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Poisson configuration spaces, von Neumann algebras, and harmonic forms

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Abstract

We give a short review of recent results on L^2-cohomology of infinite configuration spaces equipped with Poisson measures.

1 Introduction

Let X be a complete, connected, oriented Riemannian manifold of infinite volume without boundary, with curvature bounded below. The configuration space Γ_X over X is defined as the set of all locally finite subsets (configurations) in X:

$$\Gamma_X := \{\gamma \subset X : \gamma \cap \Lambda \text{ is finite for each compact } \Lambda \subset X\} . \tag{1.1}$$

The theory of configuration spaces has received a great interest in recent time. This can be explained by various important applications in statistical mechanics and quantum field theory, as well as by the rich and interesting intrinsic structure of Γ_X. Γ_X represents a bright example of an infinite dimensional "manifold-like" space. However, it does not possess any proper structure of a Hilbert or Banach manifold. On the other hand, many important geometrical objects, like differential forms, connections and the de Rham complex over Γ_X can be introduced in a specific way. A crucial role here is played by a probability measure on Γ_X (in particular, a Poisson or Gibbs measure), which is quasi-invariant with respect to the action of a group of diffeomorphisms of X. This philosophy, inspired by the pioneering works [1] and [2], has been initiated and developed in [3], [4] and has lead to many interesting and important results in the field of stochastic analysis on configuration spaces and its applications, see also references in [5], [6].

In this note, we give a short review of recent results on the L^2-cohomology of Γ_X equipped with the Poisson measure π. In Section 2, we define and study the de Rham complex of π-square-integrable differential forms over Γ_X and the corresponding Laplacian acting in this complex. We describe the structure of the spaces of harmonic forms. In Section 3, we consider the case where X is an infinite covering of a compact manifold, and compute the L^2-Betti numbers of Γ_X. That is, we introduce a natural von Neumann algebra containing projections onto the spaces of harmonic forms, and compute their traces. Further, we introduce and compute a regularized index of the Dirac operator associated with the de Rham differential on Γ_X. For a more detailed exposition, see [5], [11], [7], [8] and references given there.

Let us remark that the situation changes dramatically if the Poisson measure π is replaced by a different (for instance Gibbs) measure. From the physical point of view, this describes a passage from a system of particles without interaction (free gas) to an interacting particle system, see [4] and references within. For a wide class of measures, including Gibbs measures of Ruelle type and Gibbs measures in low activity- high temperature regime, the de Rham complex was introduced and studied in [6]. The structure of the corresponding Laplacian is much more complicated in this case, and the spaces of harmonic forms have not been studied yet.

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2 De Rham complex over a configuration space

Following [2], [3], we define the tangent space to Γ_X at a point γ as the Hilbert space

$$T_\gamma \Gamma_X = \bigoplus_{x\in\gamma} T_x X \tag{2.1}$$

(we fix a Riemannian structure on X). Under a differential form W of order n over Γ_X, we will understand a mapping

$$\Gamma_X \ni \gamma \mapsto W(\gamma) \in (T_\gamma \Gamma_X)^{\wedge n}. \tag{2.2}$$

Let $\gamma \in \Gamma_X$ and $x \in \gamma$. By $\mathcal{O}_{\gamma,x}$ we will denote an arbitrary open neighborhood of x in X such that $\mathcal{O}_{\gamma,x} \cap (\gamma \setminus \{x\}) = \varnothing$. We define the mapping

$$\mathcal{O}_{\gamma,x} \ni y \mapsto W_x(\gamma, y) := W(\gamma_y) \in (T_{\gamma_y}\Gamma_X)^{\wedge n},\ \gamma_y := (\gamma \setminus \{x\}) \cup \{y\}. \tag{2.3}$$

This is a section of the Hilbert bundle $(T_{\gamma_y}\Gamma_X)^{\wedge n} \mapsto y \in \mathcal{O}_{\gamma,x}$.The Levi–Civita connection on TX generates in a natural way a connection on this bundle. We denote by $\nabla^X_{\gamma,x}$ the corresponding covariant derivative and use the notation

$$\nabla^\Gamma W(\gamma) := (\nabla^X_{\gamma,x} W_x(\gamma, x))_{x\in\gamma} \in T_\gamma\Gamma_X \otimes (T_\gamma\Gamma_X)^{\wedge n}. \tag{2.4}$$

Let π be the Poisson measure on Γ_X with intensity given by the volume measure. We define the Hilbert space $L^2_\pi\Omega^n$ of complex n-forms which are π-square integrable.

We will now give an isomorphic description of the space $L^2_\pi\Omega^n$ via the space $L^2_\pi\Omega^0$ of π-square integrable functions on Γ_X and spaces $L^2\Omega^n(X^m)$ of square-integrable complex forms on X^m, $m = 1, \ldots, n$. We have

$$(T_\gamma\Gamma_X)^{\wedge n} = \bigoplus_{m=1}^{n} \bigoplus_{\eta\subset\gamma, |\eta|=m} \mathbb{T}^{(n)}_\eta X^m, \tag{2.5}$$

where $|\eta|$ means the number of points in η, and $\mathbb{T}_\eta^{(n)} X^m := \bigoplus_{k_1+\cdots+k_m=n} (T_{x_1}X)^{\wedge k_1} \wedge \cdots \wedge (T_{x_m}X)^{\wedge k_m}$ for any finite configuration $\eta = \{x_1, ..., x_m\}$. We denote by $W_m(\gamma;\eta)$ the projection of $W(\gamma) \in (T_\gamma \Gamma_X)^{\wedge n}$ onto the subspace $\mathbb{T}_\eta^{(n)} X^m$.

Theorem 1. [5] *Setting for $m = 1, \ldots, n$*

$$(I_n W)(\gamma; x_1, \ldots, x_m) := \frac{1}{\sqrt{m!}} W_m(\gamma \cup \{x_1, \ldots, x_m\}; \{x_1, \ldots, x_m\}), \tag{2.6}$$

we obtain the isometry

$$I_n : L^2_\pi \Omega^n \to L^2_\pi \Omega^0 \bigotimes \left[\bigoplus_{m=1}^{n} L^2 \Omega^n (X^m) \right]. \tag{2.7}$$

We define the de Rham differential $\mathbf{d}_n \colon L^2_\pi \Omega^n \to L^2_\pi \Omega^{n+1}$ by the formula

$$(\mathbf{d}_n W)(\gamma) := (n+1)^{1/2} \, \mathrm{AS}_{n+1}(\nabla^\Gamma W(\gamma)), \tag{2.8}$$

where $\mathrm{AS}_{n+1} \colon (T_\gamma \Gamma_X)^{\otimes (n+1)} \to (T_\gamma \Gamma_X)^{\wedge (n+1)}$ is the anti-symmetrization. Let $\mathbf{d}_n^*$ be the adjoint operator.

Theorem 2. [5] $\mathbf{d}_n^* : L^2_\pi \Omega^{n+1} \to L^2_\pi \Omega^n$ *is a densely defined operator.*

Thus the operator $\mathbf{d}_n$ is closable. We denote its closure by $\bar{\mathbf{d}}_n$ and introduce an infinite Hilbert complex

$$L^2_\pi \Omega^0 \xrightarrow{\bar{\mathbf{d}}_0} \cdots \xrightarrow{\bar{\mathbf{d}}_{n-1}} L^2_\pi \Omega^n \xrightarrow{\bar{\mathbf{d}}_n} L^2_\pi \Omega^{n+1} \xrightarrow{\bar{\mathbf{d}}_{n+1}} \cdots . \tag{2.9}$$

Define in the standard way $\mathbf{H}_\pi^{(n)} := \bar{\mathbf{d}}_{n-1} \mathbf{d}_{n-1}^* + \mathbf{d}_n^* \bar{\mathbf{d}}_n$, which is a self-adjoint operator in $L^2_\pi \Omega^n$. It will be called the Hodge-de Rham Laplacian of the Poisson measure π. Standard operator arguments show that the Hilbert spaces $\mathcal{H}_\pi^n := \mathrm{Ker}\, \bar{\mathbf{d}}_n / \mathrm{clo}\, \{\mathrm{Im}\, \mathbf{d}_{n-1}\}$ and $\mathrm{Ker}\, \mathbf{H}_\pi^{(n)}$ are canonically isomorphic, and we identify them.

Theorem 3. [5] *1) Let $H_{X^m}^{(n)}$ be the Hodge-de Rham Laplacian in $L^2 \Omega^n (X^m)$. Then:*

$$I_n \mathbf{H}_\pi^{(n)} = \left(\mathbf{H}_\pi^{(0)} \otimes \mathbf{1} + \mathbf{1} \otimes \left(\bigoplus_{m=1}^{n} H_{X^m}^{(n)} \right) \right) I_n, \tag{2.10}$$

2) The isometry I_n generates the unitary isomorphism of Hilbert spaces

$$\mathcal{H}_\pi^n = \mathrm{Ker}\, \mathbf{H}_\pi^{(n)} \simeq \bigoplus_{\substack{s_1, \ldots, s_d = 0, 1, 2 \ldots \\ s_1 + 2s_2 + \ldots + ds_d = n}} (\mathcal{H}^1(X))^{\overset{1}{\diamond} s_1} \otimes \cdots \otimes (\mathcal{H}^d(X))^{\overset{d}{\diamond} s_d}, \tag{2.11}$$

where $\mathcal{H}^m(X) := \mathrm{Ker}\, H_X^{(m)}$, $m = 1, 2, ..., d$, $d = \dim X - 1$, and $\overset{m}{\diamond} s$ means the symmetric tensor power when m is even and skew-symmetric tensor power when m is odd.

Remark 1. $\mathbf{H}_\pi^{(0)}$ is the Dirichlet operator of the Poisson measure π introduced in [3]. In particular, it was shown there that $\mathrm{Ker}\, \mathbf{H}_\pi^{(0)}$ consists of constant functions, which together with (2.10) implies (2.11).

Remark 2. Formula (2.11) holds also for spaces of finite configurations, see [8]. In fact, it is a “symmetrized” version of the Künneth formula.

We see from (2.11) that all spaces $\mathcal{H}^n_\pi$, $n \in \mathbb{N}$, are finite dimensional provided the spaces $\mathcal{H}^m(X)$, $m = 1, ..., d$, are so. In this case,

$$\dim \mathcal{H}^n_\pi = \sum_{\substack{s_1, ..., s_d = 0,1,2... \\ s_1 + 2s_2 + ... + ds_d = n}} \beta_1^{(s_1)} \cdots \beta_d^{(s_d)}, \tag{2.12}$$

where $\beta_k^{(s)} := \begin{pmatrix} \beta_k \\ s \end{pmatrix}$, when k is odd, $\beta_k^{(s)} := \begin{pmatrix} \beta_k + s - 1 \\ s \end{pmatrix}$, when k is even, and $\beta_k^{(0)} := 1$. Here $\beta_k := \dim \mathcal{H}^k(X)$.

3 L^2 Betti numbers of configuration spaces of coverings

An important example of a manifold X with infinite dimensional spaces $\mathcal{H}^p(X)$ is given by an infinite covering of a compact Riemannian manifold (say M). In this case, an infinite discrete group G acts freely by isometries on X and consequently on all spaces $L^2\Omega^p(X)$, and $X/G = M$. The orthogonal projection

$$\mathcal{P}_p : L^2\Omega^p(X) \to \mathcal{H}^p(X) \tag{3.1}$$

commutes with the action of G and thus belongs to the commutant $\mathcal{A}_p$ of this action which is a semifinite von Neumann algebra. The corresponding von Neumann trace $b_p := \mathrm{Tr}_{\mathcal{A}_p}\mathcal{P}_p$ gives a regularized dimension of the space $\mathcal{H}^p(X)$ and is called the L^2-Betti number of X (or M). L^2-Betti numbers were introduced in [9] and have been studied by many authors (see e.g. [10] and references given there). It is known [9] that $b_p < \infty$.

It is natural to ask whether the notion of L^2-Betti numbers can be extended to configuration spaces over infinite coverings. It particular, is formula (2.12) valid in this case (with β_k replaced by b_k)? In what follows, we construct a natural von Neumann algebra containing the orthogonal projection

$$\mathbb{P}_n : L^2_\pi\Omega^n \to \mathcal{H}^n_\pi, \tag{3.2}$$

and compute its von Neumann trace. We assume that G is an ICC group (that is, all nontrivial classes of conjugate elements are infinite). Under this condition, the von Neumann algebra $\mathcal{A}_p$ is a II_∞ factor. In the sequel, $\mathrm{Tr}_\mathcal{N}$ denotes the faithful normal semifinite trace on a II_∞ factor $\mathcal{N}$.

The following general statement holds. Let $\mathcal{M}$ be a II_∞ factor and H be a separable $\mathcal{M}$-module. Let $S_m \ni g \longmapsto U_g$ be the natural action of the symmetric group S_m in $H^{\otimes m}$ by permutations, and let $P_s = \frac{1}{m!}\sum_{g\in S_m} U_g$ and $P_a = \frac{1}{m!}\sum_{g\in S_m}(-1)^{sign(g)}U_g$ be projections in $H^{\otimes m}$ onto symmetric tensor product $H^{\hat{\otimes} m}$ and antisymmetric tensor powers $H^{\wedge m}$ respectively. Denote $\mathcal{M}_s^{(m)} := \{\mathcal{M}^{\otimes m}, P_s\}''$, $\mathcal{M}_a^{(m)} := \{\mathcal{M}^{\otimes m}, P_a\}''$. Let $W^*(\mathcal{M}^{\otimes m}, S_m)$ be the cross-product of $\mathcal{M}^{\otimes m}$ and S_m.

Theorem 4. [8] *$\mathcal{M}_s^{(m)}$ and $\mathcal{M}_a^{(m)}$ are II_∞ factors isomorphic to $W^*(\mathcal{M}^{\otimes n}, S_m)$, and for any $A \in \mathcal{M}$*

$$\mathrm{Tr}_{\mathcal{M}_s^{(m)}}(A^{\otimes m} P_s) = \mathrm{Tr}_{\mathcal{M}_a^{(m)}}(A^{\otimes m} P_a) = \frac{(\mathrm{Tr}_{\mathcal{M}} A)^m}{m!}. \tag{3.3}$$

Let us consider the orthogonal projection $\mathcal{P}_p^{(m)} : \left(L^2\Omega^p(X)\right)^{\otimes m} \to (\mathcal{H}^p(X))^{\overset{p}{\diamond} m}$. Theorem 4 shows that $\mathcal{A}_p^{(m)} := \{\mathcal{A}_p^{\otimes m}, \mathcal{P}_p^{(m)}\}'' = W^*(\mathcal{A}_p^{\otimes m}, S_m)$, and $\mathrm{Tr}_{\mathcal{A}_p^{(m)}} \mathcal{P}_p^{(m)} = \frac{(b_p)^m}{m!}$.

Next, we introduce the von Neumann algebra

$$\mathbf{A}_n := \prod_{\substack{s_1,\dots,s_d = 0,1,2\dots \\ s_1 + 2s_2 + \dots + ds_d = n}} \mathcal{A}_1^{(s_1)} \otimes \dots \otimes \mathcal{A}_d^{(s_d)}. \tag{3.4}$$

Since all algebras $\mathcal{A}_k^{(s_k)}$ are II_∞ factors, so is $\mathbf{A}_n$, with the trace defined by the traces in $\mathcal{A}_k^{(s_k)}$.

Theorem 5. [11] *Let $\mathbf{P}_n = I_n \mathbb{P}_n I_n^{-1}$. Then: $\mathbf{P}_n \in \mathbf{A}_n$ and*

$$\mathrm{Tr}_{\mathbf{A}_n} \mathbf{P}_n = \sum_{\substack{s_1,\dots,s_d = 0,1,2\dots \\ s_1 + 2s_2 + \dots + ds_d = n}} \frac{(b_1)^{s_1}}{s_1!} \dots \frac{(b_d)^{s_d}}{s_d!}, \tag{3.5}$$

where $b_1, \dots, b_d$ are the L^2-Betti numbers of X.

Theorem 4 implies that $\mathbf{A}_n$ is the minimal natural von Neumann algebra containing $\mathbf{P}_n$. We will use the notation $\mathbf{b}_n := \mathrm{Tr}_{\mathbf{A}_n} \mathbf{P}_n$ and call $\mathbf{b}_n$ the n-th L^2-Betti number of Γ_X.

Example. Let $X = \mathbb{H}^d$, the hyperbolic space of dimension d. It is known that the only non-zero L^2-Betti number of $\mathbb{H}^d$ is $b_{d/2}$ (provided d is even). Then

$$\mathbf{b}_n = \begin{cases} \frac{(b_{d/2})^k}{k!}, \ n = \frac{kd}{2}, \ k \in \mathbb{N} \\ 0, \ n \neq \frac{kd}{2}, \ k \in \mathbb{N} \end{cases}. \tag{3.6}$$

Let us introduce a regularized index $\mathrm{ind}_{\Gamma_X}(\mathbf{d} + \mathbf{d}^*)$ of the Dirac operator associated with the de Rham differential $\mathbf{d} : L^2\Omega_\pi^{even} \to L^2\Omega_\pi^{odd}$ setting

$$\mathrm{ind}_{\Gamma_X}(\mathbf{d} + \mathbf{d}^*) := \sum_{k=0}^{\infty} (-1)^k \mathbf{b}_k. \tag{3.7}$$

Theorem 6. [11] *The series on the right hand side of (3.7) converges absolutely, and*

$$\mathrm{ind}_{\Gamma_X}(\mathbf{d} + \mathbf{d}^*) = e^{\chi(M)}, \tag{3.8}$$

where $\chi(M)$ is the Euler characteristic of M.

Corollary. The L^2-cohomology of Γ_X is infinite provided $\chi(M) \neq 0$.

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