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# Vortex Trains in Super-Alfvénic Magnetogasdynamics. Application of Reciprocal-Bäcklund Transformations

*C ROGERS and W K SCHIEF*

*Australian Research Council Centre of Excellence for Mathematics and Statistics of Complex Systems*
*School of Mathematics, University of New South Wales*
*UNSW, Sydney NSW 2052, Australia*
*E-mail: colinr@maths.unsw.edu.au, schief@maths.unsw.edu.au*

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**Abstract**

A multi-parameter class of reciprocal transformations is coupled with the action of a Bäcklund transformation to construct periodic solutions of breather-type in plane, aligned, super-Alfvénic magnetogasdynamics. The constitutive law adopts a generalised Kármán-Tsien form.

## 1 Introduction

Reciprocal transformations have a long history and a diversity of applications. Haar [11], in 1928, in a paper concerned with adjoint variational problems, set down a reciprocal-type invariance of a plane, potential gasdynamic system. Bateman [2] in subsequent work on the lift and drag functions in gasdynamics introduced a cognate class of invariant transformations that have come to be termed reciprocal relations. The latter were elaborated upon by Tsien [32], Power and Smith [19].

Invariance of nonlinear gasdynamic and magnetogasdynamic systems under reciprocal-type transformations has been extensively studied [20]–[24]. Reciprocal transformations have also been applied to provide exact solutions to both stationary and moving boundary value problems of practical interest, in particular, in soil mechanics and nonlinear heat conduction. In recent work, reciprocal transformations have been applied in the analysis of a nonlinear moving boundary problem that arises in the migration of methacrylate during wood saturation processes [9].

The importance of reciprocal transformations in the context of hydrodynamic systems has been established in a series of papers by Ferapontov [3]–[6]. In geometric terms, the classification of weakly nonlinear Hamiltonian systems of hydrodynamic type up to reciprocal transformations has been shown to be completely equivalent to the classification of Dupin hypersurfaces up to Lie sphere transformations [7, 8]. Connections between the classical theory of congruences and systems of conservation laws of the Temple class [31] invariant under reciprocal transformations have also been uncovered [1].

Reciprocal transformations have been shown to play an interesting role in soliton theory and, in particular, provide a connection between the Ablowitz-Kaup-Newell-Segur (AKNS) and Wadati-Konno-Ichikawa (WKI) inverse scattering schemes [29] and their constituent integrable equations. Thus, they connect potential KdV and loop soliton hierarchies and the Heisenberg spin equation with a base member of the WKI system [27]. Likewise the Camassa-Holm equation of water wave theory is connected via a reciprocal transformation to the first negative flow of the KdV hierarchy [12]. Invariance under reciprocal transformations has been investigated in [14, 15]. In particular, invariance under reciprocal transformations of the Dym hierarchy induce unadorned auto-Bäcklund transformations in associated integrable hierarchies [27]. The combined action of reciprocal and gauge transformations on $2+1$-dimensional integrable hierarchies has been investigated by Oevel and Rogers [17]. In geometric terms, reciprocal transformations provide a natural change of coordinate system on soliton surfaces [27].

Here, we return to reciprocal transformations in their original gasdynamic context and show how they may be used to generate breather-type solutions in aligned magnetogasdynamics. This procedure involves a novel composition of reciprocal and Bäcklund transformations analogous to that recently exploited in subsonic gasdynamics in the context of the Tzitzéica equation [28].

## 2 The Magnetogasdynamic System

The governing equations of steady, non-dissipative magnetogasdynamics are

$$\begin{aligned}
&\operatorname{div}(\rho \boldsymbol{q}) = 0,\\
&\rho(\boldsymbol{q}\cdot\nabla)\boldsymbol{q} - \mu(\boldsymbol{H}\cdot\nabla)\boldsymbol{H} + \nabla\Pi = \boldsymbol{0},\\
&\operatorname{div}\boldsymbol{H} = 0,\\
&\operatorname{curl}(\boldsymbol{q}\times\boldsymbol{H}) = \boldsymbol{0},\\
&\boldsymbol{q}\cdot\nabla\eta = 0.
\end{aligned} \tag{2.1}$$

together with an appropriate constitutive law

$$\Phi(p,\rho,\eta) = 0, \quad \left.\frac{\partial p}{\partial \rho}\right|_{\eta} > 0. \tag{2.2}$$

Here, $\Pi = p + \frac{1}{2}\mu H^2$ is the total pressure, $\mu$ is the magnetic permeability (assumed constant), while $\boldsymbol{q}$ and $\boldsymbol{H}$ denote, respectively, the velocity and magnetic fields. In the usual notation, $p, \rho$ and $\eta$ designate, in turn, the gas pressure, density and entropy.

### 2.1 Aligned Magnetogasdynamics

In the following, attention is restricted to aligned magnetogasdynamics [10, 13]. Thus,

$$\boldsymbol{q} = q\boldsymbol{t}, \quad \boldsymbol{H} = H\boldsymbol{t}, \tag{2.3}$$

where $\boldsymbol{t}$ is the shared unit tangent and $q, H$ are the gas speed and magnetic intensity respectively. Insertion into the magnetogasdynamic system (2.1) produces the intrinsic decomposition

$$
\begin{aligned}
&\frac{\delta}{\delta s}(\rho q) + \rho q \operatorname{div} \boldsymbol{t} = 0, \\
&\rho q \frac{\delta q}{\delta s} - \mu H \frac{\delta H}{\delta s} + \frac{\delta \Pi}{\delta s} = 0, \\
&\rho q^2 \kappa - \mu H^2 \kappa + \frac{\delta \Pi}{\delta n} = 0, \\
&\frac{\delta \Pi}{\delta b} = 0, \\
&\frac{\delta H}{\delta s} + H \operatorname{div} \boldsymbol{t} = 0, \\
&\frac{\delta \eta}{\delta s} = 0,
\end{aligned}
\tag{2.4}
$$

where $\delta/\delta s = \boldsymbol{t} \cdot \nabla$, $\delta/\delta n = \boldsymbol{n} \cdot \nabla$, $\delta/\delta b = \boldsymbol{b} \cdot \nabla$ and $\boldsymbol{n}, \boldsymbol{b}$ are the principal normal and binormal to a generic streamline with curvature $\kappa$.

Combination of the continuity equation $(2.4)_1$, and the magnetic induction equation $(2.4)_3$ shows that

$$
H = k \rho q, \tag{2.5}
$$

where $\delta \kappa/\delta s = 0$. Insertion of (2.5) into $(2.4)_{2,3}$ yields

$$
\begin{aligned}
&\rho q \left[ \frac{\delta q}{\delta s} - \mu k^2 \frac{\delta}{\delta s}(\rho q) \right] + \frac{\delta \Pi}{\delta s} = 0, \\
&\rho q^2 (1 - \mu k^2 \rho) \kappa + \frac{\delta \Pi}{\delta n} = 0,
\end{aligned}
$$

whence if the quantities $q^*, \rho^*, \kappa^*$ and $p^*$ are introduced according to

$$
\rho q \left[ \frac{\delta q}{\delta s} - \mu k^2 \frac{\delta}{\delta s}(\rho q) \right] = \rho^* q^* \frac{\delta q^*}{\delta s}, \tag{2.6}
$$

$$
\rho q^2 (1 - \mu k^2 \rho) \kappa = \rho^* q^{*2} \kappa^*, \tag{2.7}
$$

$$
\begin{aligned}
&\Pi = p^*, \\
&\kappa = \kappa^*,
\end{aligned}
$$

then, on integration, it is seen that

$$
q^* = m q (1 - \mu k^2 \rho), \quad \rho^* = \frac{\rho}{m^2 (1 - \mu k^2 \rho)}, \tag{2.8}
$$

where $\delta m/\delta s = 0$. Moreover, $\rho^* q^* = \rho q/m$, whence the continuity equation $(2.4)_1$ becomes

$$
\frac{\delta}{\delta s}(\rho^* q^*) + \rho^* q^* \operatorname{div} \boldsymbol{t} = 0. \tag{2.9}
$$

Thus, in summary, under the transformation

$$\begin{aligned} &\boldsymbol{q}^* = m\boldsymbol{q}(1-\mu k^2\rho), \quad \rho^* = \frac{\rho}{m^2(1-\mu k^2\rho)}\,, \\ &p^* = \Pi, \quad \eta^* = \eta \end{aligned} \tag{2.10}$$

where $\boldsymbol{q}\cdot\nabla m = 0$, the governing equations of aligned magnetogasdynamics reduce to those of conventional gasdynamics, namely

$$\begin{aligned} &\operatorname{div}(\rho^*\boldsymbol{q}^*) = 0 \\ &\rho^*(\boldsymbol{q}^*\cdot\nabla)\boldsymbol{q}^* + \nabla p^* = \mathbf{0} \\ &\boldsymbol{q}^*\cdot\nabla\eta^* = 0. \end{aligned} \tag{2.11}$$

The intrusion of the parameter $m$ corresponds to the routine application of a substitution principle [18]. The basic reduction with $m = 1$ was originally obtained independently by Grad [10] and Iur'ev [13].

It is noted that, if it is required that the gasdynamic reduction be real then the necessary condition $\rho^* > 0$ shows that

$$1 - \mu k^2\rho > 0. \tag{2.12}$$

The Alfvén number $\mathcal{A}$ of the magnetogasdynamic flow is given by

$$\mathcal{A}^2 = q^2/b^2 = 1/\mu k^2\rho, \tag{2.13}$$

where $b = B/(\mu\rho)^{1/2}$ is the Alfvén speed and $B = \mu H$. Accordingly, it is seen that the condition (2.12) requires that the conducting flow be super-Alfvénic, that is $\mathcal{A}^2 > 1$.

## 2.2 A Bernoulli Integral

The intrinsic equation of motion $(2.4)_2$ admits, in view of the isentropic condition $(2.4)_6$, the Bernoulli integral

$$\frac{q^2}{2} + \int_0^p \frac{d\sigma}{\rho(\sigma,\eta)} = \int_0^{p_0} \frac{d\sigma}{\rho(\sigma,\eta)}, \tag{2.14}$$

where $p_0$ is the stagnation pressure which is constant along streamlines. It proves convenient to introduce a function $\mathcal{H}$ according to

$$\mathcal{H} = \int_{p_0}^p \frac{d\sigma}{\rho(\sigma,\eta)} = h - h_0 \tag{2.15}$$

where

$$h = \int_0^p \frac{d\sigma}{\rho(\sigma,\eta)}, \quad h_0 = \int_0^{p_0} \frac{d\sigma}{\rho(\sigma,\eta)} \tag{2.16}$$

are, respectively, the specific enthalpy and total energy. The Bernoulli integral then admits the compact representation

$$q^2 + 2\mathcal{H}(p,\eta) = 0. \tag{2.17}$$

# 3 Planar Magnetogasdynamics

In spatial magnetogasdynamics, the Faraday equation $(2.1)_4$ shows that

$$\boldsymbol{q} \times \boldsymbol{H} = \nabla\Phi, \tag{3.1}$$

whence there exist what are termed Maxwellian surfaces $\Phi = \text{const}$ containing the streamlines and magnetic lines. If the Maxwellian surfaces coincide with the total pressure surfaces $\Pi = \text{const}$ then it has been established in [30] that the governing magnetohydrodynamic equations reduce to the integrable Pohlmeyer-Lund-Regge system, subject to a volume preserving constraint. This generalises a result originally obtained in [26] linking the governing equations of non-conducting spatial gasdynamics to the integrable Heisenberg spin equation, subject to a geometric constraint.

In plane magnetogasdynamics, the condition (3.1) reduces to

$$\boldsymbol{q} \times \boldsymbol{H} = a\boldsymbol{k} \tag{3.2}$$

where $a$ is constant. This relation shows that if $\boldsymbol{q}$ and $\boldsymbol{H}$ are aligned at one point then they are everywhere aligned. This privileged situation has led to extensive study of plane, aligned magnetogasdynamics, most notably through the hodograph system that may be derived in the case of conducting flow with uniform stagnation enthalpy [13, 25].

In the two-dimensional case, the continuity equation is conveniently embodied in the relation

$$d\psi = -\rho v dx + \rho u dy, \tag{3.3}$$

where $\psi$ is a streamfunction and the isentropic condition then requires that $\eta = \eta(\psi)$, whence $\mathcal{H} = \mathcal{H}(p, \psi)$. The Bernoulli integral (2.13) now shows, in particular, that

$$uu_x + vv_x + \mathcal{H}_p p_x + \mathcal{H}_\psi \psi_x = 0 \tag{3.4}$$

which together with the equation of motion

$$\rho(uu_x + vu_y) + \mu(H_{2x} - H_{1y}) + p_x = 0 \tag{3.5}$$

produces, on elimination of $p_x$ and use of the relation $\mathcal{H}_p = \rho^{-1}$,

$$\Omega = v_x - u_y = \rho\mathcal{H}_\psi - \mu k[(k\psi_x)_x + (k\psi_y)_y] \tag{3.6}$$

where $\Omega$ is the vorticity magnitude. The latter relation together with the Bernoulli integral now provide a compact encapsulation of planar, aligned magnetogasdynamics via the pair of relations

$$\begin{aligned} &\left(\frac{\psi_x}{\rho}\right)_x + \left(\frac{\psi_y}{\rho}\right)_y - \mu k([k\psi_x)_x + (k\psi_y)_y] = -\rho\mathcal{H}_\psi \ , \\ &\left(\frac{\psi_x}{\rho}\right)^2 + \left(\frac{\psi_y}{\rho}\right)^2 = -2\mathcal{H} \ , \end{aligned} \tag{3.7}$$

augmented by the equation of state (2.2) which determines $\mathcal{H}(p, \psi)$.

In the magnetohydrodynamic reduction with $\rho = 1$, if $k$ is constant then the pair of relations (3.7) reduce to

$$\nabla^2\psi = \psi_{xx} + \psi_{yy} = \frac{\mathcal{A}^2\mathcal{H}_\psi}{1-\mathcal{A}^2}, \tag{3.8}$$

together with the Bernoulli integral

$$\psi_x^2 + \psi_y^2 = 2(\mathbb{B}(\psi) - p), \tag{3.9}$$

where $\mathbb{B}(\psi)$ is the stagnation pressure, so that

$$\mathcal{H}(p,\psi) = p - \mathbb{B}(\psi). \tag{3.10}$$

Thus, (3.8) adopts the form

$$\psi_{xx} + \psi_{yy} = \frac{\mathbb{B}_\psi}{1-\mu k^2} \tag{3.11}$$

and the vorticity magnitude is given by

$$\Omega = \frac{\mathbb{B}_\psi}{1-\mu k^2}. \tag{3.12}$$

The well-known vorticity equation of planar gasdynamics is retrieved via (3.8) in the limit as the Alfvén number $\mathcal{A} \to \infty$ or, equivalently, $k \to 0$.

# 4 The Reciprocal Relations

It may be established that the governing equations of plane steady gasdynamics are invariant, up to the equation of state under the 4-parameter class of reciprocal transformations [19]

$$\begin{aligned} &u^{*\prime} = \frac{\beta_1 u^*}{p^* + \beta_2}, \quad v^{*\prime} = \frac{\beta_1 v^*}{p^* + \beta_2}, \\ &p^{*\prime} = \beta_4 - \frac{\beta_1^2\beta_3}{p^* + \beta_2}, \quad \rho^{*\prime} = \frac{\beta_3\rho^*(p^* + \beta_2)}{p^* + \beta_2 + \rho^* q^{*2}}, \end{aligned} \tag{4.1}$$

together with the change of independent variables $(x, y) \to (x', y')$, where

$$\begin{aligned} &dx' = \beta_1^{-1}[(p^* + \beta_2 + \rho^* v^{*2})dx - \rho^* u^* v^* dy], \\ &dy' = \beta_1^{-1}[-\rho^* u^* v^* dx + (p^* + \beta_2 + \rho^* u^{*2})dy], \end{aligned} \tag{4.2}$$

subject to the requirement that $0 < |J(x', y'; x, y)| < \infty$ so that

$$0 < |(p^* + \beta_2)(p^* + \beta_2 + \rho^* q^{*2})| < \infty. \tag{4.3}$$

Importantly, one may verify that $\psi^{*\prime} = \psi^*$, where the stream function $\psi^*$ is defined by the starred version of (3.3) and hence $\psi^* = \psi/m$. The reduction relations (2.10) invert to show that the $u, v, p, \rho$ and $\eta$ of the original aligned magnetogasdynamic system are given by

$$u = \frac{u^*}{m}(1+\mu m^2k^2\rho^*), \quad v = \frac{v^*}{m}(1+\mu m^2k^2\rho^*),$$
$$p = p^* - \frac{1}{2}\mu m^2k^2\rho^{*2}q^{*2}, \quad \rho = \frac{m^2\rho^*}{(1+\mu m^2k^2\rho^*)}, \quad \eta^* = \eta. \tag{4.4}$$

Thus, the aligned magnetogasdynamic system is invariant under the multi-parameter transformations

$$u' = \frac{u^{*\prime}}{m}(1+\mu m^2k^2\rho^{*\prime}), \quad v' = \frac{v^{*\prime}}{m}(1+\mu m^2k^2\rho^{*\prime}),$$
$$p' = p^{*\prime} - \frac{1}{2}\mu m^2k^2\rho^{*\prime}q^{*\prime 2}, \quad \rho' = \frac{m^2\rho^{*\prime}}{1+\mu m^2k^2\rho^{*\prime}}, \quad \eta' = \eta, \tag{4.5}$$

where

$$u^{*\prime} = \frac{\beta_1 mu(1-\mu k^2\rho)}{p+\frac{1}{2}\mu k^2\rho^2q^2+\beta_2}, \quad v^{*\prime} = \frac{\beta_1 mv(1-\mu k^2\rho)}{p+\frac{1}{2}\mu k^2\rho^2q^2+\beta_2},$$
$$p^{*\prime} = \beta_4 - \frac{\beta_1^2\beta_3}{p+\frac{1}{2}\mu k^2\rho^2q^2+\beta_2}, \quad \rho^{*\prime} = \frac{\beta_3\rho(p+\frac{1}{2}\mu k^2\rho^2q^2+\beta_2)}{p-\frac{1}{2}\mu k^2\rho^2q^2+\rho q^2+\beta_2} \tag{4.6}$$

augmented by the change of independent variables (4.2).

## 5 A Generalised Kármán-Tsien Gas Law

The aligned magnetohydrodynamic reduction with $\rho = 1$ has corresponding hydrodynamic reduction with

$$\rho^* = \frac{1}{m^2(1-\mu k^2)}. \tag{5.1}$$

The reciprocal pressure and density in this hydrodynamic reduction are given by

$$p^{*\prime} = \beta_4 - \frac{\beta_1^2\beta_3}{p^*+\beta_2}, \quad \rho^{*\prime} = \frac{\beta_3\rho^*(p^*+\beta_2)}{p^*+\rho^*q^{*2}+\beta_2}, \tag{5.2}$$

while the Bernoulli integral yields, by virtue of (5.1),

$$q^{*2} = 2[\mathbb{B}^*(\psi^*) - m^2(1-\mu k^2)p^*], \tag{5.3}$$

where

$$\mathbb{B}^*(\psi^*) = m^2(1-\mu k^2)\mathbb{B}(\psi). \tag{5.4}$$

Accordingly,

$$\rho^{*\prime} = \frac{\beta_3\rho^*(p^*+\beta_2)}{p^*+2\rho^*[\mathbb{B}^*-m^2(1-\mu k^2)p^*]+\beta_2} = \frac{\beta_3\rho^*(p^*+\beta_2)}{-p^*+\beta_2+2\rho^*\mathbb{B}^*} \tag{5.5}$$

and elimination of $p^*$ between (5.2) and (5.5) produces the reciprocal Kármán-Tsien law

$$p^{*\prime} = A^* + \frac{B^*}{\rho^{*\prime}}, \tag{5.6}$$

where

$$A^* = \beta_4 - \frac{\beta_1^2\beta_3}{2(\rho^*\mathbb{B}^* + \beta_2)}, \quad B^* = -\frac{\beta_1^2\beta_3^2\rho^*}{2(\rho^*\mathbb{B}^* + \beta_2)}. \tag{5.7}$$

The reciprocal constitutive law associated with the original magnetogasdynamic system is given by the expressions $(4.5)_{3,4}$ for the reciprocal pressure $p'$ and density $\rho'$ together with (3.7) and the associated Bernoulli integral

$$\frac{q^{*\prime 2}}{2} = \mathbb{B}^{*\prime} - \frac{p^{*\prime 2}}{2B^*} + \frac{A^* p^{*\prime}}{B^*}, \tag{5.8}$$

where it may be shown that

$$\mathbb{B}^{*\prime} = -\frac{\beta_1^2}{4\rho^*(\beta_2 + \rho^*\mathbb{B}^*)}. \tag{5.9}$$

Thus, the reciprocal constitutive law is given by

$$\begin{aligned} p' &= A^* + \frac{B^*}{\rho^{*\prime}} - \mu m^2 k^2 \rho^{*\prime 2}\left[\mathbb{B}^{*\prime} - \frac{p^{*\prime 2}}{2B^*} + A^*\frac{p^{*\prime}}{B^*}\right] \\ &= A^* + \frac{B^*}{\rho^{*\prime}} - \mu m^2 k^2 \rho^{*\prime 2}\left[\mathbb{B}^{*\prime} + \frac{1}{2B^*}\left(A^{*2} - \frac{B^{*2}}{\rho^{*\prime 2}}\right)\right], \end{aligned} \tag{5.10}$$

where

$$\rho^{*\prime} = \left[m^2\left(\frac{1}{\rho'} - \mu k^2\right)\right]^{-1}. \tag{5.11}$$

Elimination of $\rho^{*\prime}$ produces an explicit magneto-gas law of the form

$$p' = \alpha + \beta\left(\frac{1}{\rho'} - \mu k^2\right) + \frac{\gamma\mu k^2}{\left(\frac{1}{\rho'} - \mu k^2\right)^2}, \tag{5.12}$$

where $\alpha, \beta, \gamma$ are, in general, dependent upon the entropy. It is observed that this constitutive law may be written alternatively in terms of the reciprocal Alfvén number $\mathcal{A}'$ as

$$p' = \alpha + \beta\mu k^2(\mathcal{A}'^2 - 1) + \frac{\gamma}{\mu k^2(\mathcal{A}'^2 - 1)^2}. \tag{5.13}$$

Interestingly, the model equation of state (5.12) is precisely that originally obtained in [25] via systematic reduction by Loewner transformations of the hodograph system of aligned magnetogasdyamics to the Cauchy-Riemann system associated with elliptic régimes. This hodograph system was derived by Iur'ev [13] under the requirement of uniform stagnation enthalpy whence the associated gasdynamic system is necessarily irrotational, that is

$$u^*_y - v^*_x = 0. \tag{5.14}$$

The present analysis dispenses with this strong constraint.

Thus, in summary, it has been established that, subject to the generalised Kármán-Tsien gas law (5.12), the governing equations of planar, aligned magnetogasdyamics are reduced via the multi-parameter class of reciprocal transformations given by (4.5), (4.6) together with (4.2) to the seed magnetohydrodynamic system with $\rho = 1$. In particular, if $k$ is likewise set to be unity then reduction is obtained to the canonical equation

$$\psi_{xx} + \psi_{yy} = \Phi(\psi), \tag{5.15}$$

where

$$\Phi(\psi) = \frac{\mathbb{B}_\psi}{1 - \mu k^2}. \tag{5.16}$$

# 6 Properties of the Reciprocal Magnetogasdyamic System

The reduced reciprocal Mach number $\mathcal{M}^{*'}$ is given by

$$\mathcal{M}^{*\prime 2} = \frac{q^{*\prime 2}}{c^{*\prime 2}}, \tag{6.1}$$

where

$$q^{*\prime 2} = \frac{\beta_1^2 q^{*2}}{(p^* + \beta_2)^2} \tag{6.2}$$

and

$$c^{*\prime 2} = -\frac{\mathbb{B}^*}{\rho^{*\prime 2}}, \tag{6.3}$$

on use of the reduced Kármán-Tsien gas law (5.6). Thus,

$$c^{*\prime 2} = \frac{\beta_1^2 \beta_3^2 \rho^*}{2(\beta_2 + \rho^* \mathbb{B}^*)\rho^{*\prime 2}} = \frac{\beta_1^2(-p^* + \beta_2 + 2\rho^* \mathbb{B}^*)^2}{2\rho^*(\beta_2 + \rho^* \mathbb{B}^*)(p^* + \beta_2)^2}, \tag{6.4}$$

whence, if it is required that $\rho^* > 0$[1] then

$$\rho^* \mathbb{B}^* + \beta_2 > 0. \tag{6.5}$$

The requirement $\rho^* > 0$ has been seen to require the magnetogasdynamic flow to be super-Alfvénic. Use of the reciprocal relations (6.2) and (6.4) shows that

$$\begin{aligned} \mathcal{M}^{*\prime 2} &= \frac{q^{*\prime 2}}{c^{*\prime 2}} = 4\left(\mathbb{B}^* - \frac{p^*}{\rho^*}\right) \frac{(\beta_2 + \rho^* \mathbb{B}^*)\rho^*}{(-p^* + \beta_2 + 2\rho^* \mathbb{B}^*)^2} \\ &= 1 - \frac{(p^* + \beta_2)^2}{(-p^* + \beta_2 + 2\rho^* \mathbb{B}^*)^2} \\ &= 1 - \frac{\rho^{*\prime 2}}{\beta_3^2 \rho^{*2}} < 1 \end{aligned} \tag{6.6}$$

[1]If this requirement is relaxed then sub-Alfvénic magnetogasdynamic flows may be generated corresponding to non-physical reduced gasdynamics motions.

so that the reduced gasdynamic flow is necessarily subsonic.

Now, the reciprocal reduced Bernoulli integral implies that

$$q^{*\prime}\frac{\partial q^{*\prime}}{\partial \rho^{*\prime}}\bigg|_{\psi^{*\prime}}+\frac{1}{\rho^{*}}\frac{\partial p^{*\prime}}{\partial \rho^{*\prime}}\bigg|_{\psi^{*\prime}}=0,$$

whence the condition (6) yields

$$\frac{q^{*\prime 2}}{-\rho^{*\prime}q^{*\prime}\,\dfrac{\partial q^{*\prime}}{\partial \rho^{*\prime}}\bigg|_{\psi^{*\prime}}}<1. \tag{6.7}$$

Thus, since

$$q^{*\prime}=mq^{\prime}(1-\mu k^2\rho^{\prime}),\quad \rho^{*\prime}=\frac{\rho^{\prime}}{m^2(1-\mu k^2\rho^{\prime})},$$

it is seen, on insertion into (6.7) that

$$(1-\mathcal{A}^{\prime 2})(1-\mathcal{M}^{\prime 2})<0, \tag{6.8}$$

where $\mathcal{M}^{\prime}$ and $\mathcal{A}^{\prime}$ are the reciprocal Mach number and Alfvén number respectively. Hence, the reciprocal magnetogasdynamic flow régimes are either subsonic and super-Alfvénic or supersonic and sub-Alfvénic. It is remarked that, in [25], reduction of the planar, aligned magnetogasdyamic hodograph system to the Cauchy-Riemann system was obtained subject to the condition

$$\frac{(1-\mathcal{M}^2)(1-\mu k^2\rho)}{1-\mu k^2\rho(1-\mathcal{M}^2)}>0. \tag{6.9}$$

In the present more general context, it is seen that the condition (6.8) implies that both

$$(1-\mathcal{M}^{\prime 2})(1-\mu k^2\rho^{\prime})>0 \tag{6.10}$$

and

$$1-\mu k^2\rho^{\prime}(1-\mathcal{M}^{\prime 2})>1-\mathcal{M}^{\prime 2}-\mu k^2\rho^{\prime}(1-\mathcal{M}^{\prime 2})>0 \tag{6.11}$$

so that the condition (6.9) indeed applies for the reciprocal magnetogasdynamic flows.

In the sequel, attention is restricted to the case $1-\mu k^2\rho^{\prime}>0$ so that the reciprocal magnetogasdynamic motions, like the seed magnetohydrodynamic motions, are super-Alfvénic.

# 7 Generation of Vortex Trains in Super-Alfvénic Magnetogasdynamics

Here, we take as seed solution of the reduced gasdynamic system corresponding to $\rho=1$, the Mallier-Maslowe vortex train [16]

$$\psi^{*}=-2\tanh^{-1}\left(\frac{\epsilon\cos x}{\cosh \epsilon y}\right)=\ln\left(\frac{\cosh \epsilon y-\epsilon\cos x}{\cosh \epsilon y+\epsilon\cos x}\right) \tag{7.1}$$

of

$$\psi^*_{xx} + \psi^*_{yy} = -\frac{1-\epsilon^2}{2}\sinh 2\psi^* = \rho^{*2}\mathbb{B}^*_{\psi^*}. \tag{7.2}$$

Thus,

$$\mathbb{B}^*(\psi^*) = \frac{1}{\rho^{*2}}\left[K - \frac{1}{4}(1-\epsilon^2)\cosh 2\psi^*\right] < \frac{1}{\rho^{*2}}\left[K - \frac{1}{4}(1-\epsilon^2)\right], \tag{7.3}$$

where $\rho^* = 1/(1-\mu k^2)$.

In what follows, we set $m = 1$. The reduced gasdynamic variables are then given by

$$u^* = \frac{2\epsilon^2 \cos x \sinh \epsilon y}{\rho^*\Delta}, \quad v^* = -\frac{2\epsilon \sin x \cosh \epsilon y}{\rho^*\Delta}, \tag{7.4}$$

where

$$\Delta = \cosh^2 \epsilon y - \epsilon^2 \cos^2 x \tag{7.5}$$

and the Bernoulli integral provides the pressure distribution

$$\begin{aligned} p^* &= \rho^*\left[\mathbb{B}^*(\psi^*) - \frac{q^{*2}}{2}\right] \\ &= \frac{1}{\rho^*}\left[K - \frac{1}{4}(1-\epsilon^2) - \frac{2\epsilon^2}{\Delta^2}(\epsilon^2\cos^2 x \sinh^2 \epsilon y + \sin^2 x \cosh^2 \epsilon y)\right]. \end{aligned} \tag{7.6}$$

The reciprocal magnetogasdynamic variables are now given by the relations (4.5) together with

$$\begin{aligned} u^{*\prime} &= \frac{2\beta_1\epsilon^2 \cos x \sinh \epsilon y}{\rho^*(p^*+\beta_2)\Delta}, \quad v^{*\prime} = -\frac{2\beta_1\epsilon \sin x \cosh \epsilon y}{\rho^*(p^*+\beta_2)\Delta}, \\ p^{*\prime} &= \beta_4 - \frac{\beta_1^2\beta_3}{p^*+\beta_2}, \quad \rho^{*\prime} = \frac{\beta_3\rho^*(p^*+\beta_2)}{-p^*+\beta_2+2\rho^*\mathbb{B}^*} \end{aligned} \tag{7.7}$$

and the reciprocal variables $x', y'$ are given by

$$\begin{aligned} \beta_1 dx' &= \left(p^* + \beta_2 + \frac{4\epsilon^2\beta_1^2 \sin^2 x \cosh^2 \epsilon y}{\rho^*\Delta^2}\right)dx + \frac{4\epsilon^3\beta_1^2 \sin x \cos x \sinh \epsilon y \cosh \epsilon y}{\rho^*\Delta^2}dy, \\ \beta_1 dy' &= \frac{4\epsilon^3\beta_1^2 \sin x \cos x \sinh \epsilon y \cosh \epsilon y}{\rho^*\Delta^2}dx + \left(p^* + \beta_2 + \frac{4\epsilon^4\beta_1^2 \cos^2 x \sinh^2 \epsilon y}{\rho^*\Delta^2}\right)dy. \end{aligned} \tag{7.8}$$

Integration of the latter pair of reciprocal relations with $\beta_1 = 1$ yields

$$x' = c_1 x - \frac{\epsilon^2 \sin(2x)}{\rho^*\Delta}, \quad y' = c_2 y - \frac{\epsilon \sinh(2\epsilon y)}{\rho^*\Delta}, \tag{7.9}$$

where

$$c_1 = c_2 = \frac{K}{\rho^*} + \beta_2 - \frac{1-\epsilon^2}{4\rho^*}.$$

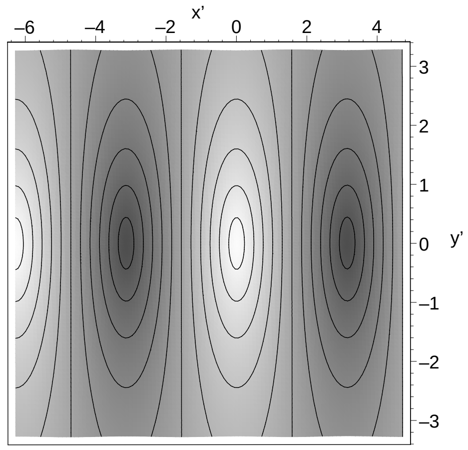


**Figure 1.** The streamlines $\psi' = \text{const}$ corresponding to (7.10)

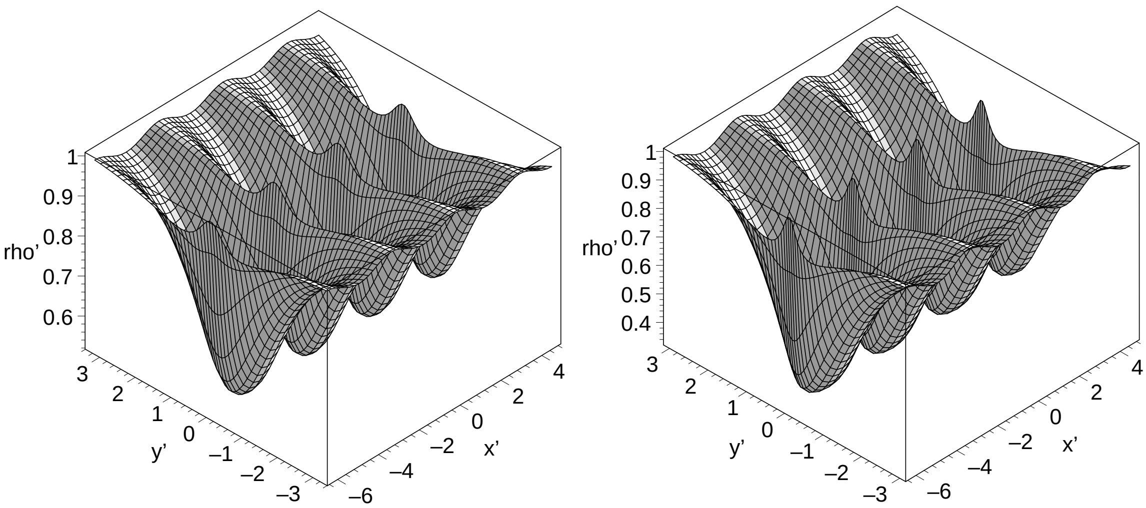


**Figure 2.** The density $\rho'$ corresponding to (7.10) and $k = 1/2$, $k = 0$ respectively

In Figures 1-3, various aspects of the reciprocal solutions are displayed with the parameter values

$$\beta_1 = \beta_2 = \beta_3 = \mu = m = 1, \quad \epsilon = \frac{1}{2}. \tag{7.10}$$

The reciprocal streamlines $\psi' = \text{const}$ corresponding to the seed Mallier-Maslowe solution are exhibited in Figure 1. They represent vortex train patterns valid in super-Alfvénic flow régimes of the generalised Kármán-Tsien gas (5.13). In Figure 2, the reciprocal density is exhibited when $k = 1/2$ and in the gasdynamic limit $k = 0$. Figure 3 displays the corresponding pressure distributions in these cases. In Figures 2, 3, $\beta_4 = 2, 3$ respectively in the magnetogasdynamic case $k = 1/2$ and the non-conducting limit $k = 0$. It is observed that the presence of the magnetic field acts to reduce the sharpness of the pressure and density peaks compared to the gasdynamic limit.

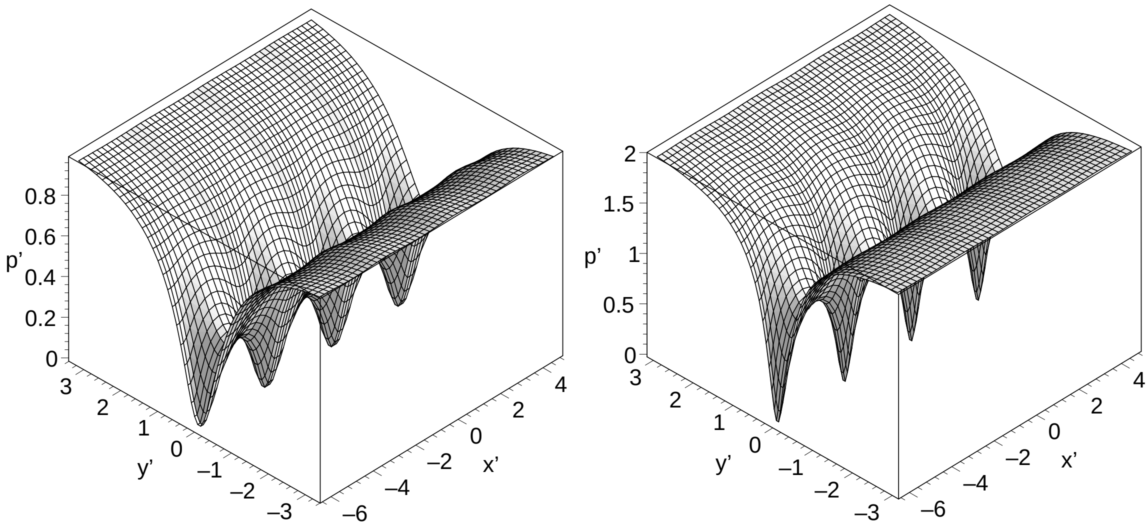


**Figure 3.** The pressure distribution $p'$ corresponding to (7.10) and $k = 1/2$, $k = 0$ respectively

# 8 Application of a Bäcklund Transformation

The Mallier-Maslowe vortex train solution is but a particular member of a large class of solutions of the sinh-Gordon equation (7.2) which may be generated by means of an analogue of the classical Bäcklund transformation for the sine-Gordon equation. In the following, it is demonstrated that for any such solution the associated reciprocal independent variables may be found in a purely algebraic manner so that no explicit integration of the reciprocal relations (4.2) is required. Thus, if we write the sinh-Gordon equation as

$$\omega_{z\bar{z}} = -\frac{\alpha^2}{2}\sinh 2\omega \tag{8.1}$$

with

$$\omega = \psi^*, \quad z = \frac{x + \mathrm{i}y}{2}, \quad \alpha^2 = 1 - \epsilon^2 \tag{8.2}$$

then the reciprocal relations (4.2) assume the compact form

$$ds = -\frac{\alpha^2}{4}\cosh 2\omega\, dz + \frac{1}{2}\omega_{\bar{z}}^2\, d\bar{z}, \qquad \beta_1 z' = \frac{s}{\rho^*} + \left(\frac{K}{\rho^*} + \beta_2\right) z. \tag{8.3}$$

It may be directly verified that $d^2 s = 0$ modulo (8.1). The analogue of the classical Bäcklund transformation [27] for the sine-Gordon equation states that the relations

$$(\omega_1 - \omega)_z = \alpha\lambda_1 \sinh(\omega_1 + \omega), \quad (\omega_1 + \omega)_{\bar{z}} = -\alpha\lambda_1^{-1}\sinh(\omega_1 - \omega) \tag{8.4}$$

are compatible if and only if $\omega$ constitutes a solution of the sinh-Gordon equation (8.1). Moreover, the function $\omega_1$ constitutes a second solution of the sinh-Gordon equation for any value of the constant Bäcklund parameter $\lambda_1$. It turns out that the corresponding potential $s_1$ obeying

$$ds_1 = -\frac{\alpha^2}{4}\cosh 2\omega_1\, dz + \frac{1}{2}\omega_{1\bar{z}}^2\, d\bar{z} \tag{8.5}$$

may be expressed explicitly in terms of $\omega, \omega_1$ and the potential $s$. Indeed, it may be directly verified that

$$s_1 = s - \frac{\alpha}{2\lambda_1}\cosh(\omega_1 - \omega) \tag{8.6}$$

modulo an irrelevant additive constant.

In general, the solution $\omega_1$ of the sinh-Gordon equation will not be real. However, the analogue of the classical Permutability Theorem for the sine-Gordon equation [27] may be exploited to generate a real solution by superposition of two complex Bäcklund transforms $\omega_1$ and $\omega_2$ corresponding to two distinct Bäcklund parameters $\lambda_1$ and $\lambda_2$. Thus, let $\omega_2$ be the Bäcklund transform of $\omega$ defined by

$$(\omega_2 - \omega)_z = \alpha\lambda_2 \sinh(\omega_2 + \omega), \quad (\omega_2 + \omega)_{\bar{z}} = -\alpha\lambda_2^{-1}\sinh(\omega_2 - \omega). \tag{8.7}$$

Then, a fourth solution of the sinh-Gordon equation (8.1) is given by the nonlinear superposition principle

$$\omega_{12} = \omega + 2\tanh^{-1}\left[\frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1}\tanh\left(\frac{\omega_2 - \omega_1}{2}\right)\right]. \tag{8.8}$$

Importantly, $\omega_{12}$ constitutes a Bäcklund transform of both $\omega_1$ with parameter $\lambda_2$ and $\omega_2$ with parameter $\lambda_1$. This implies that there exist two equivalent expressions for the potential $s_{12}$ associated with the solution $\omega_{12}$, namely

$$\begin{aligned} s_{12} &= s - \frac{\alpha}{2\lambda_1}\cosh(\omega_1 - \omega) - \frac{\alpha}{2\lambda_2}\cosh(\omega_{12} - \omega_1) \\ &= s - \frac{\alpha}{2\lambda_2}\cosh(\omega_2 - \omega) - \frac{\alpha}{2\lambda_1}\cosh(\omega_{12} - \omega_2), \end{aligned} \tag{8.9}$$

which, in turn, shows that $s_{12}$ is indeed symmetric in the indices $_1$ and $_2$. The above identity is readily shown to be a consequence of the superposition principle (8.8). Now, in order to obtain a real solution $\omega_{12}$, we make the admissible choice

$$\omega_2 = -\bar{\omega}_1, \quad \lambda_2 = -\bar{\lambda}_1^{-1} \tag{8.10}$$

so that

$$\omega_{12} = \omega + 2\tanh^{-1}\left[\frac{|\lambda_1|^2 - 1}{|\lambda_1|^2 + 1}\tanh \Re(\omega_1)\right]. \tag{8.11}$$

As an application of the above procedure, we consider the trivial seed solution $\omega = 0$ and $\lambda_1 = \mathrm{i}\nu$. In order to make contact with the Mallier-Maslowe solution (7.1), we choose

$$\alpha = \sqrt{1 - \epsilon^2}, \quad \nu = \sqrt{\frac{1+\epsilon}{1-\epsilon}}. \tag{8.12}$$

Integration of the Frobenius system (8.4) then yields

$$\omega_1 = 2\tanh^{-1}[\exp(\mathrm{i}x - \epsilon y)] \tag{8.13}$$

and hence

$$\omega_{12} = 2\tanh^{-1}\left(\frac{\epsilon\cos x}{\cosh\epsilon y}\right) \tag{8.14}$$

which indeed coincides with the Mallier-Maslowe solution on use of the invariance of (7.2) under $\psi^* \to -\psi^*$. Furthermore, evaluation of (8.9) produces

$$s_{12} = s - \frac{\epsilon^2\sin 2x}{2\Delta} - \mathrm{i}\frac{\epsilon\sinh 2\epsilon y}{2\Delta}, \quad \Delta = \cosh^2\epsilon y - \epsilon^2\cos^2 x \tag{8.15}$$

so that

$$\beta_1 x_{12} = cx - \frac{\epsilon^2\sin 2x}{\rho^*\Delta}, \quad \beta_1 y_{12} = cy - \frac{\epsilon\sinh 2\epsilon y}{\rho^*\Delta}, \quad c = \frac{K}{\rho^*} + \beta_2 - \frac{1-\epsilon^2}{4\rho^*} \tag{8.16}$$

by virtue of $(8.3)_2$. Thus, for $\beta_1 = 1$, the reciprocal variables (7.9) are retrieved.

# 9 Extension to Parallel-Transverse Magnetogasdyamics

The determination of a plane aligned magnetogasdynamic motion with velocity and density distribution $\{\boldsymbol{q}, \rho\}$ allows an associated class of *parallel-transverse* magnetogasdynamic flows to be generated with [10]

$$\boldsymbol{v} = \boldsymbol{q} + \xi\boldsymbol{k}, \quad \boldsymbol{H} = k\rho\boldsymbol{q} + \eta\boldsymbol{k}, \tag{9.1}$$

where $\xi, \eta$ are independent of $z$. Thus, insertion into the equation of motion shows that

$$\xi - \mu k\eta = \alpha(\psi), \tag{9.2}$$

while the continuity and induction equations hold automatically. The residual Faraday equation shows that

$$\eta - \xi k\rho = \beta(\psi)\rho. \tag{9.3}$$

Accordingly, transverse components may be superposed with

$$\xi = \frac{\alpha(\psi) + \mu k\beta(\psi)\rho}{1 - \mu k^2\rho}, \quad \eta = \frac{\alpha(\psi)k + \beta(\psi)\rho}{1 - \mu k^2\rho}. \tag{9.4}$$

The total pressure is then incremented with $\Pi \to \Pi + \frac{1}{2}\eta^2$, where $\Pi$ is its value in the underlying planar motion.

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