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On the Transformations of the Sixth Painlevé Equation

Valery I GROMAK † *and Galina FILIPUK* †

† *Department of differential equations, Belarusian State University, 4 Skorina Ave., Minsk, 220050, Belarus*
E-mail: grom@bsu.by
E-mail: filipuk@bsu.by

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Abstract

In this paper we investigate relations between different transformations of the solutions of the sixth Painlevé equation. We obtain nonlinear superposition formulas linking solutions by means of the Bäcklund transformation. Algebraic solutions are also studied with the help of the Bäcklund transformation.

1 Introduction

In this paper we are concerned with relations between different transformations of the sixth Painlevé equation given by

$$\frac{d^2y}{dt^2}=\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}+$$
$$+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right), \tag{1.1}$$

Equation (1.1) has three fixed singular points $z=0,1,\infty$ and it is the most general among other Painlevé equations as they can be obtained from (1.1) by a confluence procedure [14]. In general, equation (1.1) cannot be integrated in terms of classical transcendental functions [11]. However, under suitable conditions imposed on the parameters, it may be integrated in terms of elliptic functions or it admits one-parameter families of solutions which can be expressed by means of hypergeometric functions [12]. Solutions of equation (1.1) are invariant with respect to symmetry transformations which form a group generated by transformations $S_j : y(t,\alpha,\beta,\gamma,\delta) \to y_j(t_j,\alpha_j,\beta_j,\gamma_j,\delta_j)$, $j\in\{1,2,3\}$, where $y_1(t,-\beta,-\alpha,\gamma,\delta)=y^{-1}(1/t)$; $y_2(t,-\beta,-\gamma,\alpha,\delta)=1-y^{-1}(1/(1-t))$; $y_3(t,-\beta,-\alpha,-\delta+1/2,-\gamma+1/2)=ty^{-1}(t)$. An application of the transformations $S_j,\ j\in\{1,\dots,24\}$ from the group of symmetry transformations implies a change of any two parameters in accordance with the scheme $\alpha\to-\beta\to\gamma\to 1/2-\delta\to\alpha$ [12]. Equation (1.1) also admits Bäcklund transformations [5] originally found in [21] and Schlesinger transformations which link solutions of (1.1) with different parameter values. These transformations involve derivatives of the initial solution. Depending on the choice of the branches of the parameters, the Bäcklund transformation generally generates sixteen different solutions. Nonlinear superposition formulas linking them are studied in Section 2. Although the Schlesinger transformations were extensively studied in papers [4, 20], in Section 3 we present new formulas of a connection between Schlesinger and Bäcklund transformations taking into account the parameter branching. There exist other transformations for the solutions of the sixth Painlevé equation, namely, quadratic transformations found in [16, **?**, 22] which may be considered as analogues of the quadratic transformations of hypergeometric functions. A specific feature of these transformations is that the independent variables of the solutions of (1.1) are connected by some quadratic formula. In Section 4 a new relation between different quadratic transformations is obtained. Finally, we apply the Bäcklund transformation to generate new algebraic solutions of equation (1.1).

2 The Bäcklund transformation

The Bäcklund transformation is given by the following theorem [21] (see also [5, 12]).

Theorem 1. *Let $y(t) = y(t, \alpha, \beta, \gamma, \delta)$ be a solution of* (1.1)*, such that* $R(t, y) := t(t-1)y' + (\eta_2 + \eta_3 + \eta_4 - 1)y^2 - (t\eta_2 + t\eta_3 + \eta_2 + \eta_4 - 1)y + t\eta_2 \not\equiv 0$. *Then the transformation*

$$T : y(t) \to \widetilde{y}(t) = y - (\eta_1 + \eta_2 + \eta_3 + \eta_4 - 1)y(y-1)(y-t)/R(t, y), \tag{2.1}$$

where $\eta_1^2 = 2\alpha$, $\eta_2^2 = -2\beta$, $\eta_3^2 = 2\gamma$, $\eta_4^2 = 1 - 2\delta$, *determines solution* $\widetilde{y}(t)$ *of equation* (1.1) *with parameter values* $2\widetilde{\alpha} = \widetilde{\eta}_1^2$, $2\widetilde{\beta} = -\widetilde{\eta}_2^2$, $2\widetilde{\gamma} = \widetilde{\eta}_3^2$, $1 - 2\widetilde{\delta} = \widetilde{\eta}_4^2$, *and* $\widetilde{\eta}_j = \eta_j - (\eta_1 + \eta_2 + \eta_3 + \eta_4)/2 + 1/2$, $j \in \{1, \dots, 4\}$.

It follows immediately from Theorem 1 that a given solution $y(t, \alpha, \beta, \gamma, \delta)$, which is called a solution of the zero level, generates in general case sixteen different solutions $y_i(t, \alpha_i, \beta_i, \gamma_i, \delta_i)$, $i \in \{1, \dots, 16\}$, of the first level due to the parameter branching. In repeated applications of the transformation T in (2.1) the choice of the signs of η_j is independent at each step. Let us denote the choice of the signs of η_j at the $n-$th step by $\varepsilon_j^{(n)}$, $(\varepsilon_j^{(n)})^2 = 1$, and transformation T by $T_{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4}$. The corresponding solutions obtained at the $n-$th step of the application of $T-$transformation are called the solutions of the $n-$th level. It can easily be verified that $T_{1,1,1,1} \circ T_{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4} = I$, where I is an identical transformation.

Next we obtain some nonlinear superposition formulas linking solutions of (1.1) obtained by repeated application of the Bäcklund transformation (2.1). Below we consider a general case assuming $\eta_i \neq 0$, $i \in \{1, \dots, 4\}$. When $\eta_i = 0$, $i \in \{1, \dots, 4\}$, all solutions of the first level coincide. If $\eta_1 \neq 0$, $\eta_i = 0$, $i \in \{2, 3, 4\}$, then we can apply the Bäcklund transformation of Theorem 1 provided $y(t) \neq t/(-e^{C_1}(t-1) + t)$ with $\alpha = 1/2$, C_1 being constant. In this case we get two different solutions which are related by $y_2 = (y_1 + y_1\eta_1 - 2\eta_1 y)/(1 - \eta_1)$, where $y_1 = T_{1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$, $y_2 = T_{-1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$. Observe that the nonlinear superposition formulas obtained below may be considered as an alternative version of discrete Painlevé equations, see [8, 10].

Theorem 2. *A seed solution and any two solutions of the first level are algebraically dependent.*

Proof. Let $y_k = T_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}y$ and $y_l = T_{\tilde{\varepsilon_1},\tilde{\varepsilon_2},\tilde{\varepsilon_3},\tilde{\varepsilon_4}}y$ be different solutions of the first level. Eliminating $y'(t)$ between them, we get an algebraic relation linking solutions $y,\ y_k,\ y_l$ of the zero and first levels. For instance, if $y_1 = T_{1,1,1,1}y,\ y_2 = T_{1,1,1,-1}y,\ y_3 = T_{1,-1,1,1}y,\ y_4 = T_{-1,1,1,1}y,\ y_5 = T_{-1,1,-1,-1}y$, then the following relations are true:

$$y_2 = \frac{(-y\,n_1 + t\,n_2)\ y_1 + 2\,t\,y\,\eta_4}{t\,n_1 - y\,n_2 - 2\,y_1\,\eta_4},\ y_3 = \frac{y\,n_1\,y_1}{2\,y_1\,\eta_2 + y\ (-1+\eta_1-\eta_2+\eta_3+\eta_4)},$$
$$y_4 = (2\,y\,\eta_1 + y_1\ (-1-\eta_1+\eta_2+\eta_3+\eta_4))/n_1,$$
$$y_5 = \frac{\left(-t\,n_3 + y^2\,n_5 + y\ (n_4 + t\,n_6)\right)\ y_4 + 2\,y\ (t\ (\eta_3+\eta_4) - y\ (\eta_3 + t\,\eta_4))}{-y^2\,n_3 + t\,n_5 + y\ (t\,n_4 + n_6) - 2\,y_4\ (t\,\eta_3 + \eta_4 - y\ (\eta_3+\eta_4))},$$

where $n_1 = -1+\eta_1+\eta_2+\eta_3+\eta_4,\ n_2 = -1+\eta_1+\eta_2!+\eta_3-\eta_4,\ n_3 = 1+\eta_1-\eta_2+\eta_3+\eta_4,\ n_4 = 1+\eta_1-\eta_2+\eta_3-\eta_4,\ n_5 = -1-\eta_1+\eta_2+\eta_3+\eta_4,\ n_6 = 1+\eta_1-\eta_2-\eta_3+\eta_4.$ ■

Theorem 3. *Any three solutions of the first level are algebraically dependent.*

Proof. Eliminating the function $y(t)$ from formulas which are obtained in Theorem 2 we get relations linking three arbitrary solutions of the first level. For example,

$$y_4 = \frac{2\,t\,y_2\,\eta_1 + {y_1}^2\,n_7 - y_1\ (2\,t\ (\eta_1-\eta_4) + y_2\ (1+\eta_1-\eta_2-\eta_3+\eta_4))}{y_2\,n_2 + 2\,t\,\eta_4 - y_1\,n_1},$$
$$y_6 = \frac{y_2\ (t\,y_2\,n_1 - y_1\ (t\,n_2 + 2\,y_2\,\eta_4))}{y_1\ (-2\,y_2\ (\eta_2+\eta!_4) + t\,n_8) + y_2\ (2\,y_2\,\eta_2 + t\ (-1+\eta_1-\eta_2+\eta_3+\eta_4))},$$

where $y_6 = T_{1,-1,1,-1}y,\ n_7 = 1+\eta_1-\eta_2-\eta_3-\eta_4,\ n_8 = 1-\eta_1+\eta_2-\eta_3+\eta_4.$ ■

Calculating directly we can get nonlinear superposition relations linking solutions of three successive levels obtained after the repeated application of transformation (2.1). A seed solution $y = y(t,\alpha,\beta,\gamma,\delta)$, the solution of the first level $y_1 = T_{1,1,1,1}y$ and solution of the second level $y_{1,2} = T_{1,1,1,-1}y_1$ are connected by the following algebraic relation:

$$y_{1,2} = (2t\eta_4 y + y_1(tn_2 - n_1 y))/(tn_1 - 2\eta_4 y_1 - n_2 y).$$

Moreover, some solutions of the second level coincide up to the symmetries of equation (1.1). For example, the following relations are true:

$$T_{1,1,1,-1}\circ T_{1,1,1,1} = T_{1,1,-1,1}\circ T_{1,1,-1,-1} = T_{1,-1,1,1}\circ T_{1,-1,1,-1} = T_{-1,1,1,1}\circ T_{-1,1,1,-1},$$
$$T_{1,-1,1,1}\circ T_{1,1,1,1} = T_{1,1,1,-1}\circ T_{1,-1,1,-1} = T_{1,1,-1,1}\circ T_{1,-1,-1,1} = T_{-1,1,1,1}\circ T_{-1,-1,1,1},$$
$$T_{1,-1,1,-1}\circ T_{1,1,1,1} = S_6\circ T_{1,1,-1,-1}\circ T_{1,1,1,1}\circ S_6,$$

where $S_6 : y(t,\alpha,\beta,\gamma,\delta) \to y_6(t,\alpha,-\gamma,-\beta,\delta) = 1-y(1-t)$ is a symmetry transformation.

Note, that we may apply a sequence of Bäcklund transformations to a solution $y(t,\alpha,\beta,\gamma,\delta)$. Let us assume that this solution does not belong to the family of solutions generated from solutions of the Riccati equation

$$t(t-1)y' + (\eta_2+\eta_3+\eta_4-1)y^2 - (t\eta_2 + t\eta_3 + \eta_2 + \eta_4 - 1)y + t\eta_2 = 0 \qquad (2.2)$$

with $\eta_4 = \eta_1 + 1 - \eta_2 - \eta_3$. It is clear from (2.1) that this assumption enables us to apply successive Bäcklund transformations to the solution $y(t,\alpha,\beta,\gamma,\delta)$. The general values of the parameters obtained after repeated application of transformation (2.1) are given in the following theorem [12].

Theorem 4. *Assume that numbers* $n_i \in \mathbb{Z}$, $i \in \{1,\ldots,4\}$, *are such that* $\sum_{j=1}^{4} n_j \in 2\mathbb{Z}$, *and* η_i^* *denote arbitrary permutations of* η_i. *Then, for arbitrary signs* k_i, $k_i^2 = 1$, $i \in \{1,\ldots,4\}$, *there exist compositions of transformations of* T (2.1) *and symmetries* S_j, $j \in \{1,\ldots,24\}$, *which change* $y(t,\alpha,\beta,\gamma,\delta)$ *into* $\widetilde{y}(t,\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma},\widetilde{\delta})$ *with the parameters given by*

$$\widetilde{\alpha} = (\eta_1^* + n_1)^2/2,\ -\widetilde{\beta} = (\eta_2^* + n_2)^2/2,\ \widetilde{\gamma} = (\eta_3^* + n_3)^2/2,\ 1/2 - \widetilde{\delta} = (\eta_4^* + n_4)^2/2; \tag{2.3}$$

$$\begin{aligned}
&\widetilde{\alpha} = (k_1\eta_1^* - k_2\eta_2^* - k_3\eta_3^* - k_4\eta_4^* + 2n_1 + 1)^2/8,\\
&-\widetilde{\beta} = (-k_1\eta_1^* + k_2\eta_2^* - k_3\eta_3^* - k_4\eta_4^* + 2n_2 + 1)^2/8,\\
&\widetilde{\gamma} = (-k_1\eta_1^* - k_2\eta_2^* + k_3\eta_3^* - k_4\eta_4^* + 2n_3 + 1)^2/8,\\
&1/2 - \widetilde{\delta} = (-k_1\eta_1^* - k_2\eta_2^* - k_3\eta_3^* + k_4\eta_4^* + 2n_4 + 1)^2/8.
\end{aligned} \tag{2.4}$$

It may be verified directly that transformations obtained in papers [1, 7, 9, 13] are the compositions of transformation (2.1) in Theorem 1 and trivial symmetries S_j. For instance, the transformation found by Fokas and Yortsos in [9] is given by

$$F_1 : y(t,\alpha,\beta,\gamma,\delta) \to y_1(t,\alpha_1,\beta_1,\gamma_1,\delta_1) = y + \frac{2kf((t+1)y - 2t)}{-2t(t-1)f' + (t-1)J - (t+1)kf},$$

where the choice of the branches of μ_1 and μ_2 is denoted by ε_1 and ε_2 respectively and

$$\begin{aligned}
&t(t-1)y' = -\mu_1 y^2 + \frac{1}{2}\lambda(t+1)y - \mu_2 t + (\frac{1}{2} + \frac{\mu}{4} + f)(t-1)y,\\
&J = f^2 + \frac{f\mu}{2} + \nu,\ k = \mu_2 - \mu_1 - 1 \neq 0,\ \lambda = \mu_1 + \mu_2,\\
&\mu = 4(\frac{1}{2} - \gamma - \delta),\ \nu = 2\delta - 1 + (\frac{\mu}{4} + \frac{k}{2})^2,\ \mu_1^2 = 2\alpha,\ \mu_2^2 = -2\beta,\\
&2\,\alpha_1 = (\mu_2 - 1)^2,\ 2\,\beta_1 = -(\mu_1 - 1)^2,\ 2\,\gamma_1 = 1 - 2\delta,\ 2\,\delta_1 = 1 - 2\gamma.
\end{aligned}$$

Another transformation found in [13] by Gromak and Tsegel'nik is given by

$$\begin{aligned}
&F_2 : y(t,\alpha,\beta,\gamma,\delta) \to y_1(t,\alpha_1,\beta_1,\gamma_1,\delta_1) = y + \frac{(2(t+1) - 4y)f'}{2f' + J/t + (kf(t+1))/(t(t-1))},\\
&2\,\alpha_1 = (\mu_1 + 1)^2,\ 2\,\beta_1 = -(\mu_2 - 1)^2,\ \gamma_1 = \gamma,\ \delta_1 = \delta.
\end{aligned}$$

The transformation found by Adler [1] is given by the following formula:

$$F_3 : y(t,\alpha,\beta,\gamma,\delta) \to y_1(t,\alpha_1,\beta_1,\gamma_1,\delta_1) = \frac{t}{y}\frac{\beta/2 + (u-s)^2}{u^2 - \alpha/2},$$

where

$$y' = \frac{2s}{t-1} + 2p\frac{y}{t} - s\frac{t+1}{t(t-1)} - \frac{2(y-t)(t-1)}{t(t-1)}u,$$

$$2s = \mu_3 + \mu_4 - 1,\ 4p = 1 + \mu_3 - \mu_4,\ 4q = 1 + \mu_4 - \mu_3,\ \mu_3^2 = 2\gamma,\ \mu_4^2 = 1 - 2\delta,$$
$$\alpha_1 = \alpha,\ \beta_1 = \beta,\ 2\,\gamma_1 = (\mu_3 - 1)^2,\ 2\,\delta_1 = 1 - (\mu_4 - 1)^2.$$

The explicit relation between these transformations and the Bäcklund transformation is given by the following statement.

Theorem 5. *Up to the choice of the signs of* η_j*, the transformations* $T_{\varepsilon_1,\,\varepsilon_2,\,\varepsilon_3,\,\varepsilon_4}$ *and* $F_i,\ i \in \{1,2,3\}$*, are related by* $F_1 = T_{1,1,-1,-1} \circ T_{\varepsilon_1,\varepsilon_2,1,1}$*,* $F_2 = S_3 \circ T_{1,1,-1,-1} \circ T_{\varepsilon_1,\varepsilon_2,1,1}$*,* $F_3 = T_{-1,-1,1,1} \circ T_{1,1,\varepsilon_3,\varepsilon_4}$*.*

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3 The Schlesinger transformations

It is well-known [15] that equation (1.1) can be obtained as the compatibility condition of the following linear system

$$\frac{\partial}{\partial x}Y(x,t) = \left(\frac{A_0(t)}{x} + \frac{A_1(t)}{x-1} + \frac{A_t(t)}{x-t}\right)Y(x,t),\ \frac{\partial}{\partial t}Y(x,t) = -\frac{A_t(t)}{x-t}Y(x,t), \tag{3.1}$$

where $A_0 + A_1 + A_t + A_\infty = 0$, $\pm\theta_0/2$, $\pm\theta_1/2$, $\pm\theta_t/2$ and $\pm\theta_\infty/2$ are eigenvalues of $A_0,\ A_1,\ A_t,\ A_\infty$ given by

$$A_\nu = \frac{1}{2}\begin{pmatrix} z_\nu & u_\nu(\theta_\nu - z_\nu) \\ (\theta_\nu + z_\nu)/u_\nu & -z_\nu \end{pmatrix},\quad \nu = 0;1;t,\ \ A_\infty = \begin{pmatrix} \theta_\infty/2 & 0 \\ 0 & -\theta_\infty/2 \end{pmatrix}.$$

The function $y(t)$ which is determined by means of $k = tu_0(z_0-\theta_0)-(1-t)u_1(z_1-\theta_1)$, $ky = tu_0(z_0-\theta_0)$ satisfies equation (1.1) with $\alpha = (1-\theta_\infty)^2/2$, $\beta = -\theta_0^2/2$, $\gamma = \theta_1^2/2$, $\delta = (1-\theta_t^2)/2$. Let

$$Y_1(x) = R(x)Y(x) \tag{3.2}$$

be the Schlesinger transformation of system (3.1) which leaves the monodromy data invariant. Linear system (3.1) is transformed into the following system

$$(Y_1)_x = A_1(x)Y_1,\ A_1(x) = (R(x)A(x) + R_x(x))R^{-1}(x). \tag{3.3}$$

Let

$$\theta_0^1 = \theta_0 + 1,\ \theta_1^1 = \theta_1,\ \theta_t^1 = \theta_t,\ \theta_\infty^1 = \theta_\infty + 1.$$

Then using the same procedure as in [20], we get that in this case the Schlesinger matrix is given by

$$R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x^{1/2} + \begin{pmatrix} 1 & -u_0 \\ -\frac{tu_1(\theta_0+z_0)+(t-1)u_0(\theta_1+z_1)}{2(1+\theta_\infty)u_0u_1} & \frac{tu_1(\theta_0+z_0)+(t-1)u_0(\theta_1+z_1)}{2(1+\theta_\infty)u_1} \end{pmatrix} x^{-1/2}.$$

The Schlesinger transformation (3.2) generates a transformation for $y(t)$ which we denote by SL.

Theorem 6. *The Bäcklund transformation* (2.1) *and the Schlesinger transformation generated by matrix R in* (3.2) *are related by* $SL = S_3 \circ T_{1,1,-1-1} \circ T_{-1,-1,1,1}$.

Proof. Let $y_1 = y_1(t),\ u_i^1 = u_i^1(t),\ z_i^1 = z_i^1(t),\ i = 0; 1; t$ correspond to new parameters $\theta_0^1 = \theta_0 + 1,\ \theta_1^1 = \theta_1,\ \theta_t^1 = \theta_t,\ \theta_\infty^1 = \theta_\infty + 1$ in system (3.1). Using (3.3) we get $y_1 = (z_0^1 - \theta_0^1)/(tu_0^1(z_0^1 - \theta_0^1) - (1-t)u_1^1(z_1^1 - \theta_1^1))$, where, e.g., $u_1^1 = (u_0 - u_1)/(-1 + fu_0 - fu_1),\ f = -(tu_1(\theta_0 + z_0) + (t-1)u_0(\theta_1 + z_1))/(2(1+\theta_\infty)u_0u_1)$. Calculating directly we obtain formulas linking $u_i^1,\ z_i^1$ and $u_i,\ z_i$. Using relations between functions $y,\ u_i,\ z_i,\ i = 0, 1, t$, in [17], we get the statement. ■

4 Quadratic transformations

All quadratic and cubic transformation which are valid for the hypergeometric functions [3] are also valid for one-parameter families of solutions of the sixth Painlevé equation generated by the Riccati equation (2.2) and integrated in terms of the hypergeometric functions. Let us consider, for example, a hypergeometric equation

$$z(1-z)u'' + (c - (a+b+1)z)u' - abu = 0$$

for which the following transformation is valid:

$$u(z, 3a + \frac{1}{2}, a + \frac{2}{3}, \frac{3}{2}) \to v(t, a + \frac{1}{2}, a + \frac{5}{6}, \frac{3}{2}) = \frac{(1 + z/3)^{3a+3/2}}{1 - z/9},\ t = \frac{z(z-a)^2}{(z+3)^3}.$$

Using the relation [12] between the hypergeometric equation and the Riccati equation

$$t(t-1)y' - (\eta_2 t + ((\eta_3 - \eta_2)t - (1 + \eta_2 + \eta_4))y + \eta_1 y^2) = 0,\ \eta_4 = \eta_1 - 1 - \eta_2 + \eta_3 \quad (4.1)$$

solutions of which satisfy (1.1) we get the following transformation:

$$y(t, \eta_1, \frac{2\eta_1 - 3}{6}, \frac{3 + \eta_1}{3}, \eta_1 + \frac{1}{2}) \to y_1(t_1, \frac{\eta_1 - 1}{3}, \frac{\eta_1 - 2}{3}, \frac{5 + 2\eta_1}{6}, \frac{1 + 2\eta_1}{6}) =$$
$$\frac{3(t(12 - 8\eta_1 + 3t(3\eta_1 - 4)) + (32 - 60t + 27t^2)\eta_1 y)}{(8 - 9t)^2(\eta_1 - 1)},\ t_1 = \frac{27(t-1)t^2}{(9t-8)^2},$$

where $y(t, \eta_1, \eta_2, \eta_3, \eta_4)$ is a solution of (4.1).

The quadratic transformation for solutions of the sixth Painlevé equation was first obtained in [16]. Although this transformation is rather complicated, we write it explicitly here. This transformation was obtained by means of the quadratic transformation of the spectral parameter and Schlesinger transformations of the linear system [15] related to equation (1.1)

$$\partial_\lambda \Psi = \left(\frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_t}{\lambda - t} \right) \Psi,\ \partial_t \Psi = -\frac{A_t}{\lambda - t} \Psi, \quad (4.2)$$

where

$$A_s = \begin{pmatrix} z_s + \theta_s & -u_s z_s \\ u_s^{-1}(z_s + \theta_s) & -z_s \end{pmatrix},\ s = 0, 1, t,\ A_\infty = -(A_0 + A_1 + A_t) = \begin{pmatrix} & \text{nbs!p;}\, k_1 & 0 \\ 0 & k_2 & \end{pmatrix},$$

$$k_1 - k_2 = \theta_\infty,\ k_1 + k_2 = -(\theta_0 + \theta_1 + \theta_t).$$

From system (4.2) we get that solution of (1.1) $y = y(t)$ with parameters $\alpha = (\theta_\infty - 1)^2/2,\ \beta = \theta_0^2/2,\ \gamma = \theta_1^2/2,\ \delta = (1 - \theta_t^2)/2$ and the coefficients of matrices A_s are related as follows:

$$\begin{aligned}
z_0 &= y(y(y-1)(y-t)\tilde{z}^2 + (\theta_1(y-t) + t\theta_t(y-1) - 2k_2(y-1)(y-t))\tilde{z}+ \\
&\quad + k_2^2(y-t-1) - k_2(\theta_1 + t\theta_t))/(t\theta_\infty), \\
z_1 &= -(y-1)(y(y-1)(y-t)\tilde{z}^2 + ((\theta_1+\theta_\infty)(y-t) + t\theta_t(y-1)- \\
&\quad - 2k_2(y-1)(y-t))\tilde{z} + k_2^2(y-t) - k_2(\theta_1 + t\theta_t) - k_1k_2)/(\theta_\infty(t-1)), \\
z_t &= (y-t)(y(y-1)(y-t)\tilde{z}^2 + (\theta_1(y-t) + t(\theta_t+\theta_\infty)(y-1)- \\
&\quad - 2k_2(y-1)(y-t))\tilde{z} + k_2^2(y-1) - k_2(\theta_1 + t\theta_t) - tk_1k_2)/(t(t-1\theta_\infty)), \\
u_0 &= k(t)y/(tz_0),\ u_1 = -k(t)(y-1)/(z_1(t-1)),\ u_t = k(t)(y-t)/(t(t-1)z_t),
\end{aligned} \tag{4.3}$$

where[1]

$$\begin{aligned}
\tilde{z} &= z - \theta_0/y - \theta_1/(y-1) - \theta_t/(y-t), \\
dy/dt &= y(y-1)(y-t)(2z - \theta_0/y - \theta_1/(y-1) - (\theta_t-1)/(y-t))/(t(t-1)), \\
dz/dt &= ((-3y^2 + 2(1+t)y - t)z^2 + ((2y-1-t)\theta_0+ \\
&\quad + (2y-t)\theta_1 + (2y-1)(\theta_t-1))z - k_1(k_2+1))/(t(t-1)), \\
d(\log k(t))/dt &= (\theta_\infty - 1)(y-t)/(t(t-1)).
\end{aligned}$$

Theorem 7. *Let $y(t)$ be a solution of equation* (1.1) *with parameter values $\alpha = 9/8$, $\beta = -1/8$. Then the transformation*

$$K : y(t) \to \tilde{y}(T) = \left(1 - \frac{T-1}{T}\frac{Q(1)S(1)}{Q(-1)S(-1)}\right)^{-1},\ T = \frac{(\tau+1)^2}{4\tau},\ \tau = \sqrt{t}, \tag{4.4}$$

where

$$\begin{aligned}
Q(s) &= (\tau/s+1)u_0z_0(z_t+\theta_t)/(u_{s^2}(z_0+1/2)) - u_tz_t/u_{s^2} - \tau(z_t+\theta_t)/s, \\
S(s) &= (\tau/s+1)u_0z_0(z_{s^2}+\theta_{s^2})(z_t+\theta_t)/(u_{s^2}(z_0+1/2)) - \tau(z_t+\theta_t)z_{s^2}/s- \\
&\quad u_tz_t(z_{s^2}+\theta_{s^2})/u_{s^2},
\end{aligned}$$

determines a solution $\tilde{y}(T)$ of equation (1.1)*! with parameter values $\tilde{\alpha} = (1-\sqrt{1-2\delta})^2/2$, $\tilde{\beta} = -\gamma,\ \tilde{\gamma} = \gamma,\ \tilde{\delta} = \delta$.*

Transformation (4.4) was obtained in [16] under assumption $\theta_0 = 1/2,\ \theta_\infty = -1/2$, $\theta_t \neq 0$. Another quadratic transformation for the solutions of the sixth Painlevé equation was obtained in [22].

Theorem 8. *Let $y(t)$ be a solution of equation* (1.1) *with parameters $\beta = -\alpha,\ \delta = 1/2 - \gamma$. Then the transformation*

$$R : y(t) \to \tilde{y}(t_1) = \left(\frac{y(t)+\tau}{y(t)-\tau}\right)^2,\ t_1 = \left(\frac{1+\tau}{1-\tau}\right)^2,\ \tau = \sqrt{t} \tag{4.5}$$

[1] Note that in formula (4.3) from [16] there is a slip.

determines a solution $\widetilde{y}(t_1)$ *of equation* (1.1) *with parameters* $\widetilde{\alpha} = \widetilde{\beta} = 0,\ \widetilde{\gamma} = 4\alpha,\ \widetilde{\delta} = 1/2 - 4\gamma$.

Let us show that transformation (4.4) is in fact a superposition of transformations (4.5), symmetry transformation [12]

$$S : y(t, \alpha, \beta, \gamma, \delta) \to y_1(t, \gamma, \beta, \alpha, \delta) = y(t/(t-1))(y(t/(t-1)) - 1)^{-1} \tag{4.6}$$

and the Bäcklund transformation (2.1) of equation (1.1).

Theorem 9. *The following relation is valid* $K = T_{1,1,1,1} \circ S \circ R \circ T_{1,-1,-1,-1} \circ T_{-1,1,1,1}$.

Proof. Let $y(t) = y(t, 9/8, -1/8, \gamma, \delta)$ be a solution of equation (1.1). Assume that $\eta_1 = -3/2,\ \eta_2 = 1/2,\ \eta_3 = \theta_1,\ \eta_4 = \theta_t$. Then after a repeated application of the Bäcklund transformation (2.1) we get a new solution of equation (1.1) $u(t) = u(t, (1-\theta_1-\theta_t)^2/8, -(1-\theta_1-\theta_t)^2/8, (\theta_t-\theta_1)^2/8, 1/2-(\theta_t-\theta_1)^2/8) = T_{1,-1,-1,-1} \circ T_{-1,1,1,1} y(t)$. Next we apply transformations (4.5) and (4.6) to the solution $u(t)$. Relation $S \circ R : u(t, a, -a, c, 1/2-c) \to \widetilde{u}(T, 4a, 0, 0, 1/2-4c) = (\tau + u(t))^2/(4\tau u(t)),\ T = (1+\tau)^2/(4\tau),\ \tau = \sqrt{t}$ is valid. Hence, applying transformation (2.1) $T_{1,1,1,1}$ to solution $\widetilde{u}(T, (1-\theta_1-\theta_t)^2/2, 0, 0, 1/2-(\theta_t-\theta_1)^2/2)$, we get solution $\widetilde{y}(T, (1-\theta_t)^2/2, -\theta_1^2/2, \theta_1^2/2, (1-\theta_t^2)/2)$ which proves the statement. ■

Observe that we can also choose other branches of the parameters and get more relations between the quadratic transformations.

Also observe that in papers by Painlevé and later in [18] (and the references therein) equation (1.1) is written in the canonical form by means of the Weierstrass function $(2\pi i)^2 d^2 z/d\tau^2 = \sum_{j=1}^{3} \alpha_i \rho_z(z + T_i/2, \tau)$, where $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\alpha, -\beta, \gamma, 1/2-\delta)$, $e_i(\tau) = \rho(T_i/2, \tau),\ (T_0, T_1, T_2, T_3) = (0, 1, \tau, 1+\tau)$. The Weierstrass function admits [18] a Landen transform given by $\rho_z(z, \tau/2) = \rho_z(z, \tau) + \rho_z(z + \tau/2, \tau)$. Since $t = (e_3 - e_1)/(e_2 - e_1),\ y = (\rho(z) - e_1)/(e_2 - e_1)$, then putting $e_2 = (2-t)/3,\ e_1 = (-1-t)/3,\ e_3 = (2t-1)/3$ we get $t_1 = (1+\sqrt{t})^2/(4\sqrt{t}),\ y_1 = (y+\sqrt{t})^2/(4y\sqrt{t})$ which means that the Landen transform coincides up to a symmetry with the transformation of Theorem 8.

5 Algebraic solution

In this section we apply the Bäcklund transformations to obtain exact algebraic solutions of equation (1.1). Currently there is no precise classification of the algebraic solutions. A particular case of equation (1.1) with $\beta = \gamma = 0,\ \delta = 1/2$ was studied in papers [6, 7, 19] where all algebraic solutions were classified. However, as it will be shown below, some of these solutions are also valid for the general Painlevé VI equation. Algebraic solutions are also studied in [2]. However, most of these solutions belong to the hierarchies of the known solutions which form one of the following types.

1. Solutions obtained from $y(t, \alpha, -\alpha, \gamma, 1/2-\gamma) = \sqrt{t}$ by symmetries and Bäcklund transformations. For instance, using symmetry S_2 we get solution $y(t, \alpha, \beta, -\beta, 1/2-\alpha) = t + \sqrt{t^2 - t}$. No! te that solution $y = \sqrt{t}$ is not generated by the one-parameter family of solutions (2.2) in general case. Equation (1.1) has also solutions $y(t, \alpha, -a^2\alpha, 1/8, 3/8) = a\sqrt{t},\ y(t, \alpha, -1/8, 1/8, 1/2 - a^2\alpha) = t + a\sqrt{t^2 - t}$.

2. Solutions which belong to one-parameter families (2.2), such as $y(t,0,0,a^2/2,a(2-a)/2) = Ct^a$, where a and C are constants [12].

3. Solutions in the parametric form [6, 7]. The Bäcklund transformation of Theorem 1 is easily rewritten in the parametric form. Let $y = y(s),\ t = t(s)$ be a solution of equation (1.1). Then $y_1 = y_1(s) = F(t,y,y'),\ t = t(s)$ is also a solution of (1.1). Hence, using Theorem 4 we can get the values of the parameters when equation (1.1) admits algebraic equations in the parametric form.

4. Solutions in the implicit form nb! sp; $F(t,y(t)) = 0$, where F is a polynomial in two variables. An example of the algebraic curve is given in [7]. However, when a parameterization of a curve is not known, it is also possible to use the Bäcklund transformation to obtain new solutions of equation (1.1) in the same form which is illustrated below.

Let us present some methods to obtain algebraic solutions of equation (1.1) in the implicit form using the Bäcklund transformation (2.1). Let

$$y = \frac{(1+s^4)(1+s+s^3+s^4)}{1+s-2s^2+s^3-2s^4+s^5-2s^6+s^7+s^8},$$
$$t = t(s) = \frac{(1+s)^4(1-s+s^2)^2(1-2s-2s^3+s^4)}{(s-1)^4(1+s+s^2)^2(1+2s+2s^3+s^4)}$$

be one of the parametrizations of the curve $F(t,y) := 2y^3+6yt-3y^2(1+t)-t(1+t) = 0$ of equation (1.1) with parameter values $\beta = -1/18,\ \gamma = \alpha/4,\ \delta = 1/2-\alpha/4$. Assume $y_1(t) = T_{1,1,1,1}y(t)$. Rewriting the Bäcklund transformation in the parametric form we get new solution of (1.1) given by

$$y_1 = \frac{(1-2s-2s^3+s^4)(1+s+s^3+s^4)}{(s-1)^2(1+s+s^2)(1+s^4)},$$
$$t = \frac{(1+s)^4(1-s+s^2)^2(1-2s-2s^3+s^4)}{(s-1)^4(1+s+s^2)^2(1+2s+2s^3+s^4)},$$
$$\alpha_1 = 1/18,\ \beta_1 = -\frac{(3\eta_1-2)^2}{18},\ \gamma_1 = \frac{(3\eta_1-2)^2}{72},\ \delta_1 = \frac{32+12\eta_1-9\eta_1^2}{72},\ \eta_1^2 = 2\alpha,$$

where $t = t(s)$ is the same. Substituting the inverse transformation $y(t) = T_{1,1,1,1}y_1(t)$ into the equation of the algebraic curve we get new implicit solution $F_1(t,y_1,y_1') = 0$. Let $F_2 = dF_1/dx = F_2(t,y_1,\ y_1',\ y_1'') = F_3(t,y_1,y_1') = 0$. Calculating a resultant of the algebraic equations $F_1 = 0,\ F_3 = 0$ with respect to y_1', we get that new solution $y_1(t)$ satisfies algebraic equation $-2t^2+3t(1+t)y_1-6ty_1^2+(1+t)y_1^3 = 0$. However, in this case solutions y and y_1 are related by symmetry transformation S_3, i.e., $y_1 = t/y$. By analogy we consider another example. Let $y_1(t) = T_{1,1,-1,1}y(t)$ and $\eta_1 = 0$. A new solution satisfies algebraic equation $2t^2-3t(1+t)y_1+6ty_1^2-(1+t)y_1^3 = 0$ with parameters $\alpha_1 = 1/18,\ \gamma_1 = -\beta_1/4,\ \delta_1 = \beta_1/4+1/2$.

To illustrate another method to obtain hierarchies of algebraic solutions we take solution $F(t,y) := t^3-2\,t^2\,(1+t)\,y+6\,t^2\,y^2-2\,t\,(1+t)\,y^3+\left(1-t+t^2\right)\,y^4 = 0$ with $\alpha = 1/8,\ \gamma = -\beta,\ \delta = 1/2+\beta$. Assume for simplicity $\eta_2 = 0$. Take the Bäcklund transformation in the form $y_1 = T_{1,1,1,-1}y = R_1(t,y,y')$. Differentiating the equation of the algebraic curve with respect to t and substituting $y' = R_2(t, y, y_1)$ from the Bäcklund transformation, we get algebraic equation $R_3(t, y, y_1) = 0$. Eliminating function y between initial equation $F(t, y) = 0$ and $R_3(t, y, y_1) = 0$, we get a new solution in the form of the algebraic curve. In our example solution y_1 is determined by the algebraic equation $t\,(t-1)\;t^2 + 6\;(t-1)\;t\,{y_1}^2 - 4\;\left(t^2-1\right)\;{y_1}^3 + 3\;(t-1)\;{y_1}^4 = 0$ with $\beta_1 = -\alpha_1/9,\ \gamma_1 = \alpha_1/9,\ \delta_1 = 1/2 - \alpha_1/9$.

Next we apply Theorem 4 to algebraic solutions. In order to obtain general parameter values of a hierarchy of algebraic solutions take, for instance, the solution of (1.1) given by

$$y^3(2t-1) + 3yt - 3y^2t - t^2 = 0 \tag{5.1}$$

with $\alpha = 1/18,\ \gamma = -\beta,\ \delta = (1+8\beta)/2$. Up to the symmetry transformations solution (5.1) generates solutions of equation (1.1) where parameters take one of the following forms:

$$
\begin{aligned}
&\widetilde{\alpha} = (1+3n_1)^2/18,\ -\widetilde{\beta} = (\eta_2+n_2)^2/2,\ \widetilde{\gamma} = (\eta_2+n_3)^2/2,\ 1/2-\widetilde{\delta} = (2\eta_2+n_4)^2/2;\\
&\widetilde{\alpha} = (2+3n_1-6\varepsilon\eta_2)^2/18,\ -\widetilde{\beta} = (1+3n_2-3\varepsilon\eta_2)^2/18,\\
&\qquad \widetilde{\gamma} = (1+3n_3-3\varepsilon\eta_2)^2/18,\ 1/2-\widetilde{\delta} = (1+3n_4)^2/18;\\
&\widetilde{\alpha} = (2+3n_1)^2/18,\ -\widetilde{\beta} = (1+3n_2+3\varepsilon\eta_2)^2/18,\\
&\qquad \widetilde{\gamma} = (1+3n_3+3\varepsilon\eta_2)^2/18,\ 1/2-\widetilde{\delta} = (1+3n_4-6\varepsilon\eta_2)^2/18;\\
&\widetilde{\alpha} = (2+3n_1-3\varepsilon\eta_2)^2/18,\ -\widetilde{\beta} = (1+3n_2)^2/18,\\
&\qquad \widetilde{\gamma} = (1+3n_3-6\varepsilon\eta_2)^2/18,\ 1/2-\widetilde{\delta} = (1+3n_4+3\varepsilon\eta_2)^2/18;\\
&\widetilde{\alpha} = (1+3n_1-6\varepsilon\eta_2)^2/18,\ -\widetilde{\beta} = (2+3n_2-3\varepsilon\eta_2)^2/18,\\
&\qquad \widetilde{\gamma} = (2+3n_3-3\varepsilon\eta_2)^2/18,\ 1/2-\widetilde{\delta} = (2+3n_4)^2/18;\\
&\widetilde{\alpha} = (1+3n_1)^2/18,\ -\widetilde{\beta} = (2+3n_2+3\varepsilon\eta_2)^2/18,\\
&\qquad \widetilde{\gamma} = (2+3n_3+3\varepsilon\eta_2)^2/18,\ 1/2-\widetilde{\delta} = (2+3n_4-6\varepsilon\eta_2)^2/18;\\
&\widetilde{\alpha} = (1+3n_1-3\varepsilon\eta_2)^2/18,\ -\widetilde{\beta} = (2+3n_2)^2/18,\\
&\qquad \widetilde{\gamma} = (2+3n_3-6\varepsilon\eta_2)^2/18,\ 1/2-\widetilde{\delta} = (2+3n_4+3\varepsilon\eta_2)^2/18,
\end{aligned}
$$

where $\eta_2^2 = -2\beta,\ \varepsilon^2 = 1,\ \ n_i \in \mathbb{Z},\ i \in \{1, \ldots, 4\}$, are such that $\sum_{j=1}^{4} n_j \in 2\mathbb{Z}$.

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