral equation* (2.3) *which inverts the nonlocal $\bar{\partial}$-problem* (2.1) *with kernel evolving according to equation* (2.5).

Before proving this theorem we first show the following Lemma; we use here the notation of the Appendix.

Lemma 3.

$$
\partial_i K \begin{pmatrix} z_1 & \cdots & z_m \\ z_1 & \cdots & z_m \end{pmatrix} u \Big) = \sum_{\ell=1}^{m} A_i(z_\ell) K \begin{pmatrix} \lambda_i & z_1 & \cdots & \check{z}_\ell & \cdots & z_m \\ z_\ell & z_1 & \cdots & \check{z}_\ell & \cdots & z_m \end{pmatrix} u \Big),
$$

where the symbol $\check{z}_\ell$ means that z_ℓ should be removed from the sequence.

Proof. Denote by π_m the set of all permutations of $\{1, \ldots, m\}$. Differentiation of the expression

$$
K \begin{pmatrix} z_1 & \cdots & z_m \\ z_1 & \cdots & z_m \end{pmatrix} u \Big) = \sum_{\sigma \in \pi_m} \operatorname{sgn}\sigma \, K(z_1, z_{\sigma(1)}, u) \cdot \ldots \cdot K(z_m, z_{\sigma(m)}, u)
$$

with respect to u_i gives on application of equation (2.7)

$$
\partial_i K \begin{pmatrix} z_1 & \cdots & z_m \\ z_1 & \cdots & z_m \end{pmatrix} u \Big) = \sum_{\ell=1}^{m} \sum_{\sigma \in \pi_m} \operatorname{sgn}\sigma \, K(z_1, z_{\sigma(1)}, u) \cdot \ldots
$$
$$
\ldots \cdot A_i(z_\ell) K(\lambda_i, z_{\sigma(\ell)}, u) \cdot \ldots \cdot K(z_m, z_{\sigma(m)}, u) \, ,
$$

where we have used also the following elementary formula valid for any permutation $\sigma \in \pi_m$

$$
\sum_{\ell=1}^{m} A_i(z_\ell) = \sum_{\ell=1}^{m} A_i(z_{\sigma(\ell)}).
$$

After application of an even number of transpositions in any of the m determinants we obtain the statement of the Lemma. ∎

Proof of Theorem 3. Using Lemma 3 we can derive the following formula for ith derivative of the Fredholm determinant $D_F(u)$

$$
\partial_i D_F(u) = \sum_{m=1}^{\infty} \frac{1}{(m-1)!} \int_{\mathbb{C}^m} A_i(\lambda') K \begin{pmatrix} \lambda_i & z_1 & \cdots & z_{m-1} \\ \lambda' & z_1 & \cdots & z_{m-1} \end{pmatrix} u \Big) d\lambda' \wedge d\bar{\lambda}' \ldots dz_{m-1} \wedge d\bar{z}_{m-1} \, .
$$

Comparing this with

$$
D_F(\lambda_i, \lambda', u) = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\mathbb{C}^m} K \begin{pmatrix} \lambda_i & z_1 & \cdots & z_m \\ \lambda' & z_1 & \cdots & z_m \end{pmatrix} u \Big) dz_1 \wedge d\bar{z}_1 \ldots dz_m \wedge d\bar{z}_m
$$

we obtain that

$$\partial_i D_F(u) = \int_{\mathbb{C}} D_F(\lambda_i, \lambda', u) A_i(\lambda') d\lambda' \wedge d\bar{\lambda}'.$$

Note that the solution of the Fredholm equation (2.3) with normalization $A_i(\lambda)$ is given, due to (2.11) and (5.4), by

$$\chi_i(\lambda, u) g_i(\lambda_i, u) = A_i(\lambda) - \int_{\mathbb{C}} \frac{D_F(\lambda, \lambda', u)}{D_F(u)} A_i(\lambda') d\lambda' \wedge d\bar{\lambda}' .$$

In the limit $\lambda \to \lambda_i$ (compare with (3.1)) we obtain therefore

$$\partial_i \log D_F(u) = -\beta_{ii}(u) ,$$

which allows via equation (1.5) for identification of the τ-function with the Fredholm determinant $D_F(u)$. ∎

4   Conclusion and remarks

We have shown that within the $\bar{\partial}$-dresing method the τ-function of conjugate nets can be identified with the Fredholm determinant inverting the nonlocal $\bar{\partial}$-problem. In fact this result can be extended to the full N-component KP hierarchy. To justify this opinion we should:

1. incorporate higher times of the hierarchy into the scheme and

2. give the meaning to the full set of τ-functions labelled by the root lattice vectors of the A_{N-1} root system.

The first task can be done by using the idea of [4], where the KP equation was obtained from the Darboux system (the scalar basic set of equations) for $N = 3$. In the same way the higher times can be included into evolution of the kernel. The second problem can be solved by combination of the result of [8] on the relation of the Schlesinger transformations on the A_{N-1} root lattice with the Laplace transformations of conjugate nets, with the construction of such Laplace transformations within the $\bar{\partial}$-dresing method as given in [7]. The details will be presented elsewhere.

5   Appendix: Elements of the Fredholm theory

We recall in this Appendix some standard facts (see, for example, [18]) from the theory of Fredholm integral equations which we use in the paper.

Consider the Fredholm equation of the second kind

$$f(x) = g(x) - \int_{\Omega} K(x, y) f(y) dy , \tag{5.1}$$

where $K(x, y)$ is the kernel of the integral operator and $g(x)$ is a given function. The Fredholm determinant D_F is defined by the series

$$D_F = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \int_{\Omega^m} K \begin{pmatrix} x_1 & x_2 & \dots & x_m \\ x_1 & x_2 & \dots & x_m \end{pmatrix} dx_1 \dots dx_m , \tag{5.2}$$

where

$$K \begin{pmatrix} x_1 & x_2 & \ldots & x_m \\ y_1 & y_2 & \ldots & y_m \end{pmatrix} = \det \begin{pmatrix} K(x_1,y_1) & K(x_1,y_2) & \ldots & K(x_1,y_m) \\ \vdots & \vdots & & \vdots \\ K(x_m,y_1) & K(x_m,y_2) & \ldots & K(x_m,y_m) \end{pmatrix},$$

while the Fredholm minor is defined by the series

$$D_F(x,y) = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Omega^m} K \begin{pmatrix} x & x_1 & x_2 & \ldots & x_m \\ y & x_1 & x_2 & \ldots & x_m \end{pmatrix} dx_1 \ldots dx_m . \tag{5.3}$$

For a nonvanishing Fredholm determinant D_F the solution of the integral equation (5.1) is unique and can be written in the form

$$f(x) = g(x) - \int_{\Omega} \frac{D_F(x,y)}{D_F} g(y) dy . \tag{5.4}$$

Acknowledgments

References

[1] Ablowitz M J, Bar Yaacov D and Fokas A S, On the inverse scattering problem for the Kadomtsev–Petviashvili equation, *Stud. Appl. Math.* **69** (1983), 135–143.

[2] Bianchi L, Lezioni di geometria differenziale, Zanichelli, Bologna, 1924.

[3] Bogdanov L V and Konopelchenko B G, Analytic-bilinear approach to integrable hierarchies II. Multicomponent KP and 2D Toda lattice hierarchies, *J. Math. Phys.* **39** (1998), 4701–4728.

[4] Bogdanov L V and Manakov S V, The nonlocal $\bar{\partial}$-problem and (2+1)-dimensional soliton equations, *J. Phys. A: Math. Gen.* **21** (1988), L537–L544.

[5] Darboux G, Leçons sur les systémes orthogonaux et les coordonnées curvilignes, Gauthier – Villars, Paris, 1910.

[6] Date E, Kashiwara M, Jimbo M, and Miwa T, Transformation groups for soliton equations, in Nonlinear Integrable Systems — Classical Theory and Quantum Theory, Proc. of RIMS Symposium, Editors: Jimbo M and Miwa T, World Scientific, Singapore, 1983, 39–119.

[7] Doliwa A, Santini P M and Mañas M, Transformations of quadrilateral lattices, *J. Math. Phys.* **41** (2000), 944–990.

[8] Doliwa A, Mañas M, Martínez Alonso L, Medina E and Santini P M, Multicomponent KP hierarchy and classical transformations of conjugate nets, *J. Phys. A: Math. Gen.* **32** (1999), 1197–1216.

[9] Eisenhart L P, Transformations of Surfaces, Princeton University Press, Princeton, 1923.

[10] Grinevich P G and Orlov A Yu, Flag spaces in KP theory and Virasoro action on det $\bar{\partial}_j$ and Segal–Wilson τ-function, in Problems of Modern Quantum Field Theory, Springer-Verlag, Berlin, 1989, 86–106.

[11] Kac V G and van de Leur J, The n-component KP hierarchy and representation theory, *J. Math. Phys.* **44** (2003), 3245–3293.

[12] Lamé G, Leçons sur les coordonnées curvilignes et leurs diverses applications, Mallet–Bachalier, Paris, 1859.

[13] Mañas M, Martínez Alonso L and Medina E, Dressing methods for geometric nets: I. Conjugate nets, *J. Phys. A: Math. Gen.* **33** (2000) 2871–2894.

[14] van Moerbeke P, Integrable lattices: random matrices and random permutations, in Random Matrices and Their Applications, Math. Sci. Res. Inst. Publ. **40**, Cambridge Univ. Press, Cambridge, 2001, 321–406.

[15] Pöppe Ch and Sattinger D H, Fredholm determinants and the τ function for the Kadomtsev–Petviashvili hierarchy, *Publ. RIMS, Kyoto Univ.* **24** (1988), 505–538.

[16] Sato M, Soliton equations as dynamical systems on infinite dimensional Grassman manifolds, *RIMS Kokyuroku* **439** (1981), 30–46.

[17] Segal G and Wilson G, Loop groups and equations of KdV type, *Inst. Hautes Études Sci. Publ. Math.* **61** (1985), 5–65.

[18] Smithies F, Integral Equations, Cambridge Univ. Press, Cambridge, 1965.

[19] Zakharov V E and Manakov S V, Construction of multidimensional nonlinear integrable systems and their solutions, *Funct. Anal. Appl.* **19** (1985), 11–25.

[20] Zakharov V E and Shabat A B, A scheme for integrating the nonlinar equations of mathematical physics by the method of the inverse scattering problem, *Funct. Anal. Appl.* **8** (1974), 226-235.

[21] Zakharov V E, Manakov S V, Novikov S P and Pitaevskii L P, Theory of Solitons — Inverse Scattering Metod, Nauka, Moscow, 1980.