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On the Resolution of Space-Time Singularities II

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Abstract

In previous articles it has been argued that a differential calculus over a noncommutative algebra uniquely determines a gravitational field in the commutative limit and that there is a unique metric which remains as a commutative 'shadow'. Some examples were given of metrics which resulted from a given algebra and given differential calculus. Here we aboard the inverse problem, that of constructing the algebra and the differential calculus from the commutative metric. As an example a noncommutative version of the Kasner metric is proposed which is periodic. This modified metric has a cosmological constant which can be seen to be directly related to the noncommutative structure.

1 Motivation

A definition has been given [1] of a torsion-free metric-compatible linear connection on a differential calculus $\Omega^*(\mathcal{A})$ over an algebra $\mathcal{A}$ which has certain rigidity properties provided that the center $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$ is trivial. It was argued from simple examples that a differential calculus over a noncommutative algebra uniquely determines a gravitational field in the commutative limit. Some examples have been given [2, 3, 4] of metrics which resulted from a given algebra and given differential calculus. Here we aboard the inverse problem, that of constructing the algebra and the differential calculus from the commutative metric. As an example we construct noncommutative versions of the Kasner metric and we show that it is possible to choose an algebra such that the metric is nonsingular before taking the commutative limit. The 'II' on the title alludes to a preliminary version given at the Torino Euroconference [5] on noncommutative geometry [6].

The physical idea we have in mind is that the description of space-time using a set of commuting coordinates is only valid at length scales greater than some fundamental length. At smaller scales it is impossible to localize a point and a new geometry must be used. We can use a solid-state analogy and think of the ordinary Minkowski coordinates as macroscopic order parameters obtained by 'course-graining' over regions whose size is determined by a fundamental area scale $k\!\!\!^-$, which is presumably, but not necessarily, of the order of the Planck area $G\hbar$. They break down and must be replaced by elements of a noncommutative algebra when one considers phenomena on smaller scales. A simple visualization is afforded by the orientation order parameter of nematic liquid crystals. The commutative free energy is singular in the core region of a disclination. There is of course no physical singularity; the core region can simply not be studied using the commutative order parameter.

As a concrete example we have chosen, for historical reasons, the Kasner metric; we show that its singularity can be resolved into an essentially noncommutative structure. We do not however claim that an arbitrary singularity in a metric on an arbitrary smooth manifold can be resolved using a noncommutative structure. From the point of view we are adopting a commutative geometry is a rather singular limit. The close relation between the differential calculus and the metric can at most be satisfied when the center is trivial. This manifests itself in the fact that on an ordinary manifold one can put any metric with any singularity. We argue only that those metrics which are 'physical' in some sense, for example are Ricci flat, can have resolvable singularities.

There is a similarity of the method we use to resolve the singularity with the method known in algebraic geometry as 'blowing up' a singularity [7] as well as with the method used by 't Hooft and Polyakov to resolve the monopole singularity. The regular solution found in this case can in fact be considered as the Dirac monopole solution on a noncommutative geometry which contains the 2×2 matrix algebra as extra factor.

In previous articles the algebra and the differential calculus were given and the linear connection and metric were constructed. It was argued [8, 9] that given the algebra $\mathcal{A}$ the structure of $\Omega^*(\mathcal{A})$ is intimately connected with the gravitational field which remains on V as shadow in the commutative limit $k\!\!\!^- \to 0$. Within the general framework which we here consider, the principal difference between the commutative and noncommutative cases lies in the spectrum of the operators which we use to generate the noncommutative algebra which replaces the algebra of functions. This in turn depends not only on the structure of this algebra as abstract algebra but on the representation of it which we choose to consider. Here we attempt the inverse problem, that of constructing the algebra *and* the differential calculus from the commutative linear connection. We cannot claim that the procedure is in any way unique. For a discussion of the relation of noncommutative geometry to the problem of space-time singularities from rather different points of view from the one we adopt we refer, for example, to Heller & Sasin [10], to Hawkins [11] or to Lizzi *et al.* [12]. We refer elsewhere for a description of the same 'quantization' applied to the PP wave [13] and for a possible cosmological application [14].

In the next section we introduce the general formalism of noncommutative geometry which we use and we make some general remarks concerning the problem of 'quantization' of space-time. In Section 3 we recall the commutative Kasner metric. In Section 4 we make some remarks concerning perturbative approximations to noncommutative geometry and present the Kasner solution as a perturbative solution in $k\!\!\!^-$. In Section 5 we lift our view to the differential calculus and in the last section we discuss the field equations.

Greek indices take values from 0 to 3; the first half of the alphabet is used to index (moving) frames and the second half to index generators. Latin indices a, b, *etc.* take values from 1 to 3 and the indices i, j, *etc.* values from 0 to $n-1$.

2 The general formalism

The notation is the same as that of a previous article [5] on the symplectic structure of space-time and is based on a noncommutative generalization [15, 8, 16] of the Cartan moving-frame formalism . Let $\mathcal{A} = \mathcal{C}(V)$ be the algebra of smooth real-valued functions on a space-time V which for simplicity we shall suppose parallelizable and with a metric and linear connection defined in terms of a globally defined moving frame θ^α. Let $\Omega^*(\mathcal{A})$ be the algebra of de Rham differential forms. The space $\Omega^1(\mathcal{A})$ of 1-forms is free of rank 4 as a $\mathcal{A}$-module. According to the general idea outlined above a singularity in the metric is due to the use of commuting coordinates beyond their natural domain of definition into a region where they are physically inappropriate. From this point of view the space-time V should be more properly described 'near the singularity' by a noncommutative $*$-algebra $\mathcal{A}$ over the complex numbers with four hermitian generators x^λ. The observables will be some subset of the hermitian elements of $\mathcal{A}$. We shall not discuss this problem here; we shall implicitly suppose that all hermitian elements of $\mathcal{A}$ are observables, including the 'coordinates'. We shall not however have occasion to use explicitly this fact.

We introduce 6 additional elements $J^{\mu\nu}$ of $\mathcal{A}$ by the relations

$$[x^\mu, x^\nu] = i\hbar J^{\mu\nu}. \tag{2.1}$$

The details of the structure of $\mathcal{A}$ will be contained for example in the commutation relations $[x^\lambda, J^{\mu\nu}]$. One can define recursively an infinite sequence of elements by setting for $p \geq 1$

$$[x^\lambda, J^{\mu_1\cdots\mu_p}] = i\hbar J^{\lambda\mu_1\cdots\mu_p}. \tag{2.2}$$

We shall assume that for the description of a generic (strong) gravitational field the appropriate algebra $\mathcal{A}$ has a trivial center $\mathcal{Z}(\mathcal{A})$:

$$\mathcal{Z}(\mathcal{A}) = \mathbb{C}. \tag{2.3}$$

The only argument we have in favor of this assumption is the fact that it would be difficult to interpret the meaning of the center. The x^μ will be referred to as 'position generators'. We shall suppose also that there is a set of $n(= 4)$ antihermitian 'momentum generators' λ_α and a 'Fourier transform'

$$F : x^\mu \longrightarrow \lambda_\alpha = F_\alpha(x^\mu)$$

which takes the position generators to the momentum generators.

Let ρ be a representation of $\mathcal{A}$ as an algebra of linear operators on some Hilbert space. For every $k_\mu \in \mathbb{R}^4$ one can construct a unitary element $u(k) = e^{ik_\mu x^\mu}$ of $\mathcal{A}$ and one can consider the weakly closed algebra $\mathcal{A}_\rho$ generated by the image of the $u(k)$ under ρ. The momentum operators λ_α are also unbounded but using them one can construct also a set of 'translation' operators $\hat{u}(\xi) = e^{\xi^\alpha \lambda_\alpha}$ whose image under ρ belongs also to $\mathcal{A}_\rho$. In general $\hat{u}u \neq u\hat{u}$; if the metric which we introduce is the flat metric then we shall see that $[\lambda_\alpha, x^\mu] = \delta^\mu_\alpha$ and in this case we can write the commutation relations $\hat{u}u = qu\hat{u}$ with $q = e^{ik_\mu\xi^\mu}$; the 'Fourier transform' is the simple linear transformation

$$\lambda_\alpha = \frac{1}{i\hbar}\theta^{-1}_{\alpha\mu}x^\mu$$

for some symplectic structure $\theta^{\alpha\mu}$. If the structure is degenerate then it is no longer evident that the algebra can be generated by either the position generators or the momentum generators alone. In such cases we define the algebra $\mathcal{A}$ to be the one generated by both sets. The derivations could be considered as outer derivations of the smaller algebra generated by the x^μ; they become inner in the extended algebra,

We shall suppose that $\mathcal{A}$ has a commutative limit which is an algebra $\mathcal{C}(V)$ of smooth functions on a space-time V endowed with a globally defined moving frame θ^α and thus a metric. By parallelizable we mean that the module $\Omega^1(\mathcal{A})$ has a basis θ^α which commutes with the elements of $\mathcal{A}$. For all $f \in \mathcal{A}$

$$f\theta^\alpha = \theta^\alpha f. \tag{2.4}$$

We shall see that this implies that the metric components must be constants, a condition usually imposed on a moving frame. The frame θ^α allows one [17] to construct a representation of the differential algebra from that of $\mathcal{A}$. Following strictly what one does in ordinary geometry, we shall introduce the set of derivations e_α to be dual to the frame θ^α, that is with

$$\theta^\alpha(e_\beta) = \delta^\alpha_\beta. \tag{2.5}$$

We define the differential exactly as did E. Cartan in the commutative case. If e_α is a derivation of $\mathcal{A}$ then for every element $f \in \mathcal{A}$ we define df by the constraint $df(e_\alpha) = e_\alpha f$. The differential calculus is defined as the largest one consistent with the module structure of the 1-forms so constructed. One can at this point take [18] the classical limit to obtain four functions $\tilde\lambda_\alpha(\tilde x^\mu)$ which satisfy the equations

$$\{\tilde\lambda_\alpha, \tilde x^\mu\} = \tilde e^\mu_\alpha.$$

This defines a Poisson structure directly from which one can calculate the $\{\tilde x^\mu, \tilde x^\nu\}$. In this way only at the last moment does one pass to a noncommutative algebra and most of the problem remains within the category of smooth manifolds.

It follows from general arguments that the momenta λ_α must satisfy the consistency condition

$$2\lambda_\gamma\lambda_\delta P^{\gamma\delta}{}_{\alpha\beta} - \lambda_\gamma F^\gamma{}_{\alpha\beta} - K_{\alpha\beta} = 0. \tag{2.6}$$

The $P^{\gamma\delta}{}_{\alpha\beta}$ define the product π in the algebra of forms:

$$\theta^\alpha\theta^\beta = P^{\alpha\beta}{}_{\gamma\delta}\theta^\gamma \otimes \theta^\delta. \tag{2.7}$$

This product is defined to be the one with the least relations which is consistent with the module structure of the 1-forms. The $F^\gamma{}_{\alpha\beta}$ are related to the 2-form $d\theta^\alpha$ through the structure equations:

$$d\theta^\alpha = -\frac{1}{2}C^\alpha{}_{\beta\gamma}\theta^\beta\theta^\gamma.$$

In the noncommutative case the structure elements are defined as

$$C^\alpha{}_{\beta\gamma} = F^\alpha{}_{\beta\gamma} - 2\lambda_\delta P^{(\alpha\delta)}{}_{\beta\gamma}. \tag{2.8}$$

It follows that

$$e_\alpha C^\alpha{}_{\beta\gamma} = 0. \tag{2.9}$$

This must be imposed then at the classical level and can be used as a gauge-fixing condition.

Finally, to complete the definition of the coefficients of the consistency condition (2.6) we introduce the special 1-form $\theta = -\lambda_\alpha \theta^\alpha$. In the commutative, flat limit

$$\theta \to i\partial_\alpha dx^\alpha.$$

As an (antihermitian) 1-form θ defines a covariant derivative on an associated $\mathcal{A}$-module with local gauge transformations given by the unitary elements of $\mathcal{A}$. The $K_{\alpha\beta}$ are related to the curvature of θ:

$$d\theta + \theta^2 = K, \qquad K = -\frac{1}{2} K_{\alpha\beta}\theta^\alpha\theta^\beta.$$

All the coefficients lie in the center $\mathcal{Z}(\mathcal{A})$ of the algebra.

The condition (2.6) can be expressed also in terms of a twisted commutator

$$[\lambda_\alpha, \lambda_\beta]_P = 2P^{\gamma\delta}{}_{\alpha\beta}\lambda_\gamma\lambda_\delta$$

as

$$[\lambda_\alpha, \lambda_\beta]_P = \lambda_\gamma F^\gamma{}_{\alpha\beta} + K_{\alpha\beta}.$$

It is also connected with the condition that $d^2 f = 0$. The differential df of an element $f \in \mathcal{A}$ is given by $df = e_\alpha f \theta^\alpha$. Since, in particular

$$d^2\lambda_\gamma = d([\lambda_\beta, \lambda_\gamma]\theta^\beta) = ([\lambda_\alpha, [\lambda_\beta, \lambda_\gamma]] - \tfrac{1}{2}[\lambda_\mu, \lambda_\gamma]C^\mu{}_{\alpha\beta})\theta^\alpha\theta^\beta$$

it follows that

$$P^{\alpha\beta}{}_{\gamma\delta} e_\alpha e_\beta - C^\gamma{}_{\alpha\beta} e_\gamma = 0.$$

This is the same as Equation (2.6).

Equation (2.8) is the correspondence principle which associates a differential calculus to a metric. On the left in fact the quantity $C^\alpha{}_{\beta\gamma}$ determines a moving frame, which in turn fixes a metric; on the right are the elements of the algebra which fix to a large extent the differential calculus. A 'blurring' of a geometry proceeds via this correspondence. It is evident that in the presence of curvature the 1-forms cease to anticommute. On the other hand it is possible for flat 'space' to be described by 'coordinates' which do not commute. The correspondence principle between the classical and noncommutative geometries can be also described as the map

$$\tilde{\theta}^\alpha \mapsto \theta^\alpha \tag{2.10}$$

with the product satisfying the condition

$$\tilde{\theta}^\alpha\tilde{\theta}^\beta \mapsto P^{\alpha\beta}{}_{\gamma\delta}\theta^\gamma\theta^\delta.$$

The tilde on the left is to indicate that it is the classical form. The condition can be written also as

$$\tilde{C}^\alpha{}_{\beta\gamma} \mapsto C^\alpha{}_{\eta\zeta} P^{\eta\zeta}{}_{\beta\gamma}$$

or as

$$\lim_{\kbar\to 0} C^\alpha{}_{\beta\gamma} = \tilde{C}^\alpha{}_{\beta\gamma}. \tag{2.11}$$

A solution to these equations would be a solution to the problem we have set. It would be however unsatisfactory in that no smoothness condition has been imposed. This can at best be done using the inner derivations. We shall construct therefore the set of momentum generators. The procedure we shall follow is not always valid; a counter example has been constructed [19] for the flat metric on the torus. The correspondence principle which in fact we shall actually use is a modified version of the map

$$\tilde{e}_\alpha \mapsto \lambda_\alpha$$

which is the inverse of that introduced by von Neumann to represent the Heisenberg algebra.

We introduce an involution [20] on the algebra of forms using [21] a reality condition on derivations, a procedure which is more or less a straightforward generalization of that which is used in the case of ordinary differential manifolds. The involution depends on the form of the product projection π. For general $\xi, \eta \in \Omega^1(\mathcal{A})$ it follows that

$$(\xi\eta)^* = -\eta^*\xi^*.$$

In particular

$$(\theta^\alpha\theta^\beta)^* = -\theta^\beta\theta^\alpha.$$

The product of two frame elements is hermitian then if and only if they anticommute. Recall that the product of two hermitian elements f and g of the algebra is hermitian if and only if they commute. When the frame exists one has necessarily also the relations

$$(f\xi\eta)^* = (\xi\eta)^* f^*, \qquad (f\xi\otimes\eta)^* = (\xi\otimes\eta)^* f^*$$

for arbitrary $f \in \mathcal{A}$.

We write $P^{\alpha\beta}{}_{\gamma\delta}$ in the form

$$P^{\alpha\beta}{}_{\gamma\delta} = \frac{1}{2}\delta^{[\alpha}_\gamma \delta^{\beta]}_\delta + i\kbar Q^{\alpha\beta}{}_{\gamma\delta} \tag{2.12}$$

of a standard projector plus a perturbation. If further we decompose $Q^{\alpha\beta}{}_{\gamma\delta}$ as the sum of two terms

$$Q^{\alpha\beta}{}_{\gamma\delta} = Q_-^{\alpha\beta}{}_{\gamma\delta} + Q_+^{\alpha\beta}{}_{\gamma\delta}$$

symmetric (antisymmetric) and antisymmetric (symmetric) with respect to the upper (lower) indices then the condition that $P^{\alpha\beta}{}_{\gamma\delta}$ be a projector is satisfied to first order in $\kbar$ because of the property that

$$Q^{\alpha\beta}{}_{\gamma\delta} = P^{\alpha\beta}{}_{\zeta\eta} Q^{\zeta\eta}{}_{\gamma\delta} + Q^{\alpha\beta}{}_{\zeta\eta} P^{\zeta\eta}{}_{\gamma\delta}.$$

The compatibility condition with the product

$$(P^{\alpha\beta}{}_{\zeta\eta})^* P^{\eta\zeta}{}_{\gamma\delta} = P^{\beta\alpha}{}_{\gamma\delta}$$

is satisfied provided $Q^{\alpha\beta}{}_{\gamma\delta}$ is real.

We can now write (2.6) in the form

$$[\lambda_\alpha, \lambda_\beta] + 2i\bar{k}\, [\lambda_\gamma, \lambda_\delta]\, Q_{+}^{\gamma\delta}{}_{\alpha\beta} = K_{\alpha\beta} + \lambda_\gamma (F^\gamma{}_{\alpha\beta} - 2i\bar{k}\lambda_\delta Q_{-}^{\gamma\delta}{}_{\alpha\beta}). \tag{2.13}$$

This implies that to lowest order

$$K_{+\alpha\beta} = i\bar{k} K_{-\gamma\delta} Q_{+}^{\gamma\delta}{}_{\alpha\beta}$$

so that we can rewrite (2.6) as two independent equations

$$[\lambda_\alpha, \lambda_\beta] \;\; = K_{-\alpha\beta} + \lambda_\gamma F^\gamma_{-\alpha\beta} - 2i\bar{k}\lambda_\gamma\lambda_\delta Q_{-}^{\gamma\delta}{}_{\alpha\beta}, \tag{2.14}$$

$$0 = K_{+\alpha\beta} + \lambda_\gamma F^\gamma_{+\alpha\beta} - 2i\bar{k}\lambda_\gamma\lambda_\delta Q_{+}^{\gamma\delta}{}_{\alpha\beta}. \tag{2.15}$$

This is the form which we shall use.

Under a change of frame basis the coefficients of the spin connection also change. We mention only the linear approximation. If

$$\theta'^\alpha = \theta^\alpha - H^\alpha{}_\beta \theta^\beta$$

then

$$C'^\alpha{}_{\beta\gamma} = C^\alpha{}_{\beta\gamma} + D_{[\beta} H^\alpha{}_{\gamma]}.$$

The only restriction on $H^\alpha{}_\beta$, apart from the condition that it be small and antisymmetric, is that it must leave the condition (2.9) invariant or impose it if it is not satisfied.

It is necessary [1] to introduce a flip operation

$$\sigma :\; \Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A})$$

to define the reality condition and the Leibniz rules. If we write

$$S^{\alpha\beta}{}_{\gamma\delta} = \delta^\beta_\gamma \delta^\alpha_\delta + i\bar{k} T^{\alpha\beta}{}_{\gamma\delta}$$

we find that a choice [8] of connection which is torsion-free, and satisfies all Leibniz rules is given by

$$\omega^\alpha{}_\beta = \tfrac{1}{2} F^\alpha{}_{\gamma\beta}\theta^\gamma + i\bar{k}\lambda_\gamma T^{\alpha\gamma}{}_{\delta\beta}\theta^\delta. \tag{2.16}$$

The relation

$$\pi \circ (1 + \sigma) = 0$$

must hold [8, 9] to assure that the torsion be a bilinear map.

We shall suppose that $\mathcal{A}$ has a metric

$$g :\; \Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A}) \to \mathcal{A}. \tag{2.17}$$

In terms of the frame one can define the metric by the condition that

$$g(\theta^\alpha \otimes \theta^\beta) = g^{\alpha\beta}. \tag{2.18}$$

The $g^{\alpha\beta}$ are taken to form an arbitrary complex matrix which satisfies [21] the symmetry condition

$$P^{\alpha\beta}{}_{\gamma\delta} g^{\gamma\delta} = 0 \tag{2.19}$$

as well as the reality condition

$$g^{\beta\alpha} + i\bar{k} T^{\alpha\beta}{}_{\gamma\delta} g^{\gamma\delta} = (g^{\beta\alpha})^*.$$

It is tempting to suppose that to lowest order at least, in a semi-classical approximation, there is an analogue of Darboux's lemma and that it is always possible to choose generators which satisfy commutation relations of the form (2.2) with the right-hand in the center. However the example we shall examine in detail shows that this is not always the case. Having fixed the generators, the manifestations of curvature would be found then in the form of the frame. The two sets of generators x^μ and λ_α satisfy, under the assumptions we make, three sets of equations. The commutation relations (2.2) for the position generators x^μ and the associated Jacobi identities permit one definition of the algebra. The commutation relations for the momentum generators permit a second definition. The conjugacy relations assure that the two descriptions concern the same algebra. We shall analyze these identities later using the example to show that they have interesting non-trivial solutions.

The problem of gauge invariance and the algebra of observables is a touchy one upon which we shall not dwell. It is obvious that not all of the elements of $\mathcal{A}$ are gauge invariant but not that all observables are gauge-invariant. One of the principles of the theory of general relativity is that all (regular) coordinates systems or frames are equal. In the noncommutative case one finds that some are more equal than others. If one quantize a space-time using two different moving frames one will obtain two different differential calculi, although the two underlying algebras might be the same. This is equivalent to the fact that the canonical transformations of a commutative phase space are a very special set of phase-space coordinate transformations. It can also be expressed as the fact that the Poisson structure which remains on space-time as the commutative limit of the commutation relations breaks Lorentz invariance. In the special case where the H^α_β are constants then the two quantized frames will be also equivalent. Since we have decided to work only with algebras whose centers are trivial the converse will also be true. Since we are interested in finding the 'simplest' differential calculus, one of the aspects of the problem is the choice of 'correct' moving frame to start with.

One possible method of looking for a solution is to consider a manifold V embedded in $\mathbb{R}^d$ for some d with the commutation relations

$$[y^i, y^j] = i\bar{k}\theta^{ij}, \qquad \theta^{ij} \in \mathbb{R}.$$

This will induce a symplectic structure on V which is intimately related to the one we shall exhibit in the following sections. The details of this have yet to be investigated. Let the larger algebra be $\mathcal{B}$. It has a natural differential calculus defined by imposing the condition $[y^i, dy^j] = 0$ that the differentials of the generators be a frame. It follows that the associated metric is flat. The projection

$$\Omega^*(\mathcal{B}) \longrightarrow \Omega^*(\mathcal{A})$$

would yield a solution to the problem but it is not necessarily easier to find. In fact a similar situation arises in one of the possible definitions of a differential calculus as a quotient of the universal differential calculus by a differential ideal. In that case the projection is strictly equivalent to the calculus. One could also consider the problem of finding the metric as an evolution equation in field theory in the sense that one can pass from the Schrödinger picture to the Heisenberg picture with the help of an evolution Hamiltonian.

It is interesting to notice how the old Kaluza-Klein idea of gauge transformations as coordinate transformations appears here. Gauge transformations are inner automorphisms of the algebra with respect to some unitary (pseudo-)group $\mathcal{G}_G \subset \mathcal{A}$ of elements; the complete dynamical evolution of the system can be described as an involution with respect to one unitary element $U = e^{iHt}$ of a (pseudo-)group $\mathcal{G}_H \subset \mathcal{A}$ of elements of $\mathcal{A}$, just as in quantum field theory. The difference lies in the 'size' of the subalgebra $\mathcal{A}_G$ in which $\mathcal{G}$ takes its values, as can be measured for example by the dimension of the commutant of the subalgebra generated by it; whereas in general $\dim(\mathcal{A}'_G) = \dim(\mathcal{A}')$, since gauge transformations are relatively unimportant, in general $\dim(\mathcal{A}'_H) = 0$. A topological field theory has $\dim(\mathcal{A}'_G) = \dim(\mathcal{A}'_H)$.

A Riemann-flat solution to the problem is given by choosing

$$e^\mu_\alpha = \delta^\mu_\alpha, \qquad K_{\alpha\beta} = -\frac{1}{i\hbar}\theta^{-1}_{\alpha\beta} \in \mathcal{Z}(\mathcal{A}).$$

We have introduced the inverse matrix $\theta^{-1}_{\alpha\beta}$ of $\theta^{\alpha\beta}$; we must suppose the Poisson structure to be non-degenerate: $\det\theta^{\alpha\beta} \neq 0$. The relations can be written in the form

$$\lambda_\alpha = -K_{\alpha\mu}x^\mu, \qquad [\lambda_\alpha, \lambda_\beta] = K_{\alpha\beta}. \tag{2.20}$$

This structure is flat according to our definitions.

We shall find it convenient to consider a curved geometry as a perturbation of a noncommutative flat geometry. The measure of noncommutativity is the parameter $\hbar$; the measure of curvature is the quantity μ^2. There are two special interesting limits. If we keep $\hbar\mu^2$ small but fixed then we can let $\hbar \to 0$ or $\hbar \to \infty$. The former (latter) corresponds to a 'small' ('large') universe filled with 'small' ('large') cells. The number of cells is given by $(\hbar\mu^2)^{-1}$. We can assume the flat-space limit to have commutation relations of the form (2.1) with

$$J^{\mu\nu} = \theta^{\mu\nu}(1 + o(i\hbar\mu^2)).$$

3 The Kasner metric

All quantities in this section are commutative and should have a tilde on them to emphasize this fact. For notational simplicity however we drop this symbol. Choose a symmetric matrix $p = (P^a_b)$ of real numbers. A moving frame for the Kasner metric is given by

$$\theta^0 = dt, \qquad \theta^a = dx^a - P^a_b x^b t^{-1} dt. \tag{3.1}$$

The 1-forms θ^α are dual to the derivations

$$e_0 = \partial_t + P^i_j x^j t^{-1}\partial_i, \qquad e_a = \partial_a$$

of the algebra $\mathcal{A}$. The space $\mathcal{X}$ of all derivations is free of rank 4 as an $\mathcal{A}$-module and the e_α form a basis. The Lie-algebra structure of $\mathcal{X}$ is given by the commutation relations

$$[e_a, e_0] = C^b{}_{a0}e_b, \qquad [e_a, e_b] = 0 \tag{3.2}$$

with

$$C^b{}_{a0} = P^b_a t^{-1}.$$

For fixed time it is a solvable Lie algebra which is not nilpotent. We have written the frame in coordinates which are adapted to the asymptotic condition.

The expression for $C^b{}_{a0}$ contains no parameters with dimension but it has the correct physical dimensions. Let G_N be Newton's constant and μ a mass such that $G_N\mu$ is a length scale of cosmological order of magnitude. As a first guess we would like to identify the length scale determined by $\kbar$ with the Planck scale: $\hbar G_N \sim \kbar$ and so we have $\kbar \sim 10^{-87}\mathrm{sec}^2$ and since μ^{-1} is the age of the universe we have $\mu \sim 10^{-17}\mathrm{sec}^{-1}$. The dimensionless quantity $\kbar\mu^2$ is given by $\kbar\mu^2 \sim 10^{-120}$. We saw, and we shall see below, that the spectrum of the commutator of two momenta is the sum of a constant term of order $\kbar^{-1}$ and a 'gravitational' term of order $\mu t^{-1} = \kbar^{-1} \times (\kbar\mu)t^{-1}$. So the gravitational term in the units we are using is relatively important for $t \lesssim \kbar\mu$. The existence of the constant term implies that the gravitational field is not to be identified with the noncommutativity *per se* but rather with its variation in space and time. There is of course no evidence either in favor or against this assumption. We make it for reasons of convenience: it is easier to perturb an existing noncommutative structure and the constant term affords us with a convenient starting point for a perturbative expansion.

The components of the curvature form are given by

$$\Omega^a{}_0 = (P^2 - P)^a_b t^{-2}\theta^0\theta^b, \tag{3.3}$$

$$\Omega^a{}_b = -\tfrac{1}{2}P^a_{[c}P_{d]b}t^{-2}\theta^c\theta^d. \tag{3.4}$$

The curvature form is invariant under a uniform scaling of all coordinates. The Ricci tensor has components

$$R_{00} = \mathrm{Tr}(P^2 - P)t^{-2}, \qquad R_{ab} = (\mathrm{Tr}(P) - 1)P_{ab}t^{-2}.$$

The vacuum field equations reduce then to the equations

$$\mathrm{Tr}(P) = 1, \qquad \mathrm{Tr}(P^2) = 1.$$

If p_a are the eigenvalues of the matrix P^a_b there is a 1-parameter family of solutions given by

$$p_a = \frac{1}{1+\omega+\omega^2}(1+\omega,\ \omega(1+\omega),\ -\omega). \tag{3.5}$$

The most interesting value is $\omega = 1$ in which case

$$p_a = \tfrac{1}{3}(2,\ 2,\ -1).$$

The curvature invariants are proportional to t^{-2}; they are singular at $t = 0$ and vanish as $t \to \infty$.

We shall naturally be lead to consider a family of metrics obtained from the Kasner metric by a redefinition of the matrix P which depends on a vector k^a, eigenvector of P with eigenvalue q. The previous formulae will be modified. This energy-density we interpret as due to the existence of supplementary noncommutative dimensions. We start then with a Kasner solution in dimension 4 and we add a fuzzy structure which forces us out of the 1-parameter family of vacuum solutions into a family of solutions with similar properties except for the existence of a pressure-free distribution of energy density.

4 The Kasner algebra

The Kasner metric is of Petrov type I and has four distinct principal null vectors. The limiting Poisson structure defines an additional two principle null vectors. We must also choose the frame so that it is in some way adapted to these vectors. A major problem, which has not been solved, is to possess a criterium by which one can decide if the frame is well-chosen. From the form of the principal null directions in the present case we conclude that the frame is properly aligned with respect to them except possibly for a rotation around the k-axis. We recall that the vectors k^a and l^a come from the limiting symplectic structure and P^a_d from the limiting metric. It is most convenient to describe the algebra using the momentum generators.

Following the general argument we introduce an array of polynomials $L_{\alpha\beta}$ quadratic in the momenta and write

$$[\lambda_a, \lambda_b] = K_{ab} + L_{ab}, \qquad [\lambda_0, \lambda_a] = K_{0a} + L_{0a}.$$

To stress the fact that the $K_{\alpha\beta}$ diverges when $\bar{k} \to 0$ we write

$$K_{0a} = (i\bar{k})^{-1} l_a, \qquad K_{ab} = (i\bar{k})^{-1} \epsilon_{abc} k^c$$

with two space-like vectors l_a and k^a. To determine the form of $L_{\alpha\beta}$ we must consider the commutative limit. Since the Kasner metric is a vacuum solution and noncommutativity gives rise naturally to a cosmological constant we shall only be able to recover the Kasner metric as a double limit; first we must take a limit $\bar{k} \to 0$ to recover a commutative theory and then we must take the limit $\alpha \to 0$ to recover Kasner's solution. That is we shall not be able to define a series of geometries which are strictly speaking noncommutative versions of the Kasner metric.

From the classical limit we must choose the $L_{\alpha\beta}$ so that for arbitrary $f \in \mathcal{A}$

$$[e_a, e_b]f = 0, \qquad [e_a, e_0]f - P^b_a \tau(t) e_b f = 0.$$

We have here introduced the element τ of the subalgebra of $\mathcal{A}$ generated by t which must tend to t^{-1} in the commutative limit. From the Leibniz rules we find that

$$[L_{ab}, f] = 0, \qquad [L_{a0}, f] = P^b_a \tau e_b f. \tag{4.1}$$

The integrability condition for this system is the condition

$$e_a\tau = 0. \tag{4.2}$$

From the correspondence principle and the structure of the Kasner metric we obtain the momentum-position commutation relations

$$\begin{aligned} &[\lambda_0, t] = 1, \quad [\lambda_0, x^b] = P^b_c(\tau x^c - \tfrac{1}{2}[\tau, x^c]), \\ &[\lambda_a, t] = 0, \quad [\lambda_a, x^b] = \delta^b_a. \end{aligned} \tag{4.3}$$

It follows in particular that the condition (4.2) is satisfied.

To solve (formally) Equations (4.1) it suffices to chose $f = x^\mu$. We obtain then with $f = t$ that $L_{\alpha\beta}$ is a function of t alone and with $f = x^a$ that

$$L_{ab} = 0, \qquad L_{a0} = \tau P^b_a \lambda_b.$$

To allow for a family of metrics which includes the Kasner solution we shall add to the commutation relations a term of the form

$$M_{\alpha\beta} = \alpha m^{-2}\tau^2 K_{\alpha\beta}, \qquad m^2 = \mu^2 c^2, \quad c^2 = k^a l_a$$

with α an arbitrary real number. The components M_{a0} can be absorbed into a redefinition

$$Q^b_a = P^b_a + \alpha c^{-2} k^b l_a$$

of P^b_a. We arrive finally at an algebra defined by the relations

$$[\lambda_a, \lambda_b] = (i\hbar)^{-1}(1 + \alpha\frac{\tau^2}{m^2})\epsilon_{abc}k^c, \tag{4.4}$$

$$[\lambda_0, \lambda_a] = (i\hbar)^{-1}(l_a - i\hbar\tau Q^b_a \lambda_b) \tag{4.5}$$

with

$$Q^b_a\tau \to P^b_a t^{-1}.$$

We must now find an explicit expression for τ.

If we multiply both sides of (4.5) by k^a then we find the equation

$$e_0(-i\hbar\mu^2 k^a \lambda_a) + m^2 + \tau(-i\hbar\mu^2 Q^a_b k^b \lambda_a) = 0.$$

It would seem that the only natural way to solve this equation is to choose the symplectic form such that k^a is an eigenvector of Q^a_b. Let q be the associated eigenvalue and define the element

$$\tau = -i\hbar\mu^2 k^a Q^b_a \lambda_b = q(-i\hbar\mu^2 k^a \lambda_a)$$

which must satisfy the equation

$$\dot\tau + q(m^2 + \tau^2) = 0.$$

Since we have added an extra term to Equation (4.4) we must assure that the modified Jacobi identities are also satisfied. That is we must verify that the equation

$$[\lambda_0, L_{ab}] = [L_{0a}, \lambda_b] + [\lambda_a, L_{0b}]$$

is an identity. It can be written as

$$\dot\tau + \frac{1}{2}(\mathrm{Tr}\, Q - q)(\frac{m^2}{\alpha} + \tau^2) = 0.$$

It can be considered an identity if it coincides with the previous equation for τ. This will be the case if and only if

$$\mathrm{Tr}\, Q = 3q, \qquad \alpha = 1.$$

The ansatz is such that there is no real asymptotic region. The restriction on α means that there is only one metric in the 1-parameter family around the Kasner which can be made noncommutative.

From the definition of Q^a_b we find that

$$\mathrm{Tr}\, Q = 2, \qquad q_a = \tfrac{2}{3}.$$

Therefore $p_1 = p_2 = 2/3$ and since

$$p + c^{-2}k^3 l_3 \to q$$

we conclude that

$$p = \tfrac{2}{3} - 1 = -\tfrac{1}{3}.$$

This is a consistency check since we have already assumed that $\mathrm{Tr}\, P = 1$. Having the expression for τ we can also conclude that the extra terms in the commutation relations can be written in the form used in Section 2. They correspond respectively to

$$Q^{cd}{}_{a0} = \tfrac{1}{4}\mu^2 k^{(c} Q^{d)}_a, \qquad Q^{cd}{}_{ab} = -\tfrac{1}{2}\alpha m^2 k^c k^d \epsilon_{abe} k^e.$$

We shall refer to the equation for τ as the dynamical equation. It is a direct consequence of the Jacobi identities.

For ω of the form $\omega = -1 + \epsilon$ such that one can write

$$p_a = (-\epsilon + \epsilon^2, \epsilon, 1 - \epsilon^2) + o(\epsilon^3)$$

one finds that $q = 1 - \epsilon^2$. If one choose then

$$\alpha = \epsilon^2$$

one finds the equation

$$\dot\tau + \epsilon^2\tau^2 + m^2 = 0.$$

The dynamical equation is invariant under the action of the 'duality' transformation

$$\tau \mapsto -m^2\tau^{-1}.$$

The functions

$$\tau = m\cot(qmt) \simeq \frac{1}{qt}, \qquad m^2 > 0, \tag{4.6}$$

$$\tau = |m|\coth(q|m|t) \simeq \frac{1}{qt}, \qquad m^2 < 0, \tag{4.7}$$

$$\tau = |m|\tanh(q|m|t) \simeq q|m|^2 t, \qquad m^2 < 0. \tag{4.8}$$

are solutions. The small-t expansion is a strong-field limit; the Kasner metric would be a weak-field limit for large values of μt, provided that $c \to 0$.

We end this section by considering a scaling limit wherein the mass-term vanishes We do this by scaling the kinetic terms such that they progressively dominate. We set accordingly

$$\tau = \nu^{-1}\tau_0, \qquad t = \nu t_0.$$

By the relations (4.3) we must scale the momenta as $\lambda_\alpha = \nu\lambda_{0\alpha}$ and therefore we must scale $\kappa\!\!\!{}^{-} = \nu^{-2}\kappa\!\!\!{}^{-}_0$ and $K_{\alpha\beta} = \nu^2 K_{0\alpha\beta}$ as one would expect by dimension analysis. The dynamical equation becomes then

$$\dot\tau + q(\tau^2 + (\nu m)^2) = 0.$$

Alternatively one can scale the mass parameter $m = \nu m_0$. The 'fudge' term $M_{\alpha\beta}$ is scale-invariant.

Some information can be obtained concerning the $J^{\mu\nu}$ introduced in (2.1) by simply looking at the commutation relations and the associated Jacobi relations. More information can be obtained by considering representations, as we shall do in the next subsection. From the Jacobi identities with two position operators and one momentum operator we deduce that $e_a J^{\mu\nu} = 0$ and thereby the dependence of the commutator only on the generator τ. To use a covariant derivative on $J^{\mu\nu}$ we must be able to express the latter using frame indices.

Having 'blurred' the Kasner metric and deformed the resulting algebra we can now take the 'sharp' limit and see what we obtain. With the form of P^a_b we have the metric cannot be Ricci-flat but has an induced cosmological constant due to the noncommutativity [14]. The metric is that of the flat FRW universe filled with dust. We refer elsewhere [22] for a discussion of this point. The theory we are investigating has certain similarities with theories of the type called Kaluza-Klein. That is, the additional noncommutative structure can perhaps at least to a certain extent be assimilated to an effective commutative theory in higher dimensions. This means that even if one could define a curvature tensor in a satisfactory manner there is no reason to expect the Ricci tensor to vanish. One can assume that to the lowest approximation the Ricci tensor of the total structure does vanish and use the Ricci tensor of the four dimensions to elucidate the structure of the hidden dimensions.

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