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Deformation quantization for almost-Kähler manifolds

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Abstract

On an arbitrary almost-Kähler manifold, starting from a natural affine connection with nontrivial torsion which respects the almost-Kähler structure, in joint work with A. Karabegov a Fedosov-type deformation quantization on this manifold was constructed. This contribution reports on the result and supplies an overview of the essential steps in the construction. On this way Fedosov's geometric method is explained.

1 Introduction

Nowadays we have complete results on the existence of a deformation quantization (also called a star product) for symplectic manifolds and even for general Poisson manifolds. Indeed the set of equivalence classes of star products are determined. In the symplectic case the existence was shown by De Wilde and Lecomte [1], Omori, Maeda and Yoshioka [2], and Fedosov [3]. The general Poisson case was resolved by Kontsevich [4]. In each of these approaches, different techniques and ideas were used. In the symplectic case Fedosov gave a very geometric construction procedure.

Beside the question of existence of a star product there is another important point. If the symplectic manifold carries additional structures, then the question is: is there a star product which "respect" this structure. For Kähler manifolds Alexander Karabegov introduced the notion of "separation of variables" [5]. It is equivalent to the notion of "being of Wick type" introduced by Bordemann and Waldmann [6]. Roughly speaking, this says that if one restricts the star product (let $\star$ denote the deformed multiplication) to any open subset then $f \star g = f \cdot g$, for every pair of local functions f and g, if either f is holomorphic or g is antiholomorphic. He shows the existence and gives a complete characterization of all star products with this property [7]. They are parameterized by certain formal forms, the Karabegov forms.

The examples which were considered by Berezin [8], Cahen, Gutt, and Rawnsley, and Moreno and Ortega-Navarro are of this type. Also the Berezin-Toeplitz deformation quantization [9], [10] has the "separation of variables" property but with the role of the holomorphic and antiholomorphic variables switched [11]. The method used by Karabegov in the construction is quite different from Fedosov's method. Recently, Neumaier [12] obtained a way how to construct for every Karabegov form a corresponding deformation quantization by using the technique of Fedosov. In the context of star products for Kähler manifolds the work of Reshetikhin and Takhtajan, Engliš, and Dolgushev, Lyakhovich and Sharapov should also be mentioned.

In this contribution, I want to report on results for the case of almost-Kähler manifolds [13], jointly obtained with Alexander Karabegov. We start from an almost-Kähler manifold with a natural affine connection respecting the almost-Kähler structure. In general, this connection will have nontrivial torsion. Fedosov's original construction uses a connection with vanishing torsion. But by a modification of the construction we obtain a Fedosov type star product also in this case. The zero degree part of the characteristic class of the star product is constructed. In addition to reporting on the results, it is my aim to give an overview of Fedosov's beautiful construction with this contribution.

2 Almost-Kähler manifolds

Let (M, J, ω) be an almost-Kähler manifold, i.e., a differentiable real manifold M endowed with an almost-complex structure J and a symplectic form ω which are compatible in the following sense: $\omega(JX, JY) = \omega(X, Y)$ for any vector fields X, Y on M, and $g(X, Y) := \omega(JX, Y)$ is a Riemannian metric. Recall that an almost-complex structure J is a bundle map for the tangent vector bundle, $J : \mathrm{T}M \to \mathrm{T}M$, which satisfies $J \circ J = -id$. A symplectic form w is a closed 2-form (i.e. $dw = 0$) which is non-degenerate. It is known that on any symplectic manifold (M, ω) one can choose a compatible almost-complex structure J to make it to an almost-Kähler manifold.

Associated to an almost-complex structure is the Nijenhuis tensor N given by

$$N(X, Y) := [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY], \tag{2.1}$$

where X, Y are vector fields on M. If $N = 0$ then the almost-complex structure is integrable to a complex structure, i.e. there exists an atlas of coordinates such that M becomes a complex manifold; which in our case is automatically a Kähler manifold.

Let ∇ be an affine connection, i.e. a connection defined in the tangent vector bundle $\mathrm{T}M$. For each such connection the curvature R and the torsion T is defined as

$$R(X, Y)Z := [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, \qquad T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]. \tag{2.2}$$

Here ∇_X denotes as usual the covariant derivative in direction of the vector field X.

Fedosov had chosen an affine connection which respects the symplectic form and is torsion free. Based on this connection he carried out his construction.

With our additional structure given, we want a connection which respect the metric g, the symplectic form ω and the almost-complex structure J. In the general case requiring additionally that the connection is torsion free leads to a contradiction. But

Proposition 1. *Let ∇ be the unique affine connection which respects the metric g and has torsion $T = (-1/4)N$, then ∇ also respects the symplectic form ω and the almost-complex structure J.*

This is a result due to Yano [14]. A proof can also be found in [13]. Note that if the almost-complex structure is integrable, i.e. M is indeed Kähler, the Nijenhuis tensor N vanishes, hence the connection ∇ will become torsion free. It is the Kähler connection.

3 Star product

For the convenience of the reader I recall here the definition of a star product.

Let (M, ω) be a symplectic manifold. Denote by $C^\infty(M)$ the algebra of (arbitrary often) differentiable functions. By pointwise multiplication it is an associative and commutative algebra. Using the symplectic form ω one assigns to every $f \in C^\infty(M)$ its Hamiltonian vector field X_f by $\omega(X_f, \cdot) = df(\cdot)$, and to every pair of functions f and g the Poisson bracket: $\{f, g\} := \omega(X_f, X_g)$. The Poisson bracket defines a Lie algebra structure on $C^\infty(M)$. With this structure $(C^\infty(M), \cdot, \{,\})$ becomes a Poisson algebra, i.e. we have $\{f, g\cdot h\} = \{f, g\}\cdot h + g\cdot\{f, h\}$, for all $f, g, h \in C^\infty(M)$.

The star product $\star$ is a non-commutative deformation of the multiplication "in direction of the Poisson bracket". Associativity is still required. In detail: Let $\mathcal{A} := C^\infty(M)[[\nu]]$ be the algebra of formal power series in the variable ν over the algebra $C^\infty(M)$. A product $\star$ on $\mathcal{A}$ is called a (formal) star product if it is an associative $\mathbb{C}[[\nu]]$-linear product such that

$$\mathcal{A}/\nu\mathcal{A} \cong C^\infty(M) \text{ i.e. } f \star g \bmod \nu = f \cdot g, \qquad \frac{1}{\nu}(f \star g - g \star f) \bmod \nu = -\,\mathrm{i}\,\{f, g\}, \tag{3.1}$$

where $f, g \in C^\infty(M)$. If we write $f \star g = \sum_{j=0}^{\infty} C_j(f, g)\nu^j$, with $C_j(f, g) \in C^\infty(M)$, the C_j are $\mathbb{C}$-bilinear in f and g, and the conditions above can be reformulated as $C_0(f, g) = f\cdot g$, and $C_1(f, g) - C_1(g, f) = -\,\mathrm{i}\,\{f, g\}$. A star product is called a differential star product if the C_j are bidifferential operators.

Two differential star products $\star$ and $\star'$ (for the same Poisson algebra) are called equivalent if there exists an isomorphism of algebras $B : (C^\infty(M)[[\nu]], \star) \to (C^\infty(M)[[\nu]], \star')$, where $B = Id + \nu B_1 + \nu^2 B_2 + \dots$, and $B_j, j \geq 1$, are differential operators on $C^\infty(M)$.

The equivalence classes of differential star products of a symplectic manifold M are classified by its characteristic (Fedosov-Deligne) class $cl(\star) \in \frac{1}{i\lambda}[\omega] + \mathrm{H}^2_{DR}(M, \mathbb{C})\,[[\lambda]]$. Here, $\mathrm{H}^2_{DR}(M, \mathbb{C})\,[[\lambda]]$ denotes the space of formal power series in the formal variable λ over the second de-Rham cohomology space of the manifold M.

4 An outline of the modified Fedosov construction

In this section I give an overview of the construction by pointing out the essential steps. The details of the calculations can be found in [13].

For every point $x \in M$ we take an open contractible coordinate chart U with coordinates $\{x^k,\ k = 1, \dots, n\}$. Here n is the dimension of the manifold M. Let TM be the tangent bundle of M. Restricted to U we can write $\mathrm{TM}_{|U} \cong U \times \mathbb{R}^n$. We denote the fiber coordinates in the bundle by $\{y^k,\ k = 1, \dots, n\}$.

The point-wise *Wick algebra* W_x consists of the formal series

$$a(\nu, y) := \sum_{r\geq 0, |\alpha|\geq 0} a_{r,\alpha}\nu^r y^\alpha, \qquad a_{r,\alpha} \in \mathbb{C},\ y = (y^1, \dots, y^n). \tag{4.1}$$

Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, and we use the usual multi-index notation, e.g. $y^\alpha := (y^1)^{\alpha_1} \cdots (y^n)^{\alpha_n}$. This space is equipped with the fiber-wise Wick product

$$a \circ b\,(\nu, y) := \left(\exp\left(\frac{\mathrm{i}\,\nu}{2} \sum_{j,k=1,\ldots,n} \pi^{jk} \frac{\partial^2}{\partial y^j \partial z^k} \right) a(\nu, y) b(\nu, z) \right)_{|z=y} . \tag{4.2}$$

Here we used the following notation $\partial_j := \frac{\partial}{\partial x^j}$, $\omega_{jk} := \omega(\partial_j, \partial_k)$, $(\omega^{jk}) := (\omega_{jk})^{-1}$, $g_{jk} := g(\partial_j, \partial_k)$, $(g^{jk}) := (g_{jk})^{-1}$, $\pi^{jk} := \omega^{jk} - \mathrm{i}\, g^{jk}$. The principal idea in constructing $f \star g$ is now: consider the bundle W of fiber-wise Wick algebras, find a map which maps the functions on M to sections of the Wick algebra, take the multiplication there and map the result down again to a (formal) function in such a way that all the necessary conditions are fulfilled. It was an important idea of Fedosov to increase this infinite dimensional bundle even more. This also works in the non-torsionfree setting.

Let $\mathcal{W}$ be the sheaf of smooth sections of the Wick algebra bundle W, i.e. its elements are the local sections of type (4.1) where now the $a_{r,\alpha}$ are smooth functions on U. Let Λ be the bundle of differential forms on M. Denote by $\mathcal{W} \otimes \Lambda$ the sheaf of W-valued differential forms and extend the product $\circ$ to $\mathcal{W} \otimes \Lambda$ in the natural way by

$$(a \otimes \alpha) \circ (b \otimes \beta) := (a \circ b) \otimes (\alpha \wedge \beta). \tag{4.3}$$

We introduce gradings in $(\mathcal{W} \otimes \Lambda, \circ)$ by $\deg_\nu(\nu) := 1$ (the formal degree), $\deg_s(y^k) := 1$ (the fiber degree), and $\deg_a(dx^k) := 1$ (the alternating degree), where in each case the degree evaluated at the other generators is set to zero. The total degree is defined as $\mathrm{Deg} := 2\deg_\nu + \deg_s$. One easily shows that the algebra is bigraded with respect to Deg and $\deg_a$.

The connection ∇ defined by Proposition 1 can be extended to $\mathcal{W} \otimes \Lambda$ by the following description (we use the same symbol for the extended connection)

$$\nabla(a \otimes \lambda) := \sum_j \left((\frac{\partial a}{\partial x^j} - \sum_{k,l} \Gamma^l_{jk} y^k \frac{\partial a}{\partial y^l}) \otimes (dx^j \wedge \lambda) \right) + a \otimes d\lambda. \tag{4.4}$$

The Γ^l_{jk} are the Christoffel symbols of the connection ∇ defined by $\nabla_{\partial_j}(\partial_k) = \sum_l \Gamma^l_{jk} \partial_l$. Again by direct calculations one obtains that the extended ∇ is a $\deg_a$-graded derivation of $(\mathcal{W} \otimes \Lambda, \circ)$.

The Fedosov operators δ and δ^{-1} on $\mathcal{W} \otimes \Lambda$ are defined as follows. Let a a local section of $\mathcal{W} \otimes \Lambda$ which is homogeneous with respect to $\deg_s$ and $\deg_a$ and has the degrees $deg_s(a) = p$, and $deg_a(a) = q$. We set

$$\delta(a) = \sum_j dx^j \wedge \frac{\partial a}{\partial y^j} \quad \text{and} \quad \delta^{-1} a = \begin{cases} \frac{1}{p+q} \sum_j y^j i\left(\frac{\partial}{\partial x^j}\right) a & , p+q > 0, \\ 0, & , p = q = 0. \end{cases} \tag{4.5}$$

Here $i(\frac{\partial}{\partial x^j})$ is the contraction which operates only on the differential part.

Let $\sigma : a \mapsto \sigma(a)$ be the projection on the (deg_s, deg_a)-bihomogeneous part of a of bidegree zero (i.e. $deg_s(\sigma(a)) = deg_a(\sigma(a)) = 0$). The Hodge type decomposition $a = \delta\delta^{-1} a + \delta^{-1}\delta a + \sigma(a)$ can be verified. Again the operator δ is a deg_a-graded derivation of the algebra $(\mathcal{W} \otimes \Lambda, \circ)$.

We need the following intermediate operators T and R which extend in some sense the torsion and the curvature of our affine connection we started with; hence we use the same symbols to denote them. If we write the torsion and the curvature with respect of the basis ∂_i of the vector fields as $T(\partial_k, \partial_l) = \sum_i T^i_{kl}\partial_i$ and $R(\partial_k, \partial_l)\partial_j = \sum_i R^i_{jkl}\partial_i$, then they can be expressed with the help of the Christoffel symbols as

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}, \quad R^i_{jkl} = \left(\frac{\partial \Gamma^i_{lj}}{\partial x^k} - \frac{\partial \Gamma^i_{kj}}{\partial x^l}\right) + \sum_m \left(\Gamma^m_{lj}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm}\right). \tag{4.6}$$

Now the extended T and R are defined as

$$T := \frac{1}{2}\sum_{s,j,k,l} \omega_{sj} T^j_{kl} y^s dx^k, \quad \text{and} \quad R := \frac{1}{4}\sum_{s,j,t,k,l} \omega_{sj} R^j_{tkl} y^s y^t dx^k \wedge dx^l. \tag{4.7}$$

Our next goal is to make the extended connection flat, more precisely to add elements to ∇ with the aim to obtain a D with $D^2 = 0$.

Denote by $a^{(k)}$ for $a \in \mathcal{W} \otimes \Lambda$ the $\mathrm{Deg} = 2\deg_\nu + \deg_s = k$ homogeneous part of a. We showed in [13] that there is a unique element $r \in \mathcal{W} \otimes \Lambda$ which satisfies $\deg_a(r) = 1$, $\delta^{-1} r = 0$, $r^{(0)} = r^{(1)} = 0$ such that $\delta r = T + R + \nabla r - \frac{\mathrm{i}}{\nu}\, r \circ r$. Its components can be calculated recursively, as

$$r^{(2)} = \delta^{-1} T, \qquad r^{(3)} = \delta^{-1}\left(R + \nabla r^{(2)} - \frac{\mathrm{i}}{\nu}\, r^{(2)} \circ r^{(2)}\right),$$

$$r^{(k+3)} = \delta^{-1}\left(\nabla r^{(k+2)} - \frac{\mathrm{i}}{\nu}\sum_{l=0}^{k} r^{(l+2)} \circ r^{(k-l+2)}\right), \; k \geq 1.$$

By this description the element r is adjusted in such a way that we obtain the

Proposition 2. *[13, Thm. 3.1] The operator (called the Fedosov connection)* $D := -\delta + \nabla - \frac{\mathrm{i}}{\nu}\mathrm{ad}_{Wick}(r)$ *is a flat connection, i.e. we have* $D^2 = 0$.

Here $\mathrm{ad}_{Wick}(a)$ is the operator $[a, .]$ with respect to the fiber-wise Wick product $\circ$. The map D is a $\deg_a$-graded derivation. Hence, the subspace of the flat sections with alternating degree zero part, $\mathcal{W}_D := \ker D \cap \mathcal{W}$, is a subalgebra of $(\mathcal{W}, \circ)$.

Proposition 3. *[13, Thm. 3.2] Let* $\sigma : \mathcal{W}_D \to C^\infty(M)[[\nu]]$ *be the projection on the part with no tangent direction variables* y, *then* σ *is a bijection.*

The inverse mapping τ evaluated for a given function f can be recursively calculated as

$$\tau(f)^{(0)} = f, \quad \tau(f)^{(k+1)} = \delta^{-1}\left(\nabla \tau(f)^{(k)} - \frac{\mathrm{i}}{\nu}\sum_{l=0}^{k} \mathrm{ad}_{Wick}\big(r^{(l+2)}\big)\big(\tau(f)^{(k-l)}\big)\right), k \geq 0.$$

As Fedosov did, we obtain in our case

Theorem 1. *The product*

$$f \star g := \sigma(\tau(f) \circ \tau(g)) \tag{4.8}$$

defines a star product for the almost-Kähler manifold (M, ω).

To give the leading parts of its characteristic class $cl(\star)$ we first have to recall the definition of the canonical class of an almost-complex manifold (M, J). Let $\mathrm{T}_{\mathbb{C}}M$ be the complexified tangent bundle and $\mathrm{T}'_{\mathbb{C}}M$ the subbundle of $(1,0)$ vectors. Let $\epsilon := c_1(\mathrm{T}'_{\mathbb{C}}M)$ be its first Chern class. Then we showed

$$cl(\star) = \frac{1}{\mathrm{i}\,\lambda}[\omega] - \frac{1}{2i}\epsilon + \gamma_1\lambda + \gamma_2\lambda^2 + \cdots, \ \gamma_i \in \mathrm{H}^2_{DR}(M, \mathbb{C}), i = 1, 2, \ldots \tag{4.9}$$

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