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Exact Solutions of DNLS and Derivative Reaction-Diffuson Systems

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Abstract

In this paper, we obtain some exact solutions of Derivative Reaction-Diffusion (DRD) system and, as by-products, we also show some exact solutions of DNLS via Hirota bilinearization method. At first, we review some results about two by two AKNS-ZS system, then introduce Hirota bilinearization method to find out exact N-soliton solutions of DNLS and exact N-dissipaton solutions of the Derivative Reaction-Diffusion system respectively.

1 Introduction

Motivated by the results of Pashaev et al [19] on the resonance nonlinear Schrödinger equation (RNLS) and Reaction-Diffusion system , we can write down a new version of Derivative Nonlinear Schrödinger equation (DNLS),

$$iu_t + u_{xx} + i(u^2\overline{u})_x = 2\frac{|u|_{xx}}{|u|}u, \tag{1.1}$$

called Resonance DNLS (RDNLS). RDNLS is equivalent to a Derivative Reaction-Diffusion system by reciprocity relations [see Appendix].

In this paper, we obtain one and two dissipaton solutions of Derivative Reaction-Diffusion system and, as by-products, we obtain one and two soliton solutions of DNLS via Hirota bilinearization method [5, 6, 18]. At first, we review some results about 2×2 AKNS-ZS system [8, 9, 10, 12] , then we introduce Hirota bilinearization method to find out some exact soliton solutions of DNLS and dissipaton solutions of DRD system respectively.

2 Review on 2×2 AKNS-ZS system

In this section, we review some results of 2×2 AKNS-ZS system. J.-H. Lee [8, 9, 10, 12] consider the following 2×2 system

$$\frac{dM}{dx} = z^2[J, M] + (zQ + P)M, \ \ \mathrm{Im}z^2 \neq 0, \ \ Q \in L^1(R, M_2(\mathbb{C})), \tag{2.1}$$

where $M(\cdot, z)$ is bounded and continuous and $P = Q\mathcal{J}^{-1}Q$. $M(x, \cdot)$ is a meromorphic function with jump on $\sum = \{z : \mathrm{Im}z^2 = 0\}$. Let $\mathcal{D}_z M = \frac{dM}{dx} - z^2[J, M]$ and $M_\pm(x, \cdot)$ be the limits of M on $\Omega_+ = \{z : \mathrm{Im}z^2 > 0\}$, $\Omega_- = \{z : \mathrm{Im}z^2 < 0\}$. Then $\mathcal{D}_z((M_-)^{-1}M_+) = 0$. So, there exists $v(z)$ such that

$$M_+(x, z) = M_-(x, z)e^{xz^2 J}v(z)e^{-xz^2 J}. \tag{2.2}$$

Assume $M(x, \cdot)$ has finite number of poles $z_1, z_2, \cdots, z_N$. Let $N_j(x, z)$ be the regular part of $M(x, z)$ near $z = z_j$. There exists $v(z_j)$ such that

$$Res(M(x, \cdot), z_j)) = N_j(x, z_j)e^{xz_j^2 J}v(z_j)e^{-xz_j^2 J}. \tag{2.3}$$

$(v(z), z_1, z_2, \cdots, z_N, v(z_1), v(z_2), \cdots, v(z_N))$ is called scattering data of the potential (Q, P) [8, 9, 10, 11, 12].

Remark. Let $\mathcal{J}A = [J, A]$, $A \in M_{2\times2}(C)$, we denote $\mathcal{J}^{-1}$ be the inverse operator of $\mathcal{J}$ on its range.

For the case $Q = \begin{pmatrix} 0 & q \\ \epsilon q^* & 0 \end{pmatrix}$, $\epsilon = \pm1$ and the off-diagonal part of $P = 0$, $Q^* = \epsilon Q$, $P^* = -P$, $J^* = -J$. We have the following constraints

$$v^*(-\epsilon\overline{z}) = v(z), z^2 \in R; \tag{2.4a}$$
$$v^*(-\epsilon\overline{z}_j) = -v(z_j), z_j^2 \notin R; \tag{2.4b}$$
$$v^\sigma(-z) = v(z), z^2 \in R; \tag{2.4c}$$
$$v^\sigma(-z_j) = -v(z_j), z_j^2 \notin R, \tag{2.4d}$$

where $v(z_j) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$, Im$z^2 > 0$; $v^*(\overline{z}_j) = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$, Im$z^2 < 0$, here σ is an automorphism given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\sigma = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}. \tag{2.5}$$

If $v(z, t)$, $v(z_j, t)$ evolves as

$$\begin{cases} dv(z, t)/dt = \alpha z^4[J, v(z, t)], \\ dv(z_j, t)/dt = \alpha z^4[J, v(z_j, t)], \end{cases} \tag{2.6}$$

then the associated M satisfies

$$\begin{cases} dM/dx = z^2[J, M] + (zQ + P)M, \\ dM/dt = \alpha z^4[J, M] + (z^3 A_1 + z^2 A_2 + z^1 A_3 + zA_4)M. \end{cases} \tag{2.7}$$

Let $M = \psi \exp(xz^2 J)$, ψ satisfies

$$\begin{cases} d\psi/dx = (z^2 J + zQ + P)\psi = U\psi, \\ d\psi/dt = (\alpha z^4 J + z^3 A_1 + z^2 A_2 + zA_3 + A_4)\psi = V\psi. \end{cases} \tag{2.8}$$

For $\alpha = 2$, $U_t - V_x + [U, V] = 0$ is equivalent to

$$q_t = iq_{xx} - \epsilon q^2 \overline{q}_x + \frac{i}{2} q|q|^4, \ \ \epsilon = \pm 1, \tag{2.9}$$

which can be transformed into the DNLS explicitly, i.e. we consider the transformation $u = q\exp(\int_{-\infty}^{x} i\epsilon q\overline{q})$, then u satisfies

$$u_t = iu_{xx} - \epsilon(u^2\overline{u})_x, \ \ \epsilon = \pm 1, \tag{2.10}$$

which is DNLS considered by Kaup and Newell [7].

3 Exact Multi-Soliton Solutions of DNLS via Inverse Scattering Transform

We will review some results in this section on exact N-soliton solutions of DNLS [10, 12]. Consider the 2×2 system as

$$\frac{dM}{dx} = z^2[J, M] + (zQ + P)M, \tag{3.1}$$

with $Q^* = -Q$, $P = Q\mathcal{J}^{-1}Q$. In the degenerate case, $v(z) = I$, M is of the form

$$\begin{aligned} M &= I + \sum_{k=1}^{4N} \frac{a_k}{z - z_k} \\ &= \frac{a_j}{z - z_j} + (I + \sum_{k \neq j}^{4N} \frac{a_k}{z - z_k}) \\ &= \frac{a_j}{z - z_j} + N_j(x, z), \ \ \text{near} \ \ z = z_j, \end{aligned} \tag{3.2}$$

with

$$Res(M(x, \cdot), z_j)) = N_j(x, z_j)e^{xz_j^2 J} v(z_j) e^{-xz_j^2 J}. \tag{3.3}$$

Hence

$$a_j = (I + \sum_{k \neq j}^{4N} \frac{a_k}{z - z_k}) e^{xz_j^2 J} v(z_j) e^{-xz_j^2 J}, \tag{3.4}$$

here N corresponds to N-soliton.

If we let $M \sim I + \frac{m_1}{z} + \frac{m_2}{z^2} + \frac{m_3}{z^3} + \cdots$, with $|\frac{z_k}{z}| < 1$, we conclude that

$$Q = -[J, m_1] = -[J, \sum_{k=1}^{4N} a_k], \tag{3.5}$$

then potential $q(x,t)$ is the entry $(1,2)$ of $Q = \begin{pmatrix} 0 & q \\ -\overline{q} & 0 \end{pmatrix}$.

For example, we compute 1-soliton solution for DNLS by this method. Now, we look for $M = I + \frac{a_1}{z-z_1} + \frac{a_2}{z-z_2} + \frac{a_3}{z-z_3} + \frac{a_4}{z-z_4}$, where $z_3 = \overline{z}_1 = -z_2$, $z_4 = \overline{z}_2$. We put the constraint of (2.4). Let $z_1 = \alpha + i\beta$ and by a simple calculation, we get the following equations of linear algebra:

$$a_1 = (I + \frac{a_3}{i2\beta} + \frac{a_4}{2\alpha})\mathcal{A}, \tag{3.6a}$$

$$a_2 = (I + \frac{a_3}{-2\alpha} + \frac{a_4}{-i2\beta})\mathcal{A}, \tag{3.6b}$$

$$a_3 = (I + \frac{a_1}{-i2\beta} + \frac{a_2}{2\alpha})\mathcal{B}, \tag{3.6c}$$

$$a_4 = (I + \frac{a_1}{-2\alpha} + \frac{a_2}{i2\beta})\mathcal{B}, \tag{3.6d}$$

with $\mathcal{A}$, $\mathcal{B}$ denoted by

$$\mathcal{A} = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = e^{xz_1^2 J} v(z_1) e^{-xz_1^2 J} = \begin{pmatrix} 0 & l_0 e^{-2i(xz_1^2 + tz_1^4)} \\ 0 & 0 \end{pmatrix}, \tag{3.7a}$$

$$\mathcal{B} = \begin{pmatrix} 0 & 0 \\ \overline{m} & 0 \end{pmatrix} = e^{x\overline{z}_1^2 J} \overline{v}(\overline{z}_1) e^{-x\overline{z}_1^2 J} = \begin{pmatrix} 0 & 0 \\ -\overline{l}_0 e^{2i(x\overline{z}_1^2 + t\overline{z}_1^4)} & 0 \end{pmatrix}, \tag{3.7b}$$

where l_0 is a complex constant. From the process of computation, we observe that the soliton solutions only depend on a_1 and a_2. We get a_1 and a_2 immediately from (3.6), then put a_1 and a_2 into (3.5). Hence we get 1-soliton solution of DNLS as follows

$$q(x,t) = 2i \frac{i64\alpha^3\beta^3 m}{i32\alpha^3\beta^3 + |m|^2(i8\alpha^3\beta - i8\alpha\beta^3 + 16\alpha^2\beta^2)}, \tag{3.8}$$

which can be simplified as

$$q(x,t) = 2i \frac{2l_0 e^{-2i(xz_1^2 + tz_1^4)}}{1 + (\frac{1}{2\beta} - \frac{i}{2\alpha})^2 |m|^2}, \quad m = l_0 e^{-2i(xz_1^2 + tz_1^4)}. \tag{3.9}$$

Similar to the process of computation of 1-soliton, we can write down the algebraic equations by (3.2)-(3.5). Then we can compute the N-soliton solutions of DNLS. The asymptotic behavior of N-soliton of DNLS has been obtained by C.-T. Lin [14].

4 Hirota Bilinear Method and Multi-Soliton Solutions of DNLS

Hirota introduced a new method for constructing multi-soliton solutions to integrable nonlinear evolution equations [5, 6, 18], like Korteweg-de Vries, Sine-Gordon, nonlinear Schrödinger, derivative nonlinear Schrödinger equations, $\cdots$, etc. The main idea of Hirota bilinear method was to make a transformation into new variables, so that in these new variables, multi-soliton solutions appear in a particularly simple form. We can also derive multi-soliton solutions by other methods, e.g., by the inverse scattering transform. The advantage of Hirota's method over others is that it is algebraic rather than analytic [5].

Now, we introduce new derivatives D_x, D_t, called Hirota bilinear operators (also called Hirota derivative operators) defined by

$$D_x^m D_t^n a(x,t) \cdot b(x,t) \equiv (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^m (\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^n a(x,t) b(x', t')|_{x=x', t=t'}, \tag{4.1}$$

or in another form

$$D_x^m D_t^n a(x,t) \cdot b(x,t) \equiv \lim_{\epsilon_1, \epsilon_2 \to 0} \frac{\partial^{m+n}}{\partial \epsilon_1^m \partial \epsilon_2^n} (a(x+\epsilon_1, t+\epsilon_2) b(x-\epsilon_1, t-\epsilon_2)), \tag{4.2}$$

where $a(x,t)$ and $b(x,t)$ are multi-variable functions, m, n are integers. From (2.10), we consider DNLS as follows

$$u_t = i u_{xx} + \epsilon (u^2 \overline{u})_x, \quad \epsilon = -1. \tag{4.3}$$

By the transformation $u = q \exp(\int_{-\infty}^{x} i\epsilon q\overline{q})$, then we can transform (4.3) into another type of DNLS as below:

$$q_t = i q_{xx} - \epsilon q^2 \overline{q}_x + \frac{i}{2} q |q|^4, \quad \epsilon = -1. \tag{4.4}$$

Consider the dependent variable transformation as below:

$$q = \frac{G}{F}, \tag{4.5}$$

where G and F are complex functions depend on x, t. Applying Hirota bilinear operators on (4.4), we can transform (4.4) into the following bilinear forms:

$$D_t(G \cdot F) - i D_x^2 (G \cdot F) = 0, \tag{4.6a}$$

$$D_x^2(\overline{F} \cdot F) - i\frac{\epsilon}{2} D_x (\overline{G} \cdot G) = 0, \tag{4.6b}$$

$$D_x(\overline{F} \cdot F) + i\frac{\epsilon}{2} (\overline{G} \cdot G) = 0, \quad \epsilon = -1. \tag{4.6c}$$

Since we have (4.6), we can write down soliton solutions mimicking A. Nakamura and H.-H. Chen's results [17]. Here we show the exact N-soliton solution as follows:

$$G = \sum_{\mu_i = 0,1}^{(1)} \exp\left\{ \sum_{i=1}^{2N} \mu_i (\eta_i + \phi_i) + \sum_{i,j=1, i<j}^{2N} \mu_i \mu_j \tau_{i,j} \right\}, \tag{4.7}$$

and

$$F = \sum_{\mu_i = 0,1}^{(0)} \exp\left\{ \sum_{i=1}^{2N} \mu_i (\eta_i + \phi_i) + \sum_{i,j=1, i<j}^{2N} \mu_i \mu_j \tau_{i,j} \right\}, \tag{4.8}$$

with

$$\begin{aligned}
&\eta_i \equiv k_i x + w_i t + \eta_i(0) \ \ (1 \le i \le 2N), \\
&k_{i+N} \equiv \overline{k}_i, \ \ w_{i+N} \equiv \overline{w}_i, \ \ \eta_{i+N} \equiv \overline{\eta}_i, \ \ w_i \equiv i k_i^2, \\
&e^{\phi_i} \equiv 1, \ \ e^{\phi_{i+N}} \equiv -i\overline{k}_i, \\
&e^{\tau_{i,j}} \equiv 2(k_i - k_j)^2, \\
&e^{\tau_{i+N,j+N}} \equiv 2(\overline{k}_i - \overline{k}_j)^2, \\
&e^{\tau_{i,j+N}} \equiv \frac{1}{2(k_i + \overline{k}_j)^2} \ \ (1 \le i \le N),
\end{aligned} \tag{4.9}$$

where $\sum_{\mu_i=0,1}^{(m)}$ implies that the summation over $\mu_i = 0, 1 (1 \le i \le 2N)$ should be performed under the condition $\sum_{i=1}^{N} \mu_i \equiv m + \sum_{i=1}^{N} \mu_{i+N}$, k_i is the complex constant characterizing the amplitude and the width of the i-th soliton, $\eta_i(0)$ is a complex phase constant. For $N = 2$, we get 2-soliton solutions of DNLS as follows:

$$G = e^{\eta_1} + e^{\eta_2} + \left(\frac{i\epsilon\overline{k}_1(\check{k}_{12})^2}{2(\hat{k}_{11})^2(\hat{k}_{21})^2}\right)e^{\eta_1+\overline{\eta}_1+\eta_2} + \left(\frac{i\epsilon\overline{k}_2(\check{k}_{21})^2}{2(\hat{k}_{12})^2(\hat{k}_{22})^2}\right)e^{\eta_2+\overline{\eta}_2+\eta_1}, \tag{4.10}$$

$$\begin{aligned}
F \ = \ & 1 + \left(\frac{i\epsilon\overline{k}_1}{2(\hat{k}_{11})^2}\right)e^{\eta_1+\overline{\eta}_1} + \left(\frac{i\epsilon\overline{k}_2}{2(\hat{k}_{12})^2}\right)e^{\eta_1+\overline{\eta}_2} + \left(\frac{i\epsilon\overline{k}_1}{2(\hat{k}_{21})^2}\right)e^{\overline{\eta}_1+\eta_2} \\
& + \left(\frac{i\epsilon\overline{k}_2}{2(\hat{k}_{22})^2}\right)e^{\eta_2+\overline{\eta}_2} + \left(\frac{-\overline{k}_1\overline{k}_2(\check{k}_{12})^2(\overline{\check{k}_{12}})^2}{4(\hat{k}_{11})^2(\hat{k}_{12})^2(\hat{k}_{21})^2(\hat{k}_{22})^2}\right)e^{\eta_1+\overline{\eta}_1+\eta_2+\overline{\eta}_2}, \epsilon = -1,
\end{aligned} \tag{4.11}$$

$$\text{where } \ \hat{k}_{ij} = k_i + \overline{k_j}, \check{k}_{ij} = k_i - k_j. \tag{4.12}$$

For $N = 1$, we also can write down the exact 1-soliton solution

$$G = e^{\eta_1}, \ \ F = 1 + e^{\eta_1+\overline{\eta}_1+A}, \ \ e^A = \left(\frac{-i\epsilon\overline{k}_1}{2(k_1+\overline{k}_1)^2}\right). \tag{4.13}$$

Then substitute (4.13) into $q(x,t) = G/F$ to get the following form

$$q(x,t) = \frac{e^{\eta_1}}{1 + e^{\eta_1+\overline{\eta}_1+A}} = \frac{e^{\frac{-A}{2}}}{2}\frac{e^{\pm(\frac{-v}{2}x+(k^2+\frac{v^2}{4})t)}}{\cosh[k(x-vt-x_0)]}, \tag{4.14}$$

where $k \equiv (k_1 + \overline{k}_1)/2$ and $v \equiv -(k_1 - \overline{k}_1)$. We may verify that (4.14) is equivalent to (3.9).

5 DRD system and Multi-Dissipaton Solutions

Now, we apply equivalent relations theorem [see [19], Appendix] to DNLS equation. Consider DNLS as follows:

$$iu_t + u_{xx} - \varphi(u)u = 0, \tag{5.1}$$

with complex potential

$$\varphi(u) = \varphi_R(u) + i\varphi_I(u) = -i\epsilon(u\overline{u}_x + 2\overline{u}u_x), \ \ \epsilon = \mp 1, \tag{5.2}$$

where $\varphi_R(u) = -\frac{\epsilon i}{2}(u\overline{u}_x - \overline{u}u_x)$, $\varphi_I(u) = -\frac{3\epsilon}{2}(u\overline{u})_x$. Let $u = \exp(R - iS)$, $Q^+ = \exp(R + S)$,$Q^- = \exp(R - S)$. We know that $Q^+Q^- = |u|^2$. By a simple calculation, we have a Derivative Reaction Diffusion (DRD) system

$$\begin{aligned} -Q^+_t + Q^+_{xx} - \epsilon(Q^+Q^-Q^+)_x - \Phi Q^+ = 0,\\ Q^-_t + Q^-_{xx} + \epsilon(Q^+Q^-Q^-)_x - \Phi Q^- = 0, \end{aligned} \tag{5.3}$$

where $\Phi = (\ln Q^+Q^-)_{xx} + \frac{1}{2}[(\ln Q^+Q^-)_x]^2$.

For each soliton solutions of (5.1), we can construct dissipaton solutions of (5.3). Motivated by the results of Reaction-Diffusion system of Pashaev et al [19, 20, 21], we consider a simple DRD system as follows

$$\begin{aligned} &Q^+_t - Q^+_{xx} + \epsilon(Q^+Q^-Q^+)_x = 0,\\ &Q^-_t + Q^-_{xx} + \epsilon(Q^+Q^-Q^-)_x = 0, \ \ \epsilon = \pm 1. \end{aligned} \tag{5.4}$$

By using equivalent relations, we can transform (5.4) into a modified DNLS equation

$$iu_t + u_{xx} + i\epsilon(u^2\overline{u})_x = 2\frac{|u|_{xx}}{|u|}u, \ \ \epsilon = \mp 1, \tag{5.5}$$

which is called Resonance DNLS.

Obviously, we can find dissipaton solutions of (5.4) and associate the soliton solutions of (5.5) by equivalent relations. Here we use Hirota bilinear method to find its exact solutions. We only consider $\epsilon = 1$ in (5.4) and write down DRD system in the following form

$$\begin{aligned} &Q^+_t - Q^+_{xx} + \epsilon(Q^+Q^-Q^+)_x = 0,\\ &Q^-_t + Q^-_{xx} + \epsilon(Q^+Q^-Q^-)_x = 0, \ \ \epsilon = 1, \end{aligned} \tag{5.6}$$

where $\{Q^+, Q^-\}$ is a pair of real functions. Consider a transformation as follows

$$\begin{aligned} &q^+ = Q^+ \exp(-\epsilon \textstyle\int_{-\infty}^x q^+q^-),\\ &q^- = Q^- \exp(+\epsilon \textstyle\int_{-\infty}^x q^+q^-), \ \ \epsilon = 1, \end{aligned} \tag{5.7}$$

where $\{q^+, q^-\}$ also be a pair of real functions. By this transformation (5.7), we can transform (5.6) into a new DRD system. After some computation and using the fact $q^+q^- = Q^+Q^-$, we have the following new DRD system

$$\begin{aligned} &q^+_t - q^+_{xx} - \epsilon q^+q^+q^-_x - \tfrac{1}{2}(q^+q^-)^2q^+ = 0,\\ &q^-_t + q^-_{xx} - \epsilon q^-q^-q^+_x + \tfrac{1}{2}(q^+q^-)^2q^- = 0, \ \ \epsilon = 1, \end{aligned} \tag{5.8}$$

which can be written as

$$q^\pm_t \mp q^\pm_{xx} - \epsilon q^\pm q^\pm q^\mp_x \mp \frac{1}{2}(q^+q^-)^2q^\pm = 0. \tag{5.9}$$

Remark. For the expression $q^\pm$ in (5.8), we called q^+ be positive term and q^- be negative term. If we replace q^- by $-q^-$ and ϵ by $-\epsilon$ in (5.8), the system is unchanged. So we may only consider the case $\epsilon = 1$.

Consider a dependent variable transformation similarly as (4.5).

$$q^{\pm} = \frac{G^{\pm}}{F^{\pm}}, \tag{5.10}$$

where $G^{\pm}$, $F^{\pm}$ are real functions depending on x, t. Now, we transform (5.8) into bilinear forms:

$$D_t(G^{\pm}\cdot F^{\pm}) \mp D_x^2(G^{\pm}\cdot F^{\pm}) = 0, \tag{5.11a}$$

$$D_x^2(F^{-}\cdot F^{+}) - \frac{1}{2}D_x(G^{-}\cdot G^{+}) = 0, \tag{5.11b}$$

$$D_x(F^{-}\cdot F^{+}) + \frac{1}{2}(G^{-}\cdot G^{+}) = 0. \tag{5.11c}$$

We write down the solutions of this bilinear system as follows:

$$F^{\pm} = \sum_{\mu_i=0,1}^{(0)} \exp\left\{\sum_{i=1}^{2N}\mu_i(\eta_i+\phi_i^{\pm}) + \sum_{i,j=1,i<j}^{2N}\mu_i\mu_j\tau_{i,j}\right\}, \tag{5.12a}$$

$$G^{\pm} = \pm\sum_{\mu_i=0,1}^{(\pm1)} \exp\left\{\sum_{i=1}^{2N}\mu_i(\eta_i+\phi_i^{\pm}) + \sum_{i,j=1,i<j}^{2N}\mu_i\mu_j\tau_{i,j}\right\}, \tag{5.12b}$$

with

$$\begin{aligned}
&\eta_i = \eta_i^{+} \equiv k_i^{+}x + w_i^{+}t + \eta_i^{+}(0) \ \ (1\le i\le N),\\
&\eta_{i+N} = \eta_i^{-} \equiv k_i^{-}x + w_i^{-}t + \eta_i^{-}(0) \ \ (1\le i\le N),\\
&w_i^{\pm} \equiv \pm(k_i^{\pm})^2 \ \ (1\le i\le N),\\
&e^{\phi_i^{+}} \equiv 1, \ \ e^{\phi_{i+N}^{+}} \equiv -k_i^{-} \ \ (1\le i\le N),\\
&e^{\phi_i^{-}} \equiv k_i^{+}, \ \ e^{\phi_{i+N}^{-}} \equiv 1 \ \ (1\le i\le N)\\
&e^{\tau_{i,j}} \equiv 2(k_i^{+}-k_j^{+})^2 \ \ (1\le i,j\le N)\\
&e^{\tau_{i+N,j+N}} \equiv 2(k_i^{-}-k_j^{-})^2 \ \ (1\le i,j\le N)\\
&e^{\tau_{i,j+N}} \equiv \frac{1}{2(k_i^{+}+k_j^{-})^2} \ \ (1\le i,j\le N),
\end{aligned} \tag{5.13}$$

where $\sum_{\mu_i=0,1}^{(m)}$ implies that the summation over $\mu_i = 0,1 (1\le i\le 2N)$ should be performed under the condition $\sum_{i=1}^{N}\mu_i \equiv m + \sum_{i=1}^{N}\mu_{i+N}$, $k_i^{\pm}$ are real constants characterizing the amplitude and the width of the i-th dissipaton, $\eta_i^{\pm}(0)$ are real phase constants. Now, we write down the expressions of 2-dissipaton solution of DRD system

$$G^{\pm} = \pm[e^{\eta_1^{\pm}} + e^{\eta_2^{\pm}} + (\frac{\mp k_1^{\mp}(\hat{k}_{12}^{\pm\pm})^2}{2(k_{11}^{\pm\mp})^2(k_{12}^{\mp\pm})^2})e^{\eta_1^{\pm}+\eta_1^{\mp}+\eta_2^{\pm}} + (\frac{\mp k_2^{\mp}(\hat{k}_{12}^{\pm\pm})^2}{2(k_{22}^{\pm\mp})^2(k_{12}^{\pm\mp})^2})e^{\eta_2^{\pm}+\eta_2^{\mp}+\eta_1^{\pm}}], \tag{5.14}$$

and

$$\begin{aligned}
F^{\pm} \ \ = \ \ & 1 + (\frac{\mp k_1^{\mp}}{2(k_{11}^{\pm\mp})^2})e^{\eta_1^{\pm}+\eta_1^{\mp}} + (\frac{\mp k_2^{\mp}}{2(k_{12}^{\pm\mp})^2})e^{\eta_1^{\pm}+\eta_2^{\mp}} + (\frac{\mp k_1^{\mp}}{2(k_{12}^{\mp\pm})^2})e^{\eta_1^{\mp}+\eta_2^{\pm}}\\
& + (\frac{\mp k_2^{\mp}}{2(k_{22}^{\pm\mp})^2})e^{\eta_2^{\pm}+\eta_2^{\mp}} + (\frac{+k_1^{\mp}k_2^{\mp}(\hat{k}_{12}^{\pm\pm})^2(\hat{k}_{12}^{\mp\mp})^2}{4[k_{11}^{\pm\mp}k_{12}^{\pm\mp}k_{21}^{\pm\mp}k_{22}^{\pm\mp}]^2})e^{\eta_1^{\pm}+\eta_1^{\mp}+\eta_2^{\pm}+\eta_2^{\mp}}.
\end{aligned} \tag{5.15}$$

where $k_{ij}^{st} = k_i^s + k_j^t, \hat{k}_{ij}^{st} = k_i^s - k_j^t$.

In the same way, we show the 1-dissipaton solution of DRD system

$$G^{\pm} = \pm e^{\eta_1^{\pm}}, \quad F^{\pm} = 1 + e^{\eta_1^{\pm}+\eta_1^{\mp}+A^{\pm}}, \quad e^{A^{\pm}} = \frac{\mp k_1^{\mp}}{2(k_1^{\pm}+k_1^{\mp})^2}, \tag{5.16}$$

where $\eta_1^{\pm} \equiv (k_1^{\pm})x \pm (k_1^{\pm})^2 t + \eta_1^{\pm}(0)$. Since $e^{A^{\pm}} > 0$, we must choose positive k_1^{+} and negative k_1^{-} and $k_1^{+} + k_1^{-} \neq 0$ for the regularity conditions. Moreover,

$$q^{\pm}(x,t) = \pm\frac{e^{-A^{\pm}/2}}{2}\frac{e^{\pm(\frac{-v}{2}x+(k^2+\frac{v^2}{4})t)}}{\cosh[k(x-vt)+\frac{A^{\pm}}{2}]}, \tag{5.17}$$

where $k = (k_1^{+} + k_1^{-})/2$, $v = -(k_1^{+} - k_1^{-})$. We may plot the graphs by the software Mathematica. The computing results will be published elsewhere.

6 Conclusion

We have verified N-dissipaton solutions of DRD system and N-soliton solutions of DNLS. The plots of one and two dissipatons of DRD system are similar as Reaction-Diffusion system of Pashaev et al [16]. But the resonance states are not so clear. We notice the difference, but it need further investigation in the future.

7 Appendix

In this appendix, we describe some important results of Pashaev et al on equivalent relations [19, 20, 21]. Now, we introduce equivalent relations between modified nonlinear Schrödinger equation and general Reaction-Diffusion system (RD). We consider the following Schrödinger equations

$$iu_t + u_{xx} - \varphi u = 0, \tag{7.1a}$$
$$-i\overline{u}_t + \overline{u}_{xx} - \overline{\varphi u} = 0, \tag{7.1b}$$

where $u(x,t)$ is a complex function, $\varphi = \varphi(u) = \varphi_R + i\varphi_I$ is a complex potential where φ_R is real part of φ and φ_I is imaginary part of φ. In physical interpretation, we can consider potential u with resonance and dissipation states. So, we consider the following RD system with $\{Q^+, Q^-\}$ be a pair of real-value functions without time reflection symmetry in general,

$$-Q_t^{+} + Q_{xx}^{+} + V_{+}Q^{+} = 0, \tag{7.2a}$$
$$Q_t^{-} + Q_{xx}^{-} + V_{-}Q^{-} = 0, \tag{7.2b}$$

where $V_+ = V_1 + V_2$, $V_- = V_1 - V_2$. V_1 and V_2 will be determined later.

For the equivalent relations, we state the following two lemmas.

Lemma 1. *Consider $u(x,t) \in \mathcal{C}$ and Domain $D = \{(x,t)|0 < |u(x,t)| < \infty\}$. Suppose $u(x,t) \in C^{2,1}$ and let $u(x,t) = \exp(R(x,t) - iS(x,t))$, where $R(x,t)$ and $S(x,t)$ are real functions. Moreover, consider R and S as follows*

$$R = \frac{1}{2}\ln|u|^2, S = \frac{1}{2i}\ln\frac{\overline{u}}{u}, \tag{7.3}$$

then $u(x,t)$ satisfies (7.1) in D if and only if $R(x,t)$ and $S(x,t)$ in D satisfy the following relations

$$\varphi_R(u) = S_t + R_{xx} + (R_x)^2 - (S_x)^2, \tag{7.4a}$$
$$\varphi_I(u) = R_t + S_{xx} + 2R_xS_x. \tag{7.4b}$$

Proof. Substitute $u = e^{R-iS}$ into (7.1) By simple evaluation and decompose φ into real and imaginary parts, we can get (7.4). ■

Lemma 2. *Let $\{Q^+, Q^-\}$ be a pair real functions on the domain Domain D=$\{(x,t)|0 < Q^+Q^- < \infty\}$. Suppose $Q^+ = \exp(R(x,t) + S(x,t))$, $Q^- = \exp(R(x,t) - S(x,t))$. From (7.3) we have*

$$R(x,t) = \frac{1}{2}\ln|u|^2 = \frac{1}{2}\ln Q^+Q^-, \tag{7.5a}$$
$$S(x,t) = \frac{1}{2i}\ln\frac{\overline{u}}{u} = \frac{1}{2}\ln\frac{Q^+}{Q^-}. \tag{7.5b}$$

The $\{Q^+, Q^-\}$ satisfies RD system in (7.2) if and only if R and S satisfy in D the following relations

$$V_1 = S_t - R_{xx} - (R_x)^2 - (S_x)^2, \tag{7.6a}$$
$$V_2 = R_t - S_{xx} - 2R_xS_x, \tag{7.6b}$$

where $V_1 + V_2 = V_+$, $V_1 - V_2 = V_-$.

Proof. This proof is similar as Lemma 1. ■

From Lemma 1 and Lemma 2, we look for (7.4) and (7.6). Obviously, we can get

$$\varphi_R = V_1 + 2(\ln|u|)_{xx} + [(\ln|u|)_x]^2, \tag{7.7a}$$
$$\varphi_I = V_2, \tag{7.7b}$$

or

$$V_1 = \varphi_R - (\ln Q^+Q^-)_{xx} - \frac{1}{2}[(\ln Q^+Q_-)_x]^2, \tag{7.8a}$$
$$V_2 = \varphi_I. \tag{7.8b}$$

Clearly, we find a relation between $\{\varphi_R, \varphi_I\}$ and $\{V_+, V_-\}$ directly from (7.7) and (7.8), then we state the following Theorem from Lemma 1 and Lemma 2.

Theorem 1. *(Equivalent Relations) Let $\{Q^+, Q^-\}$ be a pair of real functions satisfying RD system (7.2) in D with (7.5). Let $u(x,t) = \exp(R(x,t) - iS(x,t))$ is a complex function then it is a solution of (7.1) with a potential function $\varphi = \varphi_R + i\varphi_I$,*

$$\varphi_R = V_1 + 2(\ln|u|)_{xx} + (\ln|u|)_x^2, \tag{7.9a}$$
$$\varphi_I = V_2. \tag{7.9b}$$

On the other hand, let complex-value function $u(x,t) = exp(R(x,t)+iS(x,t))$ *be a solution of (7.1) in* D *and define* $Q^+ = \exp(R(x,t) + S(x,t))$, $Q^- = \exp(R(x,t) - S(x,t))$. *Then* $\{Q^+, Q^-\}$ *be solution of RD system of (7.2) in* D *with* $V_+ = V_1 + V_2$, $V_- = V_1 - V_2$,

$$V_1 = \varphi_R - (\ln Q^+ Q^-)_{xx} - \frac{1}{2}[(\ln Q^+ Q_-)x]^2, \tag{7.10a}$$

$$V_2 = \varphi_I. \tag{7.10b}$$

Proof. The proof of the Theorem 1 is done directly from Lemma 1 and Lemma 2. ■

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