An Error Estimate for Viscous Approximate Solutions of Degenerate Parabolic Equations

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Abstract

Relying on recent advances in the theory of entropy solutions for nonlinear (strongly) degenerate parabolic equations, we present a *direct* proof of an L^1 error estimate for viscous approximate solutions of the initial value problem for

$$\partial_t w + \operatorname{div}\big(V(x)f(w)\big) = \Delta A(w),$$

where $V = V(x)$ is a vector field, $f = f(u)$ is a scalar function, and $A'(\cdot) \geq 0$. The viscous approximate solutions are weak solutions of the initial value problem for the uniformly parabolic equation

$$\partial_t w^\varepsilon + \operatorname{div}\big(V(x)f(w^\varepsilon)\big) = \Delta\big(A(w^\varepsilon) + \varepsilon w^\varepsilon\big), \qquad \varepsilon > 0.$$

The error estimate is of order $\sqrt{\varepsilon}$.

1 Introduction

In this paper we are interested in certain "viscous" approximations of entropy solutions of the initial value problem

$$\partial_t w + \operatorname{div}\big(V(x)f(w)\big) = \Delta A(w), \qquad (x,t) \in Q_T,$$
$$w(x,0) = w_0(x), \qquad x \in \mathbb{R}^d, \tag{1.1}$$

where $Q_T = \mathbb{R}^d \times (0,T)$ with $T > 0$ fixed, $u : Q_T \to \mathbb{R}$ is the sough function, $V : \mathbb{R}^d \to \mathbb{R}$ is a (not necessarily divergence free) velocity field, $f : \mathbb{R} \to \mathbb{R}$ is the convective flux function, and $A : \mathbb{R} \to \mathbb{R}$ is the "diffusion" function. For the diffusion function the basic assumption is that $A(\cdot)$ is nonincreasing. This condition implies that (1.1) is a (strongly) *degenerate parabolic* problem. For example, the hyperbolic equation $\partial_t w + \operatorname{div}\big(V(x)f(w)\big) = 0$ is a special case of (1.1). Problems such as (1.1) occur in several important applications. We

mention here only two examples: flow in porous media (see, e.g., [8]) and sedimentation-consolidation processes [3].

Since $A(\cdot)$ is merely nondecreasing, solutions are not necessarily smooth and weak solutions must be sought. Moreover, as is well-known in the theory of hyperbolic conservation laws, weak solutions are not uniquely determined by their initial data. To have a well-posed problem we need to consider entropy solutions, i.e., weak solutions that satisfy a Kružkov–Vol'pert type entropy condition. A precise statement is given in Section 2 (see Definition 1). For purely hyperbolic equations this entropy condition was introduced by Kružkov [15] and Vol'pert [21]. For degenerate parabolic equations, it was introduced by Vol'pert and Hudjaev [22].

Following Carrillo [5], Karlsen and Risebro [13] proved that the entropy solution of (1.1) (as well as a more general equation) is unique. Moreover in the $L^\infty\big(0,T;BV\big(\mathbb{R}^d\big)\big)$ class of entropy solutions, they proved an L^1 contraction principle. Existence of an $L^\infty\big(0,T;BV\big(\mathbb{R}^d\big)\big)$ entropy solution of (1.1) follows from the results in Vol'pert and Hudjaev [22] or Karlsen and Risebro [12] (the latter deals with convergence of finite difference methods). The proof in [13] of uniqueness and stability is based on the "doubling of variables" strategy introduced in Carrillo [5] (see also Chen and DiBenedetto [6]), which in turn is a generalization of the pioneering work by Kružkov [15] on hyperbolic equations. Related papers dealing with the "doubling of variables" device for degenerate parabolic equations include, among others, Carrillo [4], Otto [19], Rouvre and Gagneux [20], Cockburn and Gripenberg [7] Bürger, Evje and Karlsen [1, 2], Ohlberger [18], Mascia, Porretta, and Terracina [17], Eymard, Gallouet, Herbin and Michel [11], and Karlsen and Ohlberger [14].

In this paper we are interested in certain approximate solutions of (1.1) coming from solving the uniformly parabolic problem

$$\partial_t w^\varepsilon + \operatorname{div}\big(V(x)f(w^\varepsilon)\big) = \Delta A^\varepsilon(w^\varepsilon), \qquad (x,t) \in Q_T,$$
$$w^\varepsilon(x,0) = w_0(x), \qquad x \in \mathbb{R}^d, \tag{1.2}$$

where $A^\varepsilon(w^\varepsilon) = A(w^\varepsilon) + \varepsilon w^\varepsilon$, $\varepsilon > 0$. We refer to w^ε as a *viscous* approximate solution of (1.1). Convergence of w^ε to the unique entropy solution w of (1.1) as $\varepsilon \downarrow 0$ follows from the results in Vol'pert and Hudjaev [22]. Our main interest here is to give an explicit rate of convergence for w^ε as $\varepsilon \downarrow 0$, i.e., an L^1 error estimate for viscous approximate solutions.

There are several ways to prove such an error estimate. One way is to view it as a consequence of a continuous dependence estimate. Combining the ideas in [13] with those in Cockburn and Gripenberg [7], who used a variant of Kružkov's "doubling of variables" device for (1.1) with $V \equiv 1$, Evje, Karlsen and Risebro [9] established an explicit "continuous dependence on the nonlinearities" estimate for entropy solutions of (1.1). A direct consequence of this estimate is the error bound $\|w^\varepsilon - w\|_{L^1(Q_T)} = \mathcal{O}\big(\sqrt{\varepsilon}\big)$, at least when w^ε, w belong to $L^\infty\big(0,T;BV\big(\mathbb{R}^d\big)\big)$ and V is sufficiently regular. Unfortunately the techniques employed in [9] require that one works with (smooth) viscous approximations of (1.1). The proof in [9] (as well as the one in [7]) did not exploit the entropy solution "machinery" developed by Carrillo [5].

The main purpose of this work is to show that one can indeed use the "doubling of variables" device to compare *directly* the entropy solution w of (1.1) against the viscous approximation w^ε of (1.2). Hence there is no need to work with approximate solutions

of (1.1). Although our proof is of independent interest, it may also shed some light on how to obtain error estimates for numerical methods. Most numerical methods (related to this class of equations) have (1.2) as a "model" problem and, in this context, the size of ε designates the amount of "diffusion" present in the numerical method. A step in the direction of obtaining error estimates for numerical methods has been taken by Ohlberger [18] with his a posteriori error estimate for a finite volume method. We will in future work use the ideas devised herein to derive error estimates a priori for finite difference methods.

The rest of this paper is organized as follows: In Section 2 we state the definition of an entropy solution and the main result (Theorem 1). Section 3 is devoted to the derivation of certain entropy inequalities for the exact entropy solution and its viscous approximation. Equipped with these entropy inequalities, we prove the error estimate (Theorem 1) in Section 4.

2   Statement of result

Following [12, 13] we start by stating sufficient conditions on $V = (V_1, \ldots, V_d), f, A$ and u_0 to ensure the existence of a unique $L^\infty(0, T; BV(\mathbb{R}^d))$ entropy solution of (1.1):

$$V \in \left(L^\infty(\mathbb{R}^d)\right)^d \cap \left(\mathrm{Lip}(\mathbb{R}^d)\right)^d; \quad \mathrm{div}V \in BV(\mathbb{R}^d);$$

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}); \quad f(0) = 0;$$

$$A \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}) \quad \text{and} \quad A(\cdot) \text{ is nondecreasing with } A(0) = 0;$$

$$u_0 \in L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d). \tag{2.1}$$

Note that the first condition in (2.1) implies

$$V \in \left(W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)\right)^d.$$

In (2.1) and elsewhere in this paper the space $BV(\mathbb{R}^d)$ is defined as

$$BV(\mathbb{R}^d) = \left\{ g \in L^1(\mathbb{R}^d) : |g|_{BV(\mathbb{R}^d)} < \infty \right\},$$

where $|g|_{BV(\mathbb{R}^d)}$ denotes the total variation of g, i.e., $g \in BV(\mathbb{R}^d)$ if and only if $g \in L^1(\mathbb{R}^d)$ and the first order distributional derivatives of g are represented by finite measures on $\mathbb{R}^d$.

Equipped with (2.1) we can state the following definition of an entropy solution:

Definition 1 (Entropy Solution). A function $w(x, t)$ is called an entropy solution of (1.1) if

(i) $w \in L^1(Q_T) \cap L^\infty(Q_T) \cap C\left(0, T; L^1(\mathbb{R}^d)\right)$,

(ii) $A(w) \in L^2\left(0, T; H^1(\mathbb{R}^d)\right)$,

(iii) $w(x, t)$ satisfies the *entropy inequality*

$$\iint\limits_{Q_T} \Big(|w - k|\partial_t \phi + \mathrm{sgn}(w - k)\big[V(x)(f(w) - f(k)) - \nabla A(w)\big] \cdot \nabla \phi$$

$$- \mathrm{sgn}(w - k)\mathrm{div}V(x)f(k)\phi \Big) \, dt \, dx \geq 0, \qquad \forall \, k \in \mathbb{R}, \tag{2.2}$$

for all nonnegative $\phi \in C_0^\infty(Q_T)$ and

(iv) $\|w(\cdot, t) - w_0\|_{L^1(\mathbb{R}^d)} \to 0$ as $t \downarrow 0$ (essentially).

Note that, if we take $k > \operatorname{ess\,sup} w(x, t)$ and $k < \operatorname{ess\,inf} w(x, t)$ in (2.2), then an approximation argument reveals that

$$\iint\limits_{Q_T} \left(w\phi_t + \big[V(x)f(w) - \nabla A(w) \big] \cdot \nabla \phi \right) dt\, dx = 0 \tag{2.3}$$

holds for all $\phi \in H^1(Q_T)$. Let $\langle \cdot, \cdot \rangle$ denote the usual pairing between $H^{-1}(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$. From (2.3) we conclude that

$$\partial_t w \in L^2\big(0, T; H^{-1}(\mathbb{R}^d)\big),$$

so that

$$-\int_0^T \langle \partial_t w, \phi \rangle \, dt + \iint\limits_{Q_T} \left(\big[V(x)f(w) - \nabla A(w) \big] \cdot \nabla \phi \right) dt\, dx = 0, \quad \forall\, \phi \in H^1(Q_T). \tag{2.4}$$

In other words an entropy solution $w(x, t)$ of (1.1) is also a *weak* solution of the same problem.

In this paper we are interested in comparing the entropy solution w of (1.1) against the weak solution w^ε of the viscous problem (1.2). From the results in Karlsen and Risebro [12] or Vol'pert and Hudjaev [22] there exists a weak solution $w^\varepsilon \in L^\infty\big(0, T; BV(\mathbb{R}^d)\big)$ of (1.2). Since $A^\varepsilon(\cdot)$ is increasing, the uniqueness result in Karlsen and Risebro [13] (see also Remark 1 herein) tells us that this weak solution is in fact a unique solution. Moreover from the energy estimate we conclude that $w^\varepsilon \in L^2\big(0, T; H^1(\mathbb{R}^d)\big)$. Of course, if V, f, A, u_0 are smooth enough, one can prove that the weak solution w^ε of (1.2) is actually a classical ($C^{2,1}$) solution. See, e.g., Vol'pert and Hudjaev [22]. Here it will be sufficient to know that w^ε belongs to $L^2\big(0, T; H^1(\mathbb{R}^d)\big)$ (not $C^{2,1}$).

We are now ready to state our main theorem:

Theorem 1 (Error Estimate). *Suppose that the conditions in* (2.1) *hold. Let* $w \in L^\infty\big(0, T; BV(\mathbb{R}^d)\big)$ *be the unique entropy solution of* (1.1) *and let* $w^\varepsilon \in L^2\big(0, T; H^1(\mathbb{R}^d)\big) \cap L^\infty\big(0, T; BV(\mathbb{R}^d)\big)$ *be the unique weak solution of* (1.2). *Then there exists a constant* C, *independent of* ε, *such that*

$$\|w^\varepsilon - w\|_{L^1(Q_T)} \le C\sqrt{\varepsilon}. \tag{2.5}$$

3   Entropy inequalities

In Section 4 we follow the uniqueness proof of Carrillo [5] to obtain an estimate of the difference between w^ε and w. To this end it will be necessary to derive two entropy inequalities for the exact solution w and two approximate entropy inequalities for the viscous solution w^ε. The purpose of this section is to derive these inequalities. (See Lemma 2 and Lemma 3 below.)

Note that, differently from the pure hyperbolic case [15], we need to operate with one additional entropy inequality (actually an equality for the exact solution w) taking into account the parabolic (dissipation) mechanism in the equation. Hence we introduce a set H corresponding to the regions where $A(\cdot)$ is "flat" and (1.1) behaves hyperbolically. More precisely, let $A^{-1} : \mathbb{R} \to \mathbb{R}$ denote the unique left-continuous function which satisfies $A^{-1}(A(u)) = u$ for all $u \in \mathbb{R}$. Then we define

$$H = \left\{ r \in \mathbb{R} \,:\, A^{-1}(\cdot) \text{ is discontinuous at } r \right\}.$$

Since $A(\cdot)$ is a monotonic function, H is at most countable. The dissipation mechanism in the equation is effective only in the (x,t) region corresponding to the complement of H.

To prove Lemma 2 and Lemma 3 below we need the following "weak" chain rule:

Lemma 1. *Let $u : Q_T \to \mathbb{R}$ be a measurable function satisfying the four conditions*

(1) $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C\big(0,T; L^1(\mathbb{R}^d)\big),$

(2) $u(0,\cdot) = u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d),$

(3) $\partial_t u \in L^2\big(0,T; H^{-1}(\mathbb{R}^d)\big)$ *and*

(4) $A(u) \in L^2\big(0,T; H^1(\mathbb{R}^d)\big).$

For every nonnegative and compactly supported $\phi \in C^\infty(Q_T)$ with $\phi|_{t=0} = \phi|_{t=T} = 0$ we have

$$-\int_0^T \Big\langle \partial_t u, \psi\big(A(u)\big)\phi \Big\rangle \, dt = \iint_{Q_T} \left(\int_k^u \psi(A(\xi)) \, dxi \right) \phi_t \, dt \, dx, \qquad k \in \mathbb{R},$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing and Lipschitz continuous function.

The proof of Lemma 1 is very similar to the proof of the "weak chain" rule in Carrillo [5] and it is therefore omitted. See instead [13].

The following lemma, which deals with entropy inequalities for the exact entropy solution w, is a direct consequence of the very definition of an entropy solution.

Lemma 2. *The unique entropy solution w of (1.1) satisfies:*

(i) *For all $k \in \mathbb{R}$ and all nonnegative $\phi \in C_0^\infty(Q_T)$ we have*

$$E^{\mathrm{hyp}}(w, k, \phi) \geq 0, \tag{3.1}$$

where

$$E^{\mathrm{hyp}}(w, k, \phi) := \iint_{Q_T} \Big(|w - k|\partial_t \phi + \mathrm{sgn}(w - k)\big[V(x)(f(w) - f(k))$$
$$- \nabla A(w)\big] \cdot \nabla\phi - \mathrm{sgn}(w - k)\mathrm{div} V(x) f(k)\phi \Big) \, dt \, dx. \tag{3.2}$$

We refer to (3.1) as a hyperbolic entropy inequality.

(ii) *For all k such that $A(k) \notin H$ and all nonnegative $\phi \in C_0^\infty(Q_T)$ we have*

$$E^{\mathrm{par}}(w, k, \phi) = 0, \tag{3.3}$$

where

$$E^{\mathrm{par}}(w, k, \phi) := \iint\limits_{Q_T} \Big(|w - k| \partial_t \phi + \mathrm{sgn}(w - k) \big[V(x)(f(w) - f(k))$$

$$- \nabla A(w) \big] \cdot \nabla \phi - \mathrm{sgn}(w - k) \mathrm{div} V(x) f(k) \phi \Big) \, dt \, dx$$

$$- \lim_{\eta \downarrow 0} \iint\limits_{Q_T} |\nabla A(w)|^2 \mathrm{sgn}_\eta'(A(w) - A(k)) \phi \, dt \, dx. \tag{3.4}$$

In (3.4) (and elsewhere in this paper) sgn_η is the approximate sign function defined by

$$\mathrm{sgn}_\eta(\tau) := \begin{cases} \mathrm{sgn}(\tau) & \text{if } |\tau| > \eta, \\ \tau/\eta & \text{if } |\tau| \leq \eta, \end{cases} \qquad \eta > 0. \tag{3.5}$$

We refer to (3.3) as a parabolic entropy inequality.

Proof. The first inequality (3.1) is nothing but the entropy condition for the entropy solution w. So there is nothing to prove. We turn to the proof of the second inequality (3.3), which borrows a lot from Carrillo [5] (see also [13]). In what follows we always let k and ϕ be as in the lemma and the approximate sign function $\mathrm{sgn}_\eta(\cdot)$ is always the one defined in (3.5).

Since w satisfies (2.4) and $\big[\mathrm{sgn}_\eta(A(w) - A(k)) \phi \big] \in L^2\big(0, T; H^1(\mathbb{R}^d)\big)$, we have

$$- \int_0^T \Big\langle \partial_t w, \mathrm{sgn}_\eta(A(w) - A(k)) \phi \Big\rangle \, dt$$

$$+ \iint\limits_{Q_T} \Big(\big[V(x)(f(w) - f(k)) - \nabla A(w) \big] \cdot \nabla \big[\mathrm{sgn}_\eta(A(w) - A(k)) \phi \big]$$

$$- \mathrm{div} V(x) f(k) \big[\mathrm{sgn}_\eta(A(w) - A(k)) \phi \big] \Big) \, dt \, dx = 0.$$

Introduce the function $\psi_\eta(z) = \mathrm{sgn}_\eta(z - A(k))$ and note that Lemma 1 can be applied so that

$$- \int_0^T \Big\langle \partial_t w, \mathrm{sgn}_\eta(A(w) - A(k)) \phi \Big\rangle \, dt = \iint\limits_{Q_T} \Big(\int_k^w \mathrm{sgn}_\eta(A(\xi) - A(k)) \, d\xi \Big) \partial_t \phi \, dt \, dx.$$

Hence

$$\iint\limits_{Q_T} \Big(\int_k^w \mathrm{sgn}_\eta(A(\xi) - A(k)) \, d\xi \Big) \partial_t \phi \, dt \, dx$$

$$+ \iint\limits_{Q_T} \Big(\big[V(x)(f(w) - f(k)) - \nabla A(w) \big] \cdot \nabla \big[\mathrm{sgn}_\eta(A(w) - A(k)) \phi \big]$$

$$- \mathrm{sgn}_\eta(A(w) - A(k)) \mathrm{div} V(x) f(k) \phi \Big) \, dt \, dx = 0. \tag{3.6}$$

Note that since $A(r) > A(k)$ if and only if $r > k$ (here we make use of the assumption that $k \in$ "parabolic region", i.e., $A(k) \notin H$), $\mathrm{sgn}_\eta(A(r) - A(k)) \to 1$ as $\eta \downarrow 0$ for any $r > k$. Similarly for $r < k$. Consequently, as $\eta \downarrow 0$, $\int_k^w \mathrm{sgn}_\eta(A(\xi) - A(k))\, d\xi \to |w - k|$ a.e. in Q_T. Moreover we have $\left| \int_k^w \mathrm{sgn}_\eta(A(\xi) - A(k))\, d\xi \right| \leq |w - c| \in L^1_{\mathrm{loc}}(Q_T)$ so that by Lebesgue's dominated convergence theorem

$$\lim_{\eta \downarrow 0} \iint_{Q_T} \left(\int_k^w \mathrm{sgn}_\eta(A(\xi) - A(k))\, d\xi \right) \partial_t \phi \, dt \, dx = \iint_{Q_T} |w - k| \partial_t \phi \, dt \, dx.$$

Next we have

$$\lim_{\eta \downarrow 0} \iint_{Q_T} \left[V(x)(f(w) - f(k)) - \nabla A(w) \right] \cdot \nabla \left[\mathrm{sgn}_\eta(A(w) - A(k)) \phi \right] dt \, dx$$

$$= \lim_{\eta \downarrow 0} \iint_{Q_T} \left[V(x)(f(w) - f(k)) - \nabla A(w) \right] \cdot \nabla \mathrm{sgn}_\eta(A(w) - A(k)) \phi \, dt \, dx$$

$$+ \lim_{\eta \downarrow 0} \iint_{Q_T} \left[V(x)(f(w) - f(k)) - \nabla A(w) \right] \cdot \mathrm{sgn}_\eta(A(w) - A(k)) \nabla \phi \, dt \, dx$$

$$= \underbrace{\lim_{\eta \downarrow 0} \iint_{Q_T} V(x)(f(w) - f(k)) \mathrm{sgn}'_\eta(A(w) - A(k)) \nabla A(w) \phi \, dt \, dx}_{I_1}$$

$$- \lim_{\eta \downarrow 0} \iint_{Q_T} |\nabla A(w)|^2 \mathrm{sgn}'_\eta(A(w) - A(k)) \phi \, dt \, dx$$

$$+ \underbrace{\lim_{\eta \downarrow 0} \iint_{Q_T} \mathrm{sgn}_\eta(A(w) - A(k)) \left[V(x)(f(w) - f(k)) - \nabla A(w) \right] \cdot \nabla \phi \, dt \, dx}_{I_2}.$$

Note that I_1 can be rewritten as $I_1 = \lim\limits_{\eta \downarrow 0} \iint_{Q_T} V(x) \mathrm{div}\, \mathcal{Q}_\eta(A(w)) \phi \, dt \, dx$, where

$$\mathcal{Q}_\eta(z) := \int_0^z \mathrm{sgn}'_\eta(r - A(k)) \left(f(A^{-1}(r)) - f(A^{-1}(A(k))) \right) dr$$

$$= \frac{1}{\eta} \int_{\min(z, A(k) - \eta)}^{\min(z, A(k) + \eta)} \left(f(A^{-1}(r)) - f(A^{-1}(A(k))) \right) dr.$$

Surely $\mathcal{Q}_\eta(z)$ tends to zero as $\eta \downarrow 0$ for all $z \in \mathrm{Range}(A)$. By invoking Lebesgue's dominated convergence theorem, we conclude after an integration by parts that

$$I_1 = -\lim_{\eta \downarrow 0} \iint_{Q_T} \left(\mathcal{Q}_\eta(A(w)) V(x) \cdot \nabla \phi + \mathcal{Q}_\eta(A(w)) \mathrm{div} V(x) \phi \right) dt \, dx = 0.$$

Using that $\operatorname{sgn}(w - k) = \operatorname{sgn}\big(A(w) - A(k)\big)$ a.e. in Q_T (since $A(k) \notin H$) we have

$$I_2 = \lim_{\eta \downarrow 0} \iint_{Q_T} \operatorname{sgn}_\eta(A(w) - A(k))\big[V(x)(f(w) - f(k)) - \nabla A(w)\big] \cdot \nabla \phi \, dt \, dx$$

$$= \iint_{Q_T} \operatorname{sgn}(w - k)\big[V(x)(f(w) - f(k)) - \nabla A(w)\big] \cdot \nabla \phi \, dt \, dx.$$

For the same reason we have that

$$\lim_{\eta \downarrow 0} \iint_{Q_T} \operatorname{sgn}_\eta(A(w) - A(k))\operatorname{div}V(x)f(k)\phi \, dt \, dx = \iint_{Q_T} \operatorname{sgn}(w - k)\operatorname{div}V(x)f(k)\phi \, dt \, dx.$$

Consequently, letting $\eta \downarrow 0$ in (3.6), we obtain (3.3). $\blacksquare$

Remark 1. Observe that, if $A(\cdot)$ is increasing, then a weak solution is automatically an entropy solution and hence it is unique.

The next lemma, which deals with approximate entropy inequalities for the viscous solution w^ε, is a direct consequence of the definition of a weak solution of (1.2).

Lemma 3. *Let* E^{hyp} *and* E^{par} *be defined in* (3.2) *and* (3.4) *respectively. Furthermore define*

$$R_{\mathrm{visc}} := \varepsilon \iint_{Q_T} \big|\nabla w^\varepsilon \cdot \nabla \phi\big| \, dt \, dx. \tag{3.7}$$

The unique weak solution $w^\varepsilon \in L^2\big(0, T; H^1(\mathbb{R}^d)\big) \cap L^\infty\big(0, T; BV(\mathbb{R}^d)\big)$ *of* (1.2) *satisfies:*

(i) *For all* $k \in \mathbb{R}$ *and all nonnegative* $\phi \in C_0^\infty(Q_T)$ *we have*

$$E^{\mathrm{hyp}}(w^\varepsilon, k, \phi) \geq -R_{\mathrm{visc}}. \tag{3.8}$$

We refer to (3.8) *as an approximate hyperbolic entropy inequality.*

(ii) *For all* $k \in \mathbb{R}$ *such that* $A(k) \notin H$ *and all nonnegative* $\phi \in C_0^\infty(Q_T)$ *we have*

$$E^{\mathrm{par}}(w^\varepsilon, k, \phi) \geq -R_{\mathrm{visc}}. \tag{3.9}$$

We refer to (3.9) *as an approximate parabolic entropy inequality.*

Proof. In what follows we always let k and ϕ be as indicated by the lemma. The proof of the inequality (3.8) follows the proof of (3.1) rather closely. Since w^ε is a weak solution and $\big[\operatorname{sgn}_\eta(w^\varepsilon - k)\phi\big]$ belongs to $L^2\big(0, T; H^1(\mathbb{R}^d)\big)$, we have

$$-\int_0^T \Big\langle \partial_t w^\varepsilon, \operatorname{sgn}_\eta(w^\varepsilon - k)\phi \Big\rangle \, dt$$

$$+ \iint_{Q_T} \Big(\big[V(x)(f(w^\varepsilon) - f(k)) - \nabla A^\varepsilon(w^\varepsilon)\big] \cdot \nabla\big[\operatorname{sgn}_\eta(w^\varepsilon - k)\phi\big]$$

$$- \operatorname{div}V(x)f(k)\big[\operatorname{sgn}_\eta(w^\varepsilon - k)\phi\big] \Big) \, dt \, dx = 0.$$

By the chain rule we obviously have

$$-\int_0^T \Big\langle \partial_t w^\varepsilon, \operatorname{sgn}_\eta(w^\varepsilon - k)\phi \Big\rangle dt$$

$$= \iint_{Q_T} \left(\int_k^{w^\varepsilon} \operatorname{sgn}_\eta(\xi - k) \, d\xi \right) \partial_t \phi \, dt \, dx \xrightarrow{\eta\downarrow 0} \iint_{Q_T} |w^\varepsilon - k| \partial_t \phi \, dt \, dx$$

so that

$$\iint_{Q_T} |w^\varepsilon - k| \partial_t \phi \, dt \, dx$$

$$+ \lim_{\eta\downarrow 0} \iint_{Q_T} \Big(\big[V(x)(f(w^\varepsilon) - f(k)) - \nabla A^\varepsilon(w^\varepsilon) \big] \cdot \nabla \big[\operatorname{sgn}_\eta(w^\varepsilon - k)\phi \big]$$

$$- \operatorname{sgn}_\eta(w^\varepsilon - k)\operatorname{div}V(x)f(k)\phi \Big) dt \, dx = 0. \tag{3.10}$$

Firstly we have

$$\lim_{\eta\downarrow 0} \iint_{Q_T} \operatorname{sgn}_\eta(w^\varepsilon - k)\operatorname{div}V(x)f(k)\phi \, dt \, dx = \iint_{Q_T} \operatorname{sgn}(w^\varepsilon - k)\operatorname{div}V(x)f(k)\phi \, dt \, dx.$$

Next we have

$$\lim_{\eta\downarrow 0} \iint_{Q_T} \big[V(x)(f(w^\varepsilon) - f(k)) - \nabla A^\varepsilon(w^\varepsilon) \big] \cdot \nabla \big[\operatorname{sgn}_\eta(w^\varepsilon - k)\phi \big] \, dt \, dx$$

$$= \lim_{\eta\downarrow 0} \iint_{Q_T} \big[V(x)(f(w^\varepsilon) - f(k)) - \nabla A^\varepsilon(w^\varepsilon) \big] \cdot \nabla \operatorname{sgn}_\eta(w^\varepsilon - k)\phi \, dt \, dx$$

$$+ \lim_{\eta\downarrow 0} \iint_{Q_T} \big[V(x)(f(w^\varepsilon) - f(k)) - \nabla A^\varepsilon(w^\varepsilon) \big] \cdot \operatorname{sgn}_\eta(w^\varepsilon - k)\nabla \phi \, dt \, dx$$

$$= \underbrace{\lim_{\eta\downarrow 0} \iint_{Q_T} V(x)(f(w^\varepsilon) - f(k))\operatorname{sgn}_\eta'(w^\varepsilon - k)\nabla A^\varepsilon(w^\varepsilon)\phi \, dt \, dx}_{I_1}$$

$$- \lim_{\eta\downarrow 0} \iint_{Q_T} (A^\varepsilon)'(w^\varepsilon)|\nabla w^\varepsilon|^2 \operatorname{sgn}_\eta'(w^\varepsilon - k)\phi \, dt \, dx$$

$$+ \iint_{Q_T} \operatorname{sgn}(w^\varepsilon - k)\big[V(x)(f(w^\varepsilon) - f(k)) - \nabla A^\varepsilon(w^\varepsilon) \big] \cdot \nabla \phi \, dt \, dx.$$

Note that I_1 can be rewritten as $I_1 = \lim\limits_{\eta\downarrow 0} \iint_{Q_T} V(x)\operatorname{div}\mathcal{Q}_\eta(w^\varepsilon)\phi \, dt \, dx$, where

$$\mathcal{Q}_\eta(z) := \int_0^z \operatorname{sgn}_\eta'(r - k)\big(f(r) - f(k)\big) \, dr$$

$$= \frac{1}{\eta} \int_{\min(z,k-\eta)}^{\min(z,k+\eta)} \big(f(r) - f(k)\big) \, dr \to 0 \text{ as } \eta \downarrow 0.$$

From Lebesgue's dominated convergence theorem we conclude that

$$I_1 = -\lim_{\eta\downarrow 0}\iint_{Q_T}\Big(\mathcal{Q}_\eta(A(w^\varepsilon))V(x)\cdot\nabla\phi + \mathcal{Q}_\eta(A(w^\varepsilon))\mathrm{div}V(x)\phi\Big)\,dt\,dx = 0.$$

In conclusion we have

$$\iint_{Q_T}\Big(|w^\varepsilon - k|\phi_t + \mathrm{sgn}(w^\varepsilon - k)\big[V(x)(f(w^\varepsilon) - f(k)) - \nabla A^\varepsilon(w^\varepsilon)\big]\cdot\nabla\phi$$

$$- \mathrm{sgn}(w^\varepsilon - k)\mathrm{div}V(x)f(k)\phi\Big)\,dt\,dx$$

$$= \lim_{\varepsilon\downarrow 0}\iint_{Q_T}(A^\varepsilon)'(w^\varepsilon)|\nabla w^\varepsilon|^2\mathrm{sgn}_\eta(w^\varepsilon - k)\phi\,dt\,dx \geq 0 \tag{3.11}$$

for any $0 \leq \phi \in C_0^\infty(Q_T)$ and any $k \in \mathbb{R}$. From this we conclude easily that (3.8) holds.

It remains to prove the parabolic entropy inequality (3.9). Let $0 \leq \phi \in C_0^\infty(Q_T)$ and $k \in \mathbb{R}$ be such that $A(k) \notin H$. Starting off by choosing $\big[\mathrm{sgn}_\eta(A(w^\varepsilon) - A(k))\phi\big]$ as a test function in the weak formulation and then continuing exactly as in the proof of (3.3), we obtain

$$E^{\mathrm{par}}(w^\varepsilon, k, \phi) = \lim_{\eta\downarrow 0}\iint_{Q_T}\varepsilon\nabla w^\varepsilon \cdot \nabla\big[\mathrm{sgn}_\eta\big(A(w^\varepsilon) - A(k)\big)\phi\big]\,dt\,dx.$$

The right-hand side of this equality can be expanded into

$$\lim_{\eta\downarrow 0}\iint_{Q_T}\Big(\varepsilon A'(w^\varepsilon)|\nabla w^\varepsilon|^2\mathrm{sgn}_\eta'(A(w^\varepsilon) - A(k))\phi$$

$$+ \varepsilon\,\mathrm{sgn}_\eta(A(w^\varepsilon) - A(k))\nabla w^\varepsilon \cdot \nabla\phi\Big)\,dt\,dx$$

$$\geq \lim_{\eta\downarrow 0}\iint_{Q_T}\varepsilon\,\mathrm{sgn}_\eta(A(w^\varepsilon) - A(k))\nabla w^\varepsilon \cdot \nabla\phi\,dt\,dx \geq -\varepsilon\iint_{Q_T}|\nabla w^\varepsilon \cdot \nabla\phi|\,dt\,dx.$$

This concludes the proof of (3.9).   ■

4   Proof of Theorem 1

Following Carrillo [5] (see also [13]) in this section we use Lemma 2 and Lemma 3 to prove Theorem 1. Let $w^\varepsilon = w^\varepsilon(x,t)$ solve (1.1) and $w = w(y,s)$ solve (1.2). Following Kružkov [15] and Kuznetsov [16] we now specify a nonnegative test function $\phi = \phi(t, x, s, y)$ defined on $Q_T \times Q_T$. To this end let $\rho \in C_0^\infty(\mathbb{R})$ be a function satisfying

$$\mathrm{supp}(\rho) \subset \{\sigma \in \mathbb{R} : |\sigma| \leq 1\}, \qquad \rho(\sigma) \geq 0\,\forall\sigma \in \mathbb{R}, \qquad \int_{\mathbb{R}}\rho(\sigma)\,d\sigma = 1.$$

For $x \in \mathbb{R}^d, t \in \mathbb{R}$ and $r, r_0 > 0$, let $\omega_r(x) = \frac{1}{r}\rho\left(\frac{x_1}{r}\right)\cdots\frac{1}{r}\rho\left(\frac{x_d}{r}\right)$ and $\rho_{r_0}(t) = \frac{1}{r_0}\rho\left(\frac{t}{r_0}\right)$. Pick any two points $\nu, \tau \in (0, T), \nu < \tau$. For any $\alpha_0 > 0$ define

$$\psi_{\alpha_0}(t) = H_{\alpha_0}(t - \nu) - H_{\alpha_0}(t - \tau), \qquad H_{\alpha_0}(t) = \int_{-\infty}^t \rho_{\alpha_0}(\xi)\,d\xi.$$

With $0 < r_0 < \min(\nu, T - \tau)$ and $\alpha_0 \in \big(0, \min(\nu - r_0, T - \tau - r_0)\big)$ we set

$$\phi(x, t, y, s) := \psi_{\alpha_0}(t)\omega_r(x - y)\rho_{r_0}(t - s). \tag{4.1}$$

Note that $\operatorname{supp}(\phi(x, \cdot, y, s)) \subset (r_0, T - r_0)$ for all $x, y \in \mathbb{R}^d$, $s \in (0, T)$ and $\operatorname{supp}(\phi(x, t, y, \cdot))$ $\subset (0, T)$ for all $x, y \in \mathbb{R}^d, t \in (0, T)$. Consequently $(x, t) \mapsto \phi(x, t, y, s)$ belongs to $C_0^\infty(Q_T)$ for each fixed $(y, s) \in Q_T$ and $(y, s) \mapsto \phi(x, t, y, s)$ belongs to $C_0^\infty(Q_T)$ for each fixed $(x, t) \in Q_T$.

Observe that with the choice of ϕ as in (4.1) we have

$$\begin{aligned} \partial_t \phi + \partial_s \phi &= \big[\rho_{\alpha_0}(t - \nu) - \rho_{\alpha_0}(t - \tau)\big]\omega_r(x - y)\rho_{r_0}(t - s), \\ \nabla_x \phi + \nabla_y \phi &= 0. \end{aligned} \tag{4.2}$$

Before continuing we need to introduce the two "hyperbolic" sets

$$\mathcal{H}^\varepsilon = \Big\{(x, t) \in Q_T : A(w^\varepsilon(x, t)) \in H\Big\}, \qquad \mathcal{H} = \Big\{(y, s) \in Q_T : A(w(y, s)) \in H\Big\}$$

and note that

$$\nabla_x A(w^\varepsilon) = 0 \text{ a.e. in } \mathcal{H}^\varepsilon \quad \text{and} \quad \nabla_y A(w) = 0 \text{ a.e. in } \mathcal{H}, \tag{4.3}$$

$$\operatorname{sgn}(w^\varepsilon - w) = \operatorname{sgn}\big(A(w^\varepsilon) - A(w)\big)$$

$$\text{a.e. in } \Big[(Q_T \setminus \mathcal{H}) \times Q_T\Big] \cup \Big[Q_T \times (Q_T \setminus \mathcal{H}^\varepsilon)\Big]. \tag{4.4}$$

Using the approximate hyperbolic entropy inequality (3.8) for the viscous solution $w^\varepsilon = w^\varepsilon(x, t)$ with $k = w(y, s)$, we get for $(y, s) \in Q_T$

$$\iint\limits_{Q_T} \Big(|w^\varepsilon - w|\partial_t \phi + \operatorname{sgn}(w^\varepsilon - w)\big[V(x)(f(w^\varepsilon) - f(w)) - \nabla_x A(w^\varepsilon)\big] \cdot \nabla_x \phi$$

$$- \operatorname{sgn}(w^\varepsilon - w)\operatorname{div}_x V(x)f(w)\phi\Big) \, dt \, dx \, ds \, dy \geq -\overline{R}_{\mathrm{visc}}. \tag{4.5}$$

Using the approximate parabolic entropy inequality (3.9) for the viscous solution $w^\varepsilon = w^\varepsilon(x, t)$ with $k = w(y, s)$, we get for $(y, s) \in Q_T \setminus \mathcal{H}$

$$\iint\limits_{Q_T} \Big(|w^\varepsilon - w|\partial_t \phi + \operatorname{sgn}(w^\varepsilon - w)\big[V(x)(f(w^\varepsilon) - f(w)) - \nabla_x A(w^\varepsilon)\big] \cdot \nabla_x \phi$$

$$- \operatorname{sgn}(w^\varepsilon - w)\operatorname{div}_x V(x)f(w)\phi\Big) \, dt \, dx$$

$$\geq \lim_{\eta \downarrow 0} \iint\limits_{Q_T} \big|\nabla_x A(w^\varepsilon)\big|^2 \operatorname{sgn}'_\eta(A(w^\varepsilon) - A(w))\phi \, dt \, dx - \overline{R}_{\mathrm{visc}}. \tag{4.6}$$

Next we would like to integrate (4.5) and (4.6) over $(y, s) \in Q_T$ and $(y, s) \in Q_T \setminus \mathcal{H}$ respectively. To this end we need to know that the involved functions are (y, s) integrable. Consider first $(y, s) \mapsto \iint\limits_{Q_T} \operatorname{sgn}(v - u)\nabla_x A(w^\varepsilon) \cdot \nabla_x \phi \, dt \, dx$. We denote this function by $D(y, s)$.

To see that $D(\cdot,\cdot)$ is integrable on Q_T we observe that for each fixed $(y,s) \in Q_T$

$$\operatorname{sgn}(v - u)\nabla_x A(w^\varepsilon) = \nabla_x |A(w^\varepsilon) - A(w)| \quad \text{for a.e. } (x,t) \in Q_T$$

and hence

$$D(y,s) = \iint_{Q_T} \left[\nabla_x |A(w^\varepsilon) - A(w)|\right] \cdot \nabla_x \phi \, dt \, dx.$$

Since the function $(x,t) \mapsto \phi(x,t,y,s)$ belongs to $C_0^\infty(Q_T)$ for each fixed $(y,s) \in Q_T$, an integration by parts in x gives

$$D(y,s) = -\iint_{Q_T} |A(w^\varepsilon) - A(w)| \Delta_x \phi \, dt \, dx.$$

Integration over $(y,s) \in Q_T$ and estimation yield

$$\left|\iint_{Q_T} D(y,s) \, ds \, dy\right| \leq \iiiint_{Q_T \times Q_T} \left(|A(w^\varepsilon(x,t))| + |A(w(y,s))|\right) \Delta_x \phi(x,y,t,s) \, dt \, dx \, ds \, dy.$$

By changing the variables $(z := x - y,\ \tau = t - s)$ and taking into account that $w^\varepsilon, w \in L^1(Q_T)$ we find that

$$\left|\iint_{Q_T} D(y,s) \, ds \, dy\right| \leq \iiiint |A(w^\varepsilon(x,t))||\psi_{\alpha_0}(t)||\Delta_z \omega_r(z)|\, \rho_{r_0}(\tau) \, dt \, dx \, d\tau \, dz$$

$$+ \iiiint |A(w(x-z,t-\tau))||\psi_{\alpha_0}(t)||\Delta_z \omega_r(z)|\, \rho_{r_0}(\tau) \, dt \, dx \, d\tau \, dz$$

$$\leq \|A(w^\varepsilon)\|_{L^1(Q_T)}\|\Delta_z \omega_r\|_{L^1(\mathbb{R}^d)} + \|A(w)\|_{L^1(Q_T)}\|\Delta_z \omega_r\|_{L^1(\mathbb{R}^d)} < \infty.$$

Hence we have that $D(\cdot,\cdot)$ is integrable on Q_T.

In a similar vein one can also show the integrability of

$$(y,s) \mapsto \iint_{Q_T} |w^\varepsilon - w||\partial_t \phi \, dt \, dx,$$

$$(y,s) \mapsto \iint_{Q_T} \operatorname{sgn}(w^\varepsilon - w)V(x)(f(w^\varepsilon) - f(w)) \cdot \nabla_x \phi \, dt \, dx,$$

$$(y,s) \mapsto \iint_{Q_T} \operatorname{sgn}(w^\varepsilon - w)\operatorname{div}_x V(x)f(w)\phi \, dt \, dx, \qquad \text{and} \quad (y,s) \mapsto \overline{R}_{\text{visc}}.$$

It remains to consider the integrability of the function

$$Q_T \setminus \mathcal{H} \ni (y,s) \mapsto \lim_{\eta \downarrow 0} \iint_{Q_T} |\nabla_x A(w^\varepsilon)|^2 \operatorname{sgn}'_\eta(A(w^\varepsilon) - A(w))\phi \, dt \, dx.$$

This follows from (4.6). We have by Lebesgue's dominated convergence theorem and the first part of (4.3)

$$\iint\limits_{Q_T\setminus\mathcal{H}} \left(\lim_{\eta\downarrow 0} \iint\limits_{Q_T} |\nabla_x A(w^\varepsilon)|^2 \mathrm{sgn}'_\eta(A(w^\varepsilon)-A(w))\phi\,dt\,dx \right) ds\,dy$$

$$= \lim_{\eta\downarrow 0} \iiiint\limits_{(Q_T\setminus\mathcal{H})\times Q_T} |\nabla_x A(w^\varepsilon)|^2 \mathrm{sgn}'_\eta(A(w^\varepsilon)-A(w))\phi\,dt\,dx\,ds\,dy.$$

$$= \lim_{\eta\downarrow 0} \iiiint\limits_{(Q_T\setminus\mathcal{H})\times (Q_T\setminus\mathcal{H})} |\nabla_x A(w^\varepsilon)|^2 \mathrm{sgn}'_\eta(A(w^\varepsilon)-A(w))\phi\,dt\,dx\,ds\,dy. \qquad (4.7)$$

We now integrate (4.5) over $(y,s)\in Q_T$ and (4.6) over $(y,s)\in Q_T\setminus\mathcal{H}$. Addition of the two resulting inequalities yields

$$\iiiint\limits_{Q_T\times Q_T} \Big(|w^\varepsilon - w|\partial_t\phi + \mathrm{sgn}(w^\varepsilon - w)\big[V(x)(f(w^\varepsilon)-f(w)) - \nabla_x A(w^\varepsilon)\big]\cdot\nabla_x\phi$$

$$- \mathrm{sgn}(w^\varepsilon - w)\mathrm{div}_x V(x)f(w)\phi \Big)\,dt\,dx\,ds\,dy$$

$$= \iiiint\limits_{(Q_T\setminus\mathcal{H})\times Q_T} \Big(|w^\varepsilon - w|\partial_t\phi + \mathrm{sgn}(w^\varepsilon - w)\big[V(x)(f(w^\varepsilon)-f(w)) - \nabla_x A(w^\varepsilon)\big]\cdot\nabla_x\phi$$

$$- \mathrm{sgn}(w^\varepsilon - w)\mathrm{div}_x V(x)f(w)\phi \Big)\,dt\,dx\,ds\,dy$$

$$+ \iiiint\limits_{\mathcal{H}\times Q_T} \Big(|w^\varepsilon - w|\partial_t\phi + \mathrm{sgn}(w^\varepsilon - w)\big[V(x)(f(w^\varepsilon)-f(w)) - \nabla_x A(w^\varepsilon)\big]\cdot\nabla_x\phi$$

$$- \mathrm{sgn}(w^\varepsilon - w)\mathrm{div}_x V(x)f(w)\phi \Big)\,dt\,dx\,ds\,dy$$

$$\geq \lim_{\eta\downarrow 0} \iiiint\limits_{(Q_T\setminus\mathcal{H})\times (Q_T\setminus\mathcal{H}^\varepsilon)} |\nabla_x A(w^\varepsilon)|^2 \mathrm{sgn}'_\eta(A(w^\varepsilon)-A(w))\phi\,dt\,dx\,ds\,dy - \overline{R}_{\mathrm{visc}}, \quad (4.8)$$

where $\overline{R}_{\mathrm{visc}} := \iint\limits_{Q_T} R_{\mathrm{visc}}\,ds\,dy$ and we have used (4.7).

Similarly, using the hyperbolic, parabolic entropy inequalities (3.1), (3.3) for the exact entropy solution $w = w(y,s)$ with $k = w^\varepsilon(x,t)$ and then integrating over $(x,t)\in Q_T$, we get

$$\iiiint\limits_{Q_T\times Q_T} \Big(|w - w^\varepsilon|\partial_s\phi + \mathrm{sgn}(w - w^\varepsilon)\big[V(y)(f(w)-f(w^\varepsilon)) - \nabla_y A(w)\big]\cdot\nabla_y\phi$$

$$- \mathrm{sgn}(w - w^\varepsilon)\mathrm{div}_y V(y)f(w^\varepsilon)\phi \Big)\,dt\,dx\,ds\,dy$$

$$\geq \lim_{\eta\downarrow 0} \iiiint\limits_{(Q_T\setminus\mathcal{H}^\varepsilon)\times (Q_T\setminus\mathcal{H})} |\nabla_y A(w)|^2 \mathrm{sgn}'_\eta(A(w)-A(w^\varepsilon))\phi\,dt\,dx\,ds\,dy. \qquad (4.9)$$

Using (4.3) and (4.4) we find that

$$\iiiint\limits_{Q_T \times Q_T} \mathrm{sgn}(w_\varepsilon - w)\nabla_x A(w^\varepsilon) \cdot \nabla_y \phi \, dt \, dx \, ds \, dy$$

$$= - \iiiint\limits_{Q_T \times (Q_T \backslash \mathcal{H}^\varepsilon)} \mathrm{sgn}(A(w^\varepsilon) - A(w))\nabla_x A(w^\varepsilon) \cdot \nabla_y \phi \, dt \, dx \, ds \, dy$$

$$= - \lim_{\eta \downarrow 0} \iiiint\limits_{Q_T \times (Q_T \backslash \mathcal{H}^\varepsilon)} \mathrm{sgn}_\eta(A(w^\varepsilon) - A(w))\nabla_x A(w^\varepsilon) \cdot \nabla_y \phi \, dt \, dx \, ds \, dy$$

$$= - \lim_{\eta \downarrow 0} \iiiint\limits_{Q_T \times (Q_T \backslash \mathcal{H}^\varepsilon)} \nabla_y A(w) \cdot \nabla_x A(w^\varepsilon)\mathrm{sgn}'_\eta(A(w^\varepsilon) - A(w))\phi \, dt \, dx \, ds \, dy.$$

$$= - \lim_{\eta \downarrow 0} \iiiint\limits_{(Q_T \backslash \mathcal{H}) \times (Q_T \backslash \mathcal{H}^\varepsilon)} \nabla_y A(w) \cdot \nabla_x A(w^\varepsilon)\mathrm{sgn}'_\eta(A(w^\varepsilon) - A(w))\phi \, dt \, dx \, ds \, dy. \quad (4.10)$$

Similarly, again using (4.3) and (4.4), we find that

$$-\iiiint\limits_{Q_T \times Q_T} \mathrm{sgn}(w - w^\varepsilon)\nabla_y A(w) \cdot \nabla_x \phi \, dt \, dx \, ds \, dy$$

$$= - \lim_{\eta \downarrow 0} \iiiint\limits_{(Q_T \backslash \mathcal{H}) \times (Q_T \backslash \mathcal{H}^\varepsilon)} \nabla_x A(w^\varepsilon) \cdot \nabla_y A(w)\mathrm{sgn}'_\eta(A(w) - A(w^\varepsilon))\phi \, dt \, dx \, ds \, dy. \quad (4.11)$$

The use of the second part of (4.2) when adding (4.8) and (4.10) yields

$$\iiiint\limits_{Q_T \times Q_T} \Big(|w^\varepsilon - w|\partial_t \phi + \mathrm{sgn}(w^\varepsilon - w)\big[V(x)(f(w^\varepsilon) - f(w))\big] \cdot \nabla_x \phi$$

$$- \mathrm{sgn}(w^\varepsilon - w)\mathrm{div}_x V(x) f(w)\phi \Big) \, dt \, dx \, ds \, dy$$

$$\geq \lim_{\eta \downarrow 0} \iiiint\limits_{(Q_T \backslash \mathcal{H}^\varepsilon) \times (Q_T \backslash \mathcal{H})} \Big(|\nabla_x A(w^\varepsilon)|^2 - \nabla_y A(w) \cdot \nabla_x A(w^\varepsilon) \Big)$$

$$\times \mathrm{sgn}'_\eta(A(w^\varepsilon) - A(w))\phi \, dt \, dx \, ds \, dy - R_{\mathrm{visc}}. \quad (4.12)$$

Similarly the addition of (4.9) and (4.11) yields

$$\iiiint\limits_{Q_T \times Q_T} \Big(|w - w^\varepsilon|\partial_s \phi + \mathrm{sgn}(w - w^\varepsilon)\big[V(y)(f(w) - f(w^\varepsilon))\big] \cdot \nabla_y \phi$$

$$- \mathrm{sgn}(w - w^\varepsilon)\mathrm{div}_y V(y) f(w^\varepsilon)\phi \Big) \, dt \, dx \, ds \, dy$$

$$\geq \lim_{\eta \downarrow 0} \iiiint\limits_{(Q_T \backslash \mathcal{H}^\varepsilon) \times (Q_T \backslash \mathcal{H})} \Big(|\nabla_y A(w)|^2 - \nabla_x A(w^\varepsilon) \cdot \nabla_y A(w) \Big)$$

$$\times \mathrm{sgn}'_\eta(A(w) - A(w^\varepsilon))\phi \, dt \, dx \, ds \, dy. \quad (4.13)$$

Following Karlsen and Risebro [13] we write

$$
\begin{aligned}
\operatorname{sgn}&(w^\varepsilon - w)V(x)(f(w^\varepsilon) - f(w)) \cdot \nabla_x \phi - \operatorname{sgn}(w^\varepsilon - w)\operatorname{div}_x V(x)f(w)\phi \\
&= \operatorname{sgn}(w^\varepsilon - w)\big(V(x)f(w^\varepsilon) - V(y)f(w)\big) \cdot \nabla_x \phi \\
&\quad + \operatorname{sgn}(w^\varepsilon - w)\operatorname{div}_x\big[(V(y)f(w) - V(x)f(w))\phi\big], \\
\operatorname{sgn}&(w - w^\varepsilon)V(y)(f(w) - f(w^\varepsilon)) \cdot \nabla_y \phi - \operatorname{sgn}(w - w^\varepsilon)\operatorname{div}_y V(y)f(w^\varepsilon)\phi \\
&= \operatorname{sgn}(w^\varepsilon - w)\big(V(x)f(w^\varepsilon) - V(y)f(w)\big) \cdot \nabla_y \phi \\
&\quad - \operatorname{sgn}(w^\varepsilon - w)\operatorname{div}_y\big[(V(x)f(w^\varepsilon) - V(y)f(w^\varepsilon))\phi\big].
\end{aligned}
$$

When adding (4.12) and (4.13), we use the second part of (4.2) and the identities

$$
\operatorname{sgn}(-r) = -\operatorname{sgn}(r) \text{ a.e. in } \mathbb{R}, \qquad \operatorname{sgn}'_\eta(-r) = \operatorname{sgn}'_\eta(r) \text{ a.e. in } \mathbb{R}.
$$

The final result takes the form

$$
\begin{aligned}
-\iiiint_{Q_T \times Q_T} |w^\varepsilon - w|(\partial_t \phi + \partial_s \phi)\, dt\, dx\, ds\, dy& \\
\leq R_{\text{diss}} + \overline{R}_{\text{visc}} + R_{\text{conv}} &\leq \overline{R}_{\text{visc}} + R_{\text{conv}},
\end{aligned}
\tag{4.14}
$$

where the expression for $\partial_t \phi + \partial_s \phi$ is written out in (4.2),

$$
R_{\text{conv}} := \iiiint_{Q_T \times Q_T} I_{\text{conv}}\, dt\, dx\, ds\, dy,
$$

$$
\begin{aligned}
I_{\text{conv}} := \operatorname{sgn}(w^\varepsilon - w)\Big(&\operatorname{div}_x\big[(V(y)f(w) - V(x)f(w))\phi\big] \\
&- \operatorname{div}_y\big[(V(x)f(w^\varepsilon) - V(y)f(w^\varepsilon))\phi\big]\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
R_{\text{diss}} := -\lim_{\eta \downarrow 0} \iiiint_{(Q_T \setminus \mathcal{H}^\varepsilon) \times (Q_T \setminus \mathcal{H})} &\big|\nabla_x A(w^\varepsilon) - \nabla_y A(w)\big|^2 \\
&\times \operatorname{sgn}'_\eta\big(A(w^\varepsilon) - A(w)\big)\phi\, dt\, dx\, ds\, dy \leq 0.
\end{aligned}
$$

Having in mind the first part of (4.2), we get by the triangle inequality

$$
-\iiiint_{Q_T \times Q_T} |w^\varepsilon(x,t) - w(y,s)|(\partial_t \phi + \partial_s \phi)\, dt\, dx\, ds\, dy \leq R_{w^\varepsilon,w} + R_{w,x} + R_{w,t},
$$

where

$$
\begin{aligned}
R_{w^\varepsilon,w} := -\iiiint_{Q_T \times Q_T} &|w^\varepsilon(x,t) - w(x,t)|\big[\rho_{\alpha_0}(t - \nu) - \rho_{\alpha_0}(t - \tau)\big] \\
&\times \omega_r(x - y)\rho_{r_0}(t - s)\, dt\, dx\, ds\, dy, \\
R_{w,x} := -\iiiint_{Q_T \times Q_T} &|w(x,t) - w(y,t)|\big[\rho_{\alpha_0}(t - \nu) - \rho_{\alpha_0}(t - \tau)\big] \\
&\times \omega_r(x - y)\rho_{r_0}(t - s)\, dt\, dx\, ds\, dy,
\end{aligned}
$$

$$R_{w,t} := - \iiiint\limits_{Q_T \times Q_T} |w(y,t) - w(y,s)| \big[\rho_{\alpha_0}(t-\nu) - \rho_{\alpha_0}(t-\tau)\big]$$

$$\times \omega_r(x-y)\rho_{r_0}(t-s)\, dt\, dx\, ds\, dy.$$

Firstly a standard L^1 continuity argument gives $\lim\limits_{r_0\downarrow 0} R_{w,t} = 0$. Next

$$\lim_{\alpha_0\downarrow 0} R_{w,x} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big(|w(x,\tau) - w(y,\tau)| - |w(x,\nu) - w(y,\nu)|\Big)\omega_r(x-y)\, dx\, dy$$

$$\overset{(z:=x-y)}{\leq} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|w(y+z,\tau) - w(y,\tau)|\omega_r(z)\, dy\, dz$$

$$\leq |w|_{L^\infty(0,T;BV(\mathbb{R}^d))}\int_{\mathbb{R}^d}|z|\omega_r(z)\, dz \leq C_1 r,$$

where $C_1 := |w|_{L^\infty(0,T;BV(\mathbb{R}^d))}$. Finally we have

$$\lim_{\alpha_0\downarrow 0} R_{w^\varepsilon,w} = \int_{\mathbb{R}^d}|w^\varepsilon(x,\tau) - w(x,\tau)|\, dx - \int_{\mathbb{R}^d}|w^\varepsilon(x,\nu) - w(x,\nu)|\, dx.$$

In summary from (4.14) we obtain the following approximation inequality

$$\int_{\mathbb{R}^d}|w^\varepsilon(x,\tau) - w(x,\tau)|\, dx$$

$$\leq \int_{\mathbb{R}^d}|w^\varepsilon(x,\nu) - w(x,\nu)|\, dx + C_1 r + \lim_{r_0,\alpha_0\downarrow 0}\Big(\overline{R}_{\mathrm{visc}} + R_{\mathrm{conv}}\Big). \tag{4.15}$$

We start with the estimation of $\overline{R}_{\mathrm{visc}}$, which can be done as follows:

$$\overline{R}_{\mathrm{visc}} \leq \varepsilon \sum_{i=1}^d \iiiint\limits_{Q_T \times Q_T} |\partial_{x_i}w^\varepsilon|\,\psi_{\alpha_0}(t)\big|\partial_{x_i}\omega_r(x-y)\big|\rho_{r_0}(t-s)\, dt\, dx\, ds\, dy$$

$$\overset{\alpha_0\downarrow 0}{\longrightarrow} \sum_{i=1}^d \int_\nu^\tau \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\partial_{x_i}w^\varepsilon|\,\big|\partial_{x_i}\omega_r(x-y)\big|\, dx\, dy\, dt$$

$$\leq \varepsilon K/r \sum_{i=1}^d \int_\nu^\tau \int_{\mathbb{R}^d}\big|\partial_{x_i}w^\varepsilon\big|\, dt\, dx \leq \varepsilon TK/r|w^\varepsilon|_{L^\infty(0,T;BV(\mathbb{R}^d))} \leq C_2 T\varepsilon/r, \tag{4.16}$$

where $K := \int_{\mathbb{R}^d}\big|\delta'(\sigma)\big|\, d\sigma$ and $C_2 := K|w^\varepsilon|_{L^\infty(0,T;BV(\mathbb{R}^d))}$.

Before we continue with the estimation of R_{conv} we write $I_{\mathrm{conv}} = I_{\mathrm{conv}}^1 + I_{\mathrm{conv}}^2$, where

$$I_{\mathrm{conv}}^1 = \mathrm{sgn}(w^\varepsilon - w)\Big[\big(V(y)f(w) - V(x)f(w)\big) \cdot \nabla_x\phi$$

$$- \big(V(x)f(w^\varepsilon) - V(y)f(w^\varepsilon)\big) \cdot \nabla_y\phi\Big],$$

$$I_{\mathrm{conv}}^2 = \mathrm{sgn}(w^\varepsilon - w)\big(\mathrm{div}_y V(y)f(w^\varepsilon) - \mathrm{div}_x V(x)f(w)\big)\phi,$$

so that

$$R_{\mathrm{conv}} = R_{\mathrm{conv}}^1 + R_{\mathrm{conv}}^2,$$

$$R_{\mathrm{conv}}^1 = \iiiint\limits_{Q_T \times Q_T} I_{\mathrm{conv}}^1\, dt\, dx\, ds\, dy, \qquad R_{\mathrm{conv}}^2 = \iiiint\limits_{Q_T \times Q_T} I_{\mathrm{conv}}^2\, dt\, dx\, ds\, dy.$$

We start by estimating R^1_{conv}. To this end introduce

$$F(w^\varepsilon, w) := \text{sgn}(w^\varepsilon - w)\big[f(w^\varepsilon) - f(w)\big]$$

and observe that since $\nabla_y \phi = -\nabla_x \phi$,

$$R^1_{\text{conv}} = \iiiint\limits_{Q_T \times Q_T} \Big((V(x) - V(y))F(w^\varepsilon, w)\Big) \cdot \nabla_x \phi \, dt \, dx \, ds \, dy.$$

The function $F(\cdot, \cdot)$ is locally Lipschitz continuous in both variables and the common Lipschitz constant equals $\text{Lip}(f)$. Since $w^\varepsilon \in L^\infty(Q_T) \cap L^\infty(0, T; BV(\mathbb{R}^d))$, $\nabla_x F(w^\varepsilon, w)$ is a finite measure and

$$\iint\limits_{Q_T} \big|\partial_{x_i} F(w^\varepsilon, w)\big| \, dt \, dx \leq \text{Lip}(f) \iint\limits_{Q_T} \big|\partial_{x_i} w^\varepsilon\big| \, dt \, dx, \qquad i = 1, \ldots, d.$$

Integration by parts thus gives

$$R^1_{\text{conv}} = -\underbrace{\iiiint\limits_{Q_T \times Q_T} \Big(\text{div}_x V(x) F(w^\varepsilon, w) \psi_{\alpha_0}(t) \omega_r(x - y) \rho_{r_0}(t - s) \, dt \, dx \, ds \, dy}_{R^{1,1}_{\text{conv}}}$$

$$-\underbrace{\iiiint\limits_{Q_T \times Q_T} (V(x) - V(y)) \cdot \nabla_x F(w^\varepsilon, w) \psi_{\alpha_0}(t) \omega_r(x - y) \rho_{r_0}(t - s) \, dt \, dx \, ds \, dy}_{R^{1,2}_{\text{conv}}}.$$

For $R^{1,2}_{\text{conv}}$ we calculate as follows:

$$\big|R^{1,2}_{\text{conv}}\big| \leq \text{Lip}(f) \sum_{i=1}^d \iiiint\limits_{Q_T \times Q_T} |V_i(x) - V_i(y)| \big|\partial_{x_i} w^\varepsilon\big| \psi_{\alpha_0}(t) \omega_r(x - y) \rho_{r_0}(t - s) \, dt \, dx \, ds \, dy$$

$$\xrightarrow{\alpha_0 \downarrow 0} \text{Lip}(f) \sum_{i=1}^d \int_\nu^\tau \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V_i(x) - V_i(y)| \big|\partial_{x_i} w^\varepsilon\big| \omega_r(x - y) \, dx \, dy \, dt$$

$$\stackrel{(z:=x-y)}{=} \text{Lip}(f) \sum_{i=1}^d \int_\nu^\tau \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V_i(y + z) - V_i(y)| \big|\partial_{y_i} w^\varepsilon(y + z, t)\big| \omega_r(z) \, dz \, dy \, dt$$

$$\leq \text{Lip}(V) \text{Lip}(f) \sum_{i=1}^d \int_\nu^\tau \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |z| \big|\partial_{y_i} w^\varepsilon(y + z, t)\big| \omega_r(z) \, dz \, dy \, dt$$

$$\leq T \text{Lip}(V) \text{Lip}(f) |w^\varepsilon|_{L^\infty(0, T; BV(\mathbb{R}^d))} \int_{\mathbb{R}^d} |z| \omega_r(z) \, dz$$

$$\leq r T \text{Lip}(V) \text{Lip}(f) |w^\varepsilon|_{L^\infty(0, T; BV(\mathbb{R}^d))} \leq C_3 T r,$$

where $\text{Lip}(V) := \max\limits_{i=1,\ldots,d} \text{Lip}(V_i)$ and $C_3 := \text{Lip}(V) \text{Lip}(f) |w^\varepsilon|_{L^\infty(0, T; BV(\mathbb{R}^d))}$. Note that we have used the Lipschitz regularity of the velocity field V (see (2.1)) to get the desired result.

Regarding the term R_{conv}^2, we firstly rewrite it as

$$R_{\text{conv}}^2 = \underbrace{\iiiint\limits_{Q_T \times Q_T} \text{div}_x V(x) F(w^\varepsilon, w) \psi_{\alpha_0}(t) \omega_r(x - y) \rho_{r_0}(t - s) \, dt \, dx \, ds \, dy}_{R_{\text{conv}}^{2,1}}$$

$$+ \underbrace{\iiiint\limits_{Q_T \times Q_T} \text{sgn}(w^\varepsilon - w) \big(\text{div}_y V(y) - \text{div}_x V(x)\big) f(w^\varepsilon) \psi_{\alpha_0}(t) \omega_r(x - y) \rho_{r_0}(t - s) \, dt \, dx \, ds \, dy}_{R_{\text{conv}}^{2,2}}.$$

We set

$$D(x) = \text{div} V(x),$$

and keep in mind that $D \in BV(\mathbb{R}^d)$ by (2.1). Now we estimate $R_{\text{conv}}^{2,2}$ as follows:

$$\big|R_{\text{conv}}^{2,2}\big| \leq \|f(w^\varepsilon)\|_{L^\infty(Q_T)} \iiiint\limits_{Q_T \times Q_T} \big|D(y) - D(x)\big| \psi_{\alpha_0}(t) \omega_r(x - y) \rho_{r_0}(t - s) \, dt \, dx \, ds \, dy$$

$$\xrightarrow{\alpha_0 \downarrow 0} \|f(w^\varepsilon)\|_{L^\infty(Q_T)} \int_\nu^\tau \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \big|D(y) - D(x)\big| \omega_r(x - y) \, dx \, dy \, dt$$

$$\overset{(z = x - y)}{=} \|f(w^\varepsilon)\|_{L^\infty(Q_T)} \int_\nu^\tau \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \big|D(y) - D(y + z)\big| \omega_r(z) \, dz \, dy \, dt$$

$$\leq T \|f(w^\varepsilon)\|_{L^\infty(Q_T)} \big|D\big|_{BV(\mathbb{R}^d)} \int_{\mathbb{R}^d} |z| \omega_r(z) \, dz \leq C_4 T r,$$

where $C_4 := \|f(w^\varepsilon)\|_{L^\infty(Q_T)} \big|D\big|_{BV(\mathbb{R}^d)}$. Note that we have used the BV regularity of $\text{div} V$ to get the desired result. Since $R_{\text{conv}}^{1,1} = R_{\text{conv}}^{2,1}$, we have

$$R_{\text{conv}} = R_{\text{conv}}^1 + R_{\text{conv}}^2 \leq C_5 T r, \qquad C_5 = \max(C_3, C_4). \tag{4.17}$$

Set $C_6 = \max(C_1, C_2, C_5)$. Then from (4.15), (4.16) and (4.17) we get

$$\int_{\mathbb{R}^d} |w^\varepsilon(x, \tau) - w(x, \tau)| \, dx \leq \int_{\mathbb{R}^d} |w^\varepsilon(x, \nu) - w(x, \nu)| \, dx + C_6 \left((1 + T) r + \frac{T\varepsilon}{r} \right)$$

$$\xrightarrow{\nu \downarrow 0} C_6 \left((1 + T) r + \frac{\varepsilon}{r} \right). \tag{4.18}$$

By choosing $r = \sqrt{T\varepsilon}$ we immediately obtain

$$\int_{\mathbb{R}^d} |w^\varepsilon(x, \tau) - w(x, \tau)| \, dx \leq C_7 \sqrt{T\varepsilon} \tag{4.19}$$

for some constant C_7 independent of ε. To obtain (2.5), we simply integrate (4.19) over $\tau \in (0, T)$.

Added in process

After the main result of this paper was obtained, we became aware of a paper by Eymard, Gallouet and Herbin [10] which also proves an error estimate for viscous approximate solutions. They, however, deal with a certain boundary value problem with a divergence free velocity field and obtain an error estimate of order $\varepsilon^{\frac{1}{5}}$. As is the case herein, the proof in [10] does not rely on a continuous dependence estimate.

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