

An analytical approach to evaluating monotonic functions of fuzzy numbers

Arthur Seibel, Josef Schlattmann

Workgroup on System Technologies and Engineering Design Methodology
Hamburg University of Technology, 21073 Hamburg, Germany
E-mail: arthur.seibel@tuhh.de

Abstract

This paper presents a practical analytical approach to evaluating continuous, monotonic functions of independent fuzzy numbers. The approach is based on a parametric α -cut representation of fuzzy numbers and allows for the inclusion of parameter uncertainties into mathematical models.

Keywords: Engineering design computations with uncertain parameters, monotonic functions of fuzzy numbers, analytical fuzzy calculus

1. Introduction

There is an increasing effort in the scientific community to provide suitable methods for the inclusion of uncertainties into mathematical models. One way to do so is to introduce parametric uncertainty by representing the uncertain model parameters as fuzzy numbers [1] and evaluating the model equations by means of ZADEH's extension principle [2]. The evaluation of this *classical formulation* of the extension principle, however, turns out to be a highly complex task [3]. Fortunately, BUCKLEY & QU [4] provide an *alternative formulation* that operates on α -cuts and is applicable to continuous functions of independent fuzzy numbers. Powerful numerical techniques have been developed to implement this alternative formulation [5]. These techniques are particularly suitable for very complex simulation models [6]. In engineering design [7], however, the mathematical equations are usually less complex and hence analytical methods might be more suitable for the inclusion of parameter uncertainties into the computations. For this purpose, a practical analytical approach to evaluating continuous, monotonic functions of independent fuzzy numbers is introduced in this paper, which is based on the alternative formulation of the extension principle.

An outline of this paper is as follows: In Section 2, we give a definition of fuzzy numbers and present two important types. In Section 3, we briefly recall ZADEH's extension principle and introduce the alternative formulation based on α -cuts. In Section 4, we describe our analytical approach and give three illustrative examples. In Section 5, a practical engineering application is presented. Finally, in Section 6, some conclusions are drawn.

2. Fuzzy numbers

Fuzzy numbers [1] are a special class of fuzzy sets [8], which can be defined as follows:

A normal, convex fuzzy set \tilde{x} over the real line \mathbb{R} is called *fuzzy number* if there is exactly one $\bar{x} \in \mathbb{R}$ with $\mu_{\tilde{x}}(\bar{x}) = 1$ and the membership function is at least piecewise continuous. The value \bar{x} is called the *modal* or *peak value* of \tilde{x} .

Theoretically, an infinite number of possible types of fuzzy numbers can be defined. However, only few of them are important for engineering applications [6]. These typical fuzzy numbers shall be described in the following.

2.1. Triangular fuzzy numbers

Due to its very simple, linear membership function, the *triangular fuzzy number* (TFN) is the most frequently used fuzzy number in engineering. In order to define a TFN with the membership function

$$\mu_{\tilde{x}}(x) = \begin{cases} 1 + \frac{x - \bar{x}}{\tau^L}, & \bar{x} - \tau^L \leq x \leq \bar{x}, \\ 1 - \frac{x - \bar{x}}{\tau^R}, & \bar{x} < x \leq \bar{x} + \tau^R, \end{cases} \quad (1)$$

we use the parametric notation [6]

$$\tilde{x} = \text{tfn}(\bar{x}, \tau^L, \tau^R),$$

where \bar{x} denotes the *modal value*, τ^L the *left-hand*, and τ^R the *right-hand spread* of \tilde{x} (cf. Figure 1). If $\tau^L = \tau^R$, the TFN is called *symmetric*. Its α -cuts $x(\alpha) = [x^L(\alpha), x^R(\alpha)]$ result from the inverse functions of Eqs. (1) with respect to x :

$$\begin{aligned} x^L(\alpha) &= \bar{x} - \tau^L(1 - \alpha), & 0 < \alpha \leq 1, \\ x^R(\alpha) &= \bar{x} + \tau^R(1 - \alpha), & 0 < \alpha \leq 1. \end{aligned}$$

2.2. Gaussian fuzzy numbers

Another widely-used fuzzy number in engineering is the *Gaussian fuzzy number* (GFN), which is based on the normal distribution from probability theory. In order to define such a GFN with the membership function

$$\mu_{\tilde{x}}(x) = \begin{cases} \exp\left[-\frac{1}{2}\left(\frac{x - \bar{x}}{\sigma^L}\right)^2\right], & x \leq \bar{x}, \\ \exp\left[-\frac{1}{2}\left(\frac{x - \bar{x}}{\sigma^R}\right)^2\right], & x > \bar{x}, \end{cases}$$

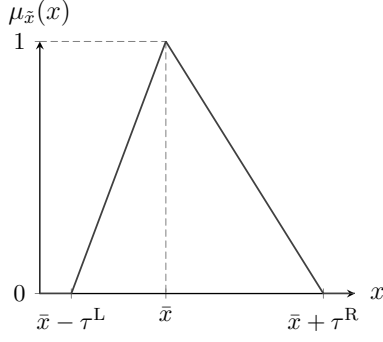


Figure 1: Triangular fuzzy number.

we use the parametric notation [6]

$$\tilde{x} = \text{gfn}(\bar{x}, \sigma^L, \sigma^R),$$

where \bar{x} denotes the *modal value*, σ^L the *left-hand*, and σ^R the *right-hand standard deviation* of \tilde{x} (cf. Figure 2). If $\sigma^L = \sigma^R$, the GFN is called *symmetric*. Its α -cuts $x(\alpha) = [x^L(\alpha), x^R(\alpha)]$ result to:

$$\begin{aligned} x^L(\alpha) &= \bar{x} - \sigma^L \sqrt{-2 \ln(\alpha)}, \quad 0 < \alpha \leq 1, \\ x^R(\alpha) &= \bar{x} + \sigma^R \sqrt{-2 \ln(\alpha)}, \quad 0 < \alpha \leq 1. \end{aligned}$$

3. Extension principle

ZADEH's extension principle [2] allows to extend any real-valued function to a function of fuzzy numbers. More specifically, let $\tilde{x}_1, \dots, \tilde{x}_n$ be n independent or *non-interactive* fuzzy numbers and let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a function with $y = f(x_1, \dots, x_n)$. The fuzzy extension $\tilde{y} = f(\tilde{x}_1, \dots, \tilde{x}_n)$ is then defined by

$$\mu_{\tilde{y}}(y) = \sup_{y=f(x_1, \dots, x_n)} \min\{\mu_{\tilde{x}_1}(x_1), \dots, \mu_{\tilde{x}_n}(x_n)\}.$$

In case of *interdependency* between $\tilde{x}_1, \dots, \tilde{x}_n$, the minimum operator should be replaced by a suitable *triangular norm* [9]. In this paper, however, we restrict ourselves to independent fuzzy numbers.

The evaluation of this *classical formulation* of the extension principle turns out to be a highly complex task [3]. Fortunately, BUCKLEY & QU [4] provide an *alternative formulation* that operates on α -cuts:

Let $x_1(\alpha), \dots, x_n(\alpha)$ denote the α -cuts of the n independent fuzzy numbers $\tilde{x}_1, \dots, \tilde{x}_n$ and let f be continuous. Then, the α -cuts $y(\alpha) = [y^L(\alpha), y^R(\alpha)]$ of \tilde{y} can be computed from

$$\begin{aligned} y^L(\alpha) &= \min\{f(x_1, \dots, x_n) \mid (x_1, \dots, x_n) \in \Omega(\alpha)\}, \\ y^R(\alpha) &= \max\{f(x_1, \dots, x_n) \mid (x_1, \dots, x_n) \in \Omega(\alpha)\}, \end{aligned}$$

where $\Omega(\alpha) = x_1(\alpha) \times \dots \times x_n(\alpha)$ represent the n -dimensional interval boxes that are spanned by the α -cuts $x_1(\alpha), \dots, x_n(\alpha)$.

The proposed analytical approach, which is presented in the next section, is based on this alternative formulation of the extension principle.

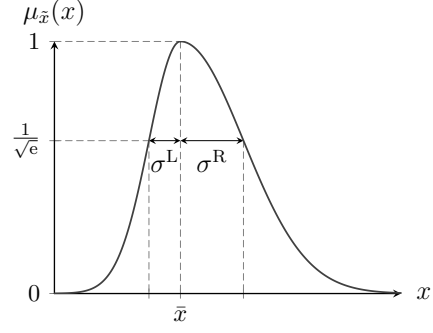


Figure 2: Gaussian fuzzy number.

4. Analytical approach

Let the continuous function f be (strictly) monotonic increasing in x_i , $i = 1, \dots, k$, and (strictly) monotonic decreasing in x_j , $j = 1, \dots, \ell$, in the domain of interest, and let $k + \ell = n$. Then, the minimum values of f inside of every sub-domain $\Omega(\alpha)$ are always found at the left boundaries of $x_i(\alpha)$ and the right boundaries of $x_j(\alpha)$, and its maximum values at the right boundaries of $x_i(\alpha)$ and the left boundaries of $x_j(\alpha)$, respectively. In such case, the α -cuts $y(\alpha) = [y^L(\alpha), y^R(\alpha)]$ of \tilde{y} become

$$\begin{aligned} y^L(\alpha) &= f(x_i^L(\alpha), x_j^R(\alpha)), \quad 0 < \alpha \leq 1, \\ y^R(\alpha) &= f(x_i^R(\alpha), x_j^L(\alpha)), \quad 0 < \alpha \leq 1, \end{aligned} \quad (2)$$

with $x_m(\alpha) = [x_m^L(\alpha), x_m^R(\alpha)]$, $m = 1, \dots, n$. If Eqs. (2) are invertible with respect to α , then the membership function of \tilde{y} yields

$$\mu_{\tilde{y}}(y) = \begin{cases} y^L(\alpha)^{-1}, & y^L(0) < y \leq y^L(1), \\ y^R(\alpha)^{-1}, & y^R(1) < y < y^R(0). \end{cases}$$

This *reduced* part of our approach can be viewed as an analytical version of the *short transformation method* [10]. Basically, it is equivalent to Lemma 3 from [11] or Corollary 2 from [12].

Example 1. The function $f_1: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ with

$$y_1 = f_1(x_1, x_2) = \frac{x_1}{x_1 + x_2}$$

shall be evaluated for the two fuzzy numbers $\tilde{x}_1 = \text{tfn}(2, 2, 3)$ and $\tilde{x}_2 = \text{tfn}(2, 2, 2)$. Since

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} &= \frac{x_2}{(x_1 + x_2)^2} > 0, \\ \frac{\partial f_1}{\partial x_2} &= \frac{-x_1}{(x_1 + x_2)^2} < 0, \end{aligned}$$

the function f_1 is (strictly) monotonic increasing in x_1 and (strictly) monotonic decreasing in x_2 in the domain $\text{supp}(\tilde{x}_1) \times \text{supp}(\tilde{x}_2) = (0, 5) \times (0, 4)$. Hence, the α -cuts $y_1(\alpha) = [y_1^L(\alpha), y_1^R(\alpha)]$ of \tilde{y}_1 are

$$y_1^L(\alpha) = f_1(x_1^L(\alpha), x_2^R(\alpha)) = \frac{1}{2}\alpha,$$

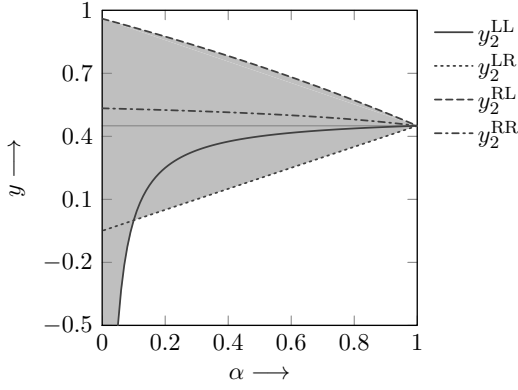


Figure 3: Solution candidates from Example 2.

$$y_1^R(\alpha) = f_1(x_1^R(\alpha), x_2^L(\alpha)) = \frac{3\alpha - 5}{\alpha - 5}.$$

With $y_1^L(0) = 0$, $y_1^L(1) = 0.5 = y_1^R(1)$, and $y_1^R(0) = 1$, the membership function of \tilde{y}_1 yields

$$\mu_{\tilde{y}_1}(y) = \begin{cases} 2y, & 0 < y \leq 0.5, \\ \frac{5(y-1)}{y-3}, & 0.5 < y < 1. \end{cases}$$

However, the above approach is only valid if the function f does not change its monotonicity within the domain of interest. We know from [13, 14] that the global extrema of any monotonic function f are always found at the corner points of $\Omega(\alpha)$. Hence, in order to compute the analytical solution, we can always proceed as follows:

1. Evaluate the function f for all the 2^n permutations of the interval boundaries of $x_m(\alpha)$; e. g., if $n = 2$, then compute

$$\begin{aligned} y^{LL}(\alpha) &= f(x_1^L(\alpha), x_2^L(\alpha)), \\ y^{LR}(\alpha) &= f(x_1^L(\alpha), x_2^R(\alpha)), \\ y^{RL}(\alpha) &= f(x_1^R(\alpha), x_2^L(\alpha)), \\ y^{RR}(\alpha) &= f(x_1^R(\alpha), x_2^R(\alpha)). \end{aligned}$$

2. Plot these solution candidates in the same diagram.
3. The analytical solution then corresponds to the maximum envelope formed by the possible solution candidates.

This *general* part of our approach can be viewed as an analytical version of the *reduced transformation method* [15]. Basically, it is equivalent to Lemma 2 from [11] or Corollary 1 from [12].

Example 2. Next, the function $f_2: \mathbb{R}_+^2 \rightarrow \mathbb{R}$ with

$$y_2 = f_2(x_1, x_2) = \frac{x_1 - \frac{1}{5}}{x_1 + x_2}$$

shall be evaluated for the two fuzzy numbers from Example 1. Since

$$\frac{\partial f_2}{\partial x_1} = \frac{x_2 + \frac{1}{5}}{(x_1 + x_2)^2} > 0, \quad 0 < x_2 < 4,$$

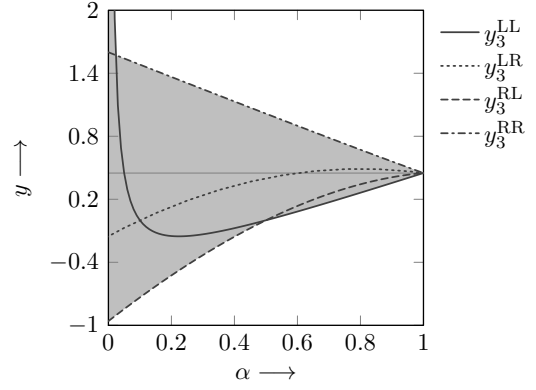


Figure 4: Solution candidates from Example 3.

$$\frac{\partial f_2}{\partial x_2} = \frac{\frac{1}{5} - x_1}{(x_1 + x_2)^2} \begin{cases} \geq 0, & 0 < x_1 \leq 0.2, \\ < 0, & 0.2 < x_1 < 5, \end{cases}$$

the function f_2 changes its monotonicity within the domain $\text{supp}(\tilde{x}_1) \times \text{supp}(\tilde{x}_2) = (0, 5) \times (0, 4)$. Hence, the general part of our approach should be applied. The solution candidates for $y_2(\alpha)$ are

$$\begin{aligned} y_2^{LL}(\alpha) &= f_2(x_1^L(\alpha), x_2^L(\alpha)) = \frac{10\alpha - 1}{20\alpha}, \\ y_2^{LR}(\alpha) &= f_2(x_1^L(\alpha), x_2^R(\alpha)) = \frac{1}{2}\alpha - \frac{1}{20}, \\ y_2^{RL}(\alpha) &= f_2(x_1^R(\alpha), x_2^L(\alpha)) = \frac{3}{5} \frac{5\alpha - 8}{\alpha - 5}, \\ y_2^{RR}(\alpha) &= f_2(x_1^R(\alpha), x_2^R(\alpha)) = \frac{3}{5} \frac{5\alpha - 8}{5\alpha - 9}. \end{aligned}$$

We can see from their plots in Figure 3 that the left branch of the maximum envelope, illustrated by the gray area, is formed by y_2^{LL} for $0 < \alpha \leq 0.1$ and by y_2^{LR} for $0.1 < \alpha \leq 1$, where the value 0.1 corresponds to their intersection point. Its right branch, on the other hand, is entirely formed by y_2^{RL} . Hence, the α -cuts $y_2(\alpha) = [y_2^L(\alpha), y_2^R(\alpha)]$ of \tilde{y}_2 are

$$y_2^L(\alpha) = \begin{cases} \frac{10\alpha - 1}{20\alpha}, & 0 < \alpha \leq 0.1, \\ \frac{1}{2}\alpha - \frac{1}{20}, & 0.1 < \alpha \leq 1, \end{cases}$$

$$y_2^R(\alpha) = \frac{3}{5} \frac{5\alpha - 8}{\alpha - 5}, \quad 0 < \alpha \leq 1.$$

With $\lim_{\alpha \rightarrow 0} y_2^L(\alpha) = -\infty$, $y_2^L(0.1) = 0$, $y_2^L(1) = 0.45 = y_2^R(1)$, and $y_2^R(0) = 0.96$, the membership function of \tilde{y}_2 yields

$$\mu_{\tilde{y}_2}(y) = \begin{cases} \frac{1}{10} \frac{1}{1 - 2y}, & -\infty < y \leq 0, \\ 2y + \frac{1}{10}, & 0 < y \leq 0.45, \\ \frac{1}{5} \frac{25y - 24}{y - 3}, & 0.45 < y < 0.96. \end{cases}$$

Example 3. Finally, the function $f_3: \mathbb{R}_+^2 \rightarrow \mathbb{R}$ with

$$y_3 = f_3(x_1, x_2) = \frac{(x_1 - \frac{1}{5})(x_2 - 1)}{x_1 + x_2}$$

shall be evaluated for the two fuzzy numbers from Example 1. Since

$$\frac{\partial f_3}{\partial x_1} = \frac{(x_2 - 1)(x_2 + \frac{1}{5})}{(x_1 + x_2)^2} \begin{cases} \leq 0, & 0 < x_2 \leq 1, \\ > 0, & 1 < x_2 < 4, \end{cases}$$

$$\frac{\partial f_3}{\partial x_2} = \frac{(x_1 - \frac{1}{5})(x_1 + 1)}{(x_1 + x_2)^2} \begin{cases} \leq 0, & 0 < x_1 \leq 0.2, \\ > 0, & 0.2 < x_1 < 5, \end{cases}$$

the function f_3 changes its monotonicity within the domain $\text{supp}(\tilde{x}_1) \times \text{supp}(\tilde{x}_2) = (0, 5) \times (0, 4)$. Hence, the general part of our approach should be applied. The solution candidates for $y_3(\alpha)$ are

$$y_3^{LL}(\alpha) = f_3(x_1^L(\alpha), x_2^L(\alpha)) = \frac{(10\alpha - 1)(2\alpha - 1)}{20\alpha},$$

$$y_3^{LR}(\alpha) = f_3(x_1^L(\alpha), x_2^R(\alpha)) = \frac{(10\alpha - 1)(3 - 2\alpha)}{20},$$

$$y_3^{RL}(\alpha) = f_3(x_1^R(\alpha), x_2^L(\alpha)) = \frac{3}{5} \frac{(5\alpha - 8)(2\alpha - 1)}{\alpha - 5},$$

$$y_3^{RR}(\alpha) = f_3(x_1^R(\alpha), x_2^R(\alpha)) = \frac{3}{5} \frac{(5\alpha - 8)(3 - 2\alpha)}{5\alpha - 9}.$$

We can see from their plots in Figure 4 that the left branch of the maximum envelope is formed by y_3^{RL} for $0 < \alpha \leq 0.5$ and by y_3^{LL} for $0.5 < \alpha \leq 1$, where the value 0.5 corresponds to the intersection point between y_3^{RL} and y_3^{LL} . Its right branch, on the other hand, is formed by y_3^{LL} for $0 < \alpha \leq 0.02$ and by y_3^{RR} for $0.02 < \alpha \leq 1$, where the value 0.02 corresponds to the intersection point between y_3^{LL} and y_3^{RR} . Note that the value 0.02 is only approximate. Hence, the α -cuts $y_3(\alpha) = [y_3^L(\alpha), y_3^R(\alpha)]$ of \tilde{y}_3 are

$$y_3^L(\alpha) = \begin{cases} \frac{3}{5} \frac{(5\alpha - 8)(2\alpha - 1)}{\alpha - 5}, & 0 < \alpha \leq 0.5, \\ \frac{(10\alpha - 1)(2\alpha - 1)}{20\alpha}, & 0.5 < \alpha \leq 1, \end{cases}$$

$$y_3^R(\alpha) = \begin{cases} \frac{(10\alpha - 1)(2\alpha - 1)}{20\alpha}, & 0 < \alpha \leq 0.02, \\ \frac{3}{5} \frac{(5\alpha - 8)(3 - 2\alpha)}{5\alpha - 9}, & 0.02 < \alpha \leq 1. \end{cases}$$

With $y_3^L(0) = -0.96$, $y_3^L(0.5) = 0$, $y_3^L(1) = 0.45 = y_3^R(1)$, $y_3^R(0.02) \approx 1.57$, and $\lim_{\alpha \rightarrow 0} y_3^R(\alpha) = \infty$, the membership function of \tilde{y}_3 yields

$$\mu_{\tilde{y}_3}(y) = \begin{cases} \frac{21}{20} + \frac{1}{12}y - \frac{1}{60}\sqrt{A}, & -0.96 < y \leq 0, \\ \frac{3}{10} + \frac{1}{2}y + \frac{1}{10}\sqrt{B}, & 0 < y \leq 0.45, \\ \frac{31}{20} - \frac{5}{12}y - \frac{1}{60}\sqrt{C}, & 0.45 < y \leq 1.57, \\ \frac{3}{10} + \frac{1}{2}y - \frac{1}{10}\sqrt{D}, & 1.57 < y < \infty, \end{cases}$$

where

$$A = 25y^2 - 2370y + 1089,$$

$$B = 25y^2 + 30y + 4,$$

$$C = 625y^2 + 750y + 9,$$

$$D = 25y^2 + 30y + 4.$$

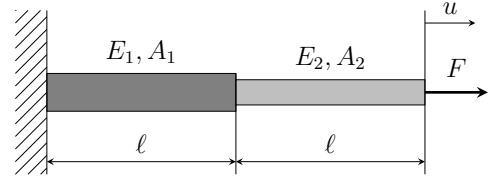


Figure 5: Two-component massless rod.

5. Engineering application

In order to illustrate the analytical approach in a more practical context, we consider a rather simple but typical example from engineering mechanics [6] consisting of a two-component massless rod under tensile load as shown in Figure 5. The components of the rod are characterized by the length ℓ , the elastic moduli E_1 and E_2 , and the cross sectional areas A_1 and A_2 . The left component of the rod is clamped to a wall, whereas the right component is subjected to a tensile force F . In order to compute the (static) displacement u of the tip of the rod, we can proceed as follows:

Every component of the rod can be viewed as a spring with the stiffness

$$c_i = \frac{E_i \cdot A_i}{\ell}, \quad i = 1, 2.$$

Since both components are placed in a row, the total stiffness c_{tot} of the rod is

$$\frac{1}{c_{\text{tot}}} = \frac{1}{c_1} + \frac{1}{c_2}.$$

According to HOOK's law, the displacement u of the tip of the rod yields

$$u = \frac{F}{c_{\text{tot}}} = \left(\frac{1}{c_1} + \frac{1}{c_2} \right) F. \quad (3)$$

At first, the displacement u shall be computed for crisp parameters c_1 and c_2 , where the first component is assumed to be made of steel and the second component of aluminum with the following material and geometry values:

$$E_1 = 2.0 \cdot 10^5 \text{ N/mm}^2, \quad A_1 = 100 \text{ mm}^2,$$

$$E_2 = 6.9 \cdot 10^4 \text{ N/mm}^2, \quad A_2 = 75 \text{ mm}^2,$$

and $\ell = 500 \text{ mm}$. The external force shall have the absolute value $F = 1000 \text{ N}$. With these values, the (crisp) solution for the displacement u yields

$$\bar{u} \approx 0.1216 \text{ mm}.$$

In reality, however, exact stiffness values for both rod components can usually not be provided due to variations in the manufacturing process. In order to include these uncertainties into the computation, the stiffness parameters c_1 and c_2 shall be modeled as symmetric GFNs with the modal values

$$\bar{c}_1 = 4.0 \cdot 10^4 \text{ N/mm},$$

$$\bar{c}_2 = 1.035 \cdot 10^4 \text{ N/mm}.$$

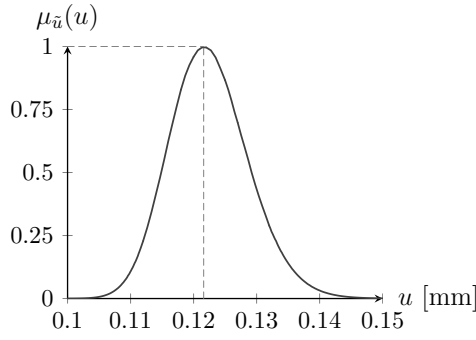


Figure 6: Fuzzy displacement \tilde{u} of the tip of the rod.

The standard deviations of \tilde{c}_1 and \tilde{c}_2 are assumed to be 5 % of their modal values. Hence, the parametric representation of the fuzzy stiffness parameters is

$$\tilde{c}_i = \text{gfn}(\bar{c}_i, 0.05 \bar{c}_i, 0.05 \bar{c}_i), \quad i = 1, 2.$$

In order to compute the α -cuts of the fuzzy displacement \tilde{u} , we consider again Eq. (3), where we can see that u is (strictly) monotonic decreasing in both c_1 and c_2 for positive values. Hence, the α -cuts $u(\alpha) = [u^L(\alpha), u^R(\alpha)]$ of \tilde{u} are

$$u^L(\alpha) = u(c_1^R(\alpha), c_2^R(\alpha)) = \frac{1007}{414 (20 + \sqrt{-2 \ln(\alpha)}),}$$

$$u^R(\alpha) = u(c_1^L(\alpha), c_2^L(\alpha)) = \frac{1007}{414 (20 - \sqrt{-2 \ln(\alpha)}),}$$

and its membership function yields

$$\mu_{\tilde{u}}(u) = \exp \left[-\frac{1}{a} \left(b - \frac{c}{u} + \frac{d}{u^2} \right) \right], \quad u > 0,$$

where

$$a = 342792, \quad b = 68558400,$$

$$c = 16675920, \quad d = 1014049.$$

The plot of $\mu_{\tilde{u}}(u)$ in the range $0.1 \leq u \leq 0.15$ is illustrated in Figure 6.

6. Conclusions

The proposed analytical approach turns out to be a very practical tool for the inclusion of parameter uncertainties into mathematical models. It is valid for continuous, monotonic functions of independent fuzzy numbers, but can also be applied to fuzzy intervals as defined, e. g., in [3].

An analytical solution has the advantage that the degrees of membership of the fuzzy output can be computed for any value within the support, whereas a numerical solution only provides a finite number of values. Furthermore, our approach also allows a symbolic processing of uncertainties.

In further research activities, this approach shall be extended to general, non-monotonic functions of independent fuzzy numbers, where the influence of interdependency shall be investigated as well.

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