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Gauge Classification, Lie Symmetries and Integrability of a Family of Nonlinear Schrödinger Equations

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Abstract

In this contribution we review and summarize recent articles on a family of nonlinear SCHRÖDINGER equations proposed by G.A. GOLDIN and one of us (HDD) [J. Phys. A. 27, 1994, 1771–1780], dealing with a gauge description of the family, a classification of its Lie symmetries in terms of gauge invariants and the integrability of certain sub-families indicated by their LIE symmetry, respectively.

1 Introduction

A classification of unitarily inequivalent representations of the *kinematic algebra* on $\mathbb{R}^n$, i.e., the semi-direct sum of the smooth vector-fields and the smooth functions,

$$S(\mathbb{R}^n) = \mathrm{Vect}(\mathbb{R}^n) {\in\!\!\!\!+} C^\infty(\mathbb{R}^n)\,, \tag{1}$$

defined by the commutator ($X_j \in \mathrm{Vect}(\mathbb{R}^n)$, $f_j \in C^\infty(\mathbb{R}^n)$)

$$\Big[(X_1, f_1), (X_2, f_2)\Big]_{S(\mathbb{R}^n)} = \Big([X_1, X_2]_{\mathrm{Vect}(\mathbb{R}^n)}, \mathcal{L}_{X_1} f_2 - \mathcal{L}_{X_2} f_1\Big)\,, \tag{2}$$

led G.A. GOLDIN and one of us (HDD) to a family of nonlinear SCHRÖDINGER equations [1, 2]. Their derivation fixed the imaginary part of $i\partial_t\psi/\psi$ and the real part was obtained by some additional physical and mathematical assumptions. In terms of probability densities $\rho = \psi\bar{\psi}$ and currents $\vec{J} = \frac{1}{2i}\left(\psi\vec{\nabla}\bar{\psi} - \bar{\psi}\vec{\nabla}\psi\right)$ these equations, that have been called DOEBNER-GOLDIN(DG)-equations [3], span an eight parameter family of homogeneous nonlinear partial differential equations (PDEs),

$$i\partial_t\psi/ = i\sum_{j=1}^{2}\nu_j R_j[\psi] + \sum_{j=1}^{5}\mu_j R_j[\psi] + \mu_0 V\,, \qquad \nu_1 \neq 0\,, \tag{3}$$

that includes the linear SCHRÖDINGER equation for $\nu_1 = -\hbar/2m$, $\mu_2 = -\hbar/4m$, $\mu_3 = \hbar/2m$, $\mu_5 = \hbar/8m$, $\mu_0 = 1/\hbar$, and $\nu_2 = \mu_1 = \mu_4 = 0$. Here V is a (real valued) potential and $R_j[\psi]$ denote real valued nonlinear functionals of ψ, complex homogeneous of degree zero:

$$\begin{gathered} R_1[\psi] := \frac{\vec{\nabla}\cdot\vec{J}}{\rho}\,, \qquad R_2[\psi] := \frac{\Delta\rho}{\rho}\,, \qquad R_3[\psi] := \frac{\vec{J}^2}{\rho^2}\,, \\ R_4[\psi] := \frac{\vec{J}\cdot\vec{\nabla}\rho}{\rho^2}\,, \qquad R_5[\psi] := \frac{(\vec{\nabla}\rho)^2}{\rho^2}\,. \end{gathered} \tag{4}$$

The family contains various nonlinear extensions of the SCHRÖDINGER equation put forward by other authors, e.g., [4, 5, 6, 7, 8, 9, 20]. Derivation, properties, and interpretation of the DG-equations have been studied to some extent; for recent results see the contributions in [10].

For the purposes of this paper it is convenient to use the decomposition

$$\psi(\vec{x},t) = \exp\left(r(\vec{x},t) + is(\vec{x},t)\right) . \tag{5}$$

This leads to a pair of coupled PDEs for the real valued functions r and s

$$\left\{ \begin{array}{lcl} \partial_t r &=& 2\nu_2\Delta r + \nu_1\Delta s + 4\nu_2(\vec{\nabla} r)^2 + 2\nu_1\vec{\nabla} r\cdot\vec{\nabla} s \\ \partial_t s &=& -2\mu_2\Delta r - \mu_1\Delta s - 4(\mu_2+\mu_5)(\vec{\nabla} r)^2 \\ && -2(\mu_1+\mu_4)\vec{\nabla} r\cdot\vec{\nabla} s - \mu_3(\vec{\nabla} s)^2 - \mu_0 V. \end{array}\right. \tag{6}$$

Note that due to the ambiguity of the phase function s in (5) the complex PDE (3) and the two real PDEs (6) are *not* fully equivalent. However, any solution $(r(\vec{x},t), s(\vec{x},t))$ of (6) yields a solution $\psi(\vec{x},t) = \exp\left(r(\vec{x},t) + is(\vec{x},t)\right)$ of (3).

In this paper we review a gauge classification of the family put forward in [11] (section 2), the maximal LIE symmetries calculated in [12] (section 3), and finally the integration of two sub-families according to [13] (section 4).

2 Gauge classification

It has been noticed [14, 15] that the sub-family

$$\mu_1 = 2\nu_2\,, \quad \mu_3 = -\nu_1\,, \quad \mu_4 = -2\nu_2\,, \quad \mu_5 = -\frac{1}{2}\mu_2\,, \qquad \mu_2 > 2\frac{\nu_2^2}{\nu_1} \tag{7}$$

of (3) may be transformed into the linear SCHRÖDINGER equation

$$i\partial_t\psi = \nu_1'\Delta\psi + \mu_0' V(\vec{x})\psi \tag{8}$$

by a nonlinear transformation of the dependent complex variable

$$N_{(\Lambda,\gamma)}(\psi) = \psi^{\frac{1}{2}(1+\Lambda+i\gamma)}\bar{\psi}^{\frac{1}{2}(1-\Lambda+i\gamma)} = |\psi|\, e^{i(\gamma\ln|\psi| + \Lambda\arg\psi)}\,, \tag{9}$$

where $\Lambda = \sqrt{\dfrac{\nu_1^2}{2\nu_1\mu_2 - \nu_2^2}}$, $\gamma = \dfrac{2\nu_2}{\nu_1}\Lambda$, and $\nu_1' = \dfrac{\nu_1}{\Lambda}$, $\mu_0' = \Lambda\mu_0$.

Obviously, these transformations leave the probability density ρ invariant. Since measurements in non-relativistic quantum mechanics are basically measurements of positions at different times (see, e.g., [16, p. 96]), these transformations are called *nonlinear gauge transformations* [11].

Again the transformation (9) of ψ is not properly defined for non-integer Λ. However, if Λ is integer, a generalized concept of 'nonlinear' observables (different from the one proposed by S. WEINBERG, [17]) *consistent* with the time evolution of the states ψ can be developed rigorously establishing full equivalence between this DG-model and linear quantum mechanics [18, 19].

Nevertheless, using the decomposition (5) it may be defined for the system (6) as a simple linear transformation of the functions r and s

$$\begin{pmatrix} r'(\vec{x},t) \\ s'(\vec{x},t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \gamma & \Lambda \end{pmatrix} \begin{pmatrix} r(\vec{x},t) \\ s(\vec{x},t) \end{pmatrix} . \tag{10}$$

Thus, $N_{(\Lambda,\gamma)}$ is a realization of the affine group $\mathit{Aff}(1)$. Furthermore, as is evident from (6), the nonlinear gauge transformations $N_{(\Lambda,\gamma)}$ leave the whole family of DG-equations invariant, changing the parameters (ν,μ) of the equations according to

$$\begin{gathered} \nu_1' = \frac{\nu_1}{\Lambda}\,, \quad \nu_2' = -\frac{\gamma}{2\Lambda}\nu_1 + \nu_2, \\ \mu_1' = -\frac{\gamma}{\Lambda}\nu_1 + \mu_1\,, \quad \mu_2' = \frac{\gamma^2}{2\Lambda}\nu_1 - \gamma\nu_2 - \frac{\gamma}{2}\mu_1 + \Lambda\mu_2\,, \quad \mu_3' = \frac{\mu_3}{\Lambda}\,, \\ \mu_4' = -\frac{\gamma}{\Lambda}\mu_3 + \mu_4\,, \quad \mu_5' = \frac{\gamma^2}{4\Lambda}\mu_3 - \frac{\gamma}{2}\mu_4 + \Lambda\mu_5\,, \quad \mu_0' = \Lambda\mu_0\,. \end{gathered} \tag{11}$$

Since this action of the two-dimensional group $\mathit{Aff}(1)$ on the eight-dimensional space of parameters is regular for $\nu_1 \neq 0$, we may choose six invariant parameters

$$\begin{gathered} \iota_1 = \nu_1\mu_2 - \nu_2\mu_1\,, \quad \iota_2 = \mu_1 - 2\nu_2\,, \quad \iota_3 = 1 + \mu_3/\nu_1\,, \quad \iota_4 = \mu_4 - \mu_1\mu_3/\nu_1\,, \\ \iota_5 = \nu_1(\mu_2 + 2\mu_5) - \nu_2(\mu_1 + 2\mu_4) + 2\nu_2^2\mu_3/\nu_1\,, \quad \iota_0 = \nu_1\mu_0\,, \end{gathered} \tag{12}$$

and two group parameters ν_1 and ν_2.

Since DG-equations connected by the gauge transformation (9) are equivalent, we choose the group parameters to be

$$\nu_1 = -1, \qquad \nu_2 = 0\,, \tag{13}$$

and we will use the gauge invariants to characterize the various sub-families of DG-equations.

3 Lie symmetries

Classifying the Lie symmetries of the free ($V \equiv 0$) DG-equations, we are led to distinguish nine different sub-families with different restrictions of the gauge invariants ι. This sub-family structure and the corresponding symmetry algebras are illustrated in Fig.1. Symmetry algebras with an upper index are infinite-dimensional, the others finite dimensional.

The finite-dimensional Lie symmetry algebras are spanned in the following way

$$sym_0(n) = \langle H, D, L_{jk}, P_j, E, R \rangle\,, \tag{14}$$
$$sym_1(n) = \langle H, D, L_{jk}, P_j, E, R, C, B_j \rangle\,, \tag{15}$$
$$sym_2(n) = \langle H, D, L_{jk}, P_j, E, R, A \rangle\,, \tag{16}$$
$$sym_3(n) = \langle H, D, L_{jk}, P_j, E, R, C, B_j, A \rangle\,, \tag{17}$$
$$sym_4(n) = \langle H, D, L_{jk}, P_j, E, R, F \rangle\,, \tag{18}$$

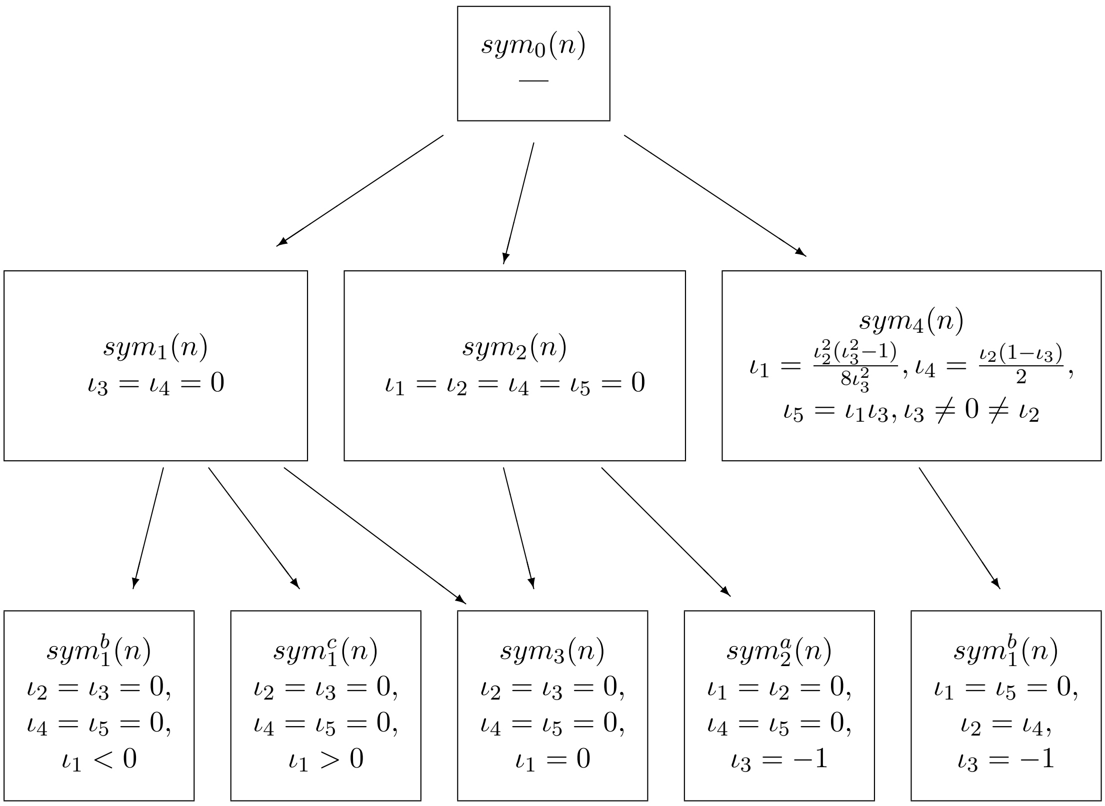

Figure 1: LIE symmetries of the DG-equations. Sub-families are characterized by gauge invariants and arrows indicate the subfamily structure.

with the following generators:

$$\begin{gathered}
H = \partial_t\,, \quad D = \sum_{j=1}^{n} x_j \partial_{x_j} + 2t\partial_t - \frac{n}{2}\partial_r - \frac{n\iota_2}{2}\partial_s\,, \quad P_j = \partial_{x_j}\,, \\
L_{jk} = x_j\partial_{x_k} - x_k\partial_{x_j}\,, P_j = \partial_{x_j}\,, \quad E = \frac{1}{2}\partial_s\,, \quad R = \partial_r\,, \\
C = \sum_{j=1}^{n} x_j t\partial_{x_j} + t^2\partial_t - \frac{n}{2}t\partial_r + \left(\frac{1}{4}\,\vec{x}^2 - \frac{n\iota_2}{2}\,t\right)\partial_s\,, \quad B_j = t\partial_{x_j} + \frac{1}{2}x_j\partial_s\,, \\
A = -t\partial_t + s\partial_s\,, \quad F = e^{(\iota_3-1)r - \frac{2\iota_3}{\iota_2}s}\left(\partial_r - \frac{\iota_2(1+\iota_3)}{2\iota_3}\partial_s\right)\,.
\end{gathered} \tag{19}$$

The generators of $sym_3(n) \supset sym_j(n)$, $j = 0, 1, 2$ obey the following nontrivial commutation relations ($j, k, l, m = 1, \ldots, n$):

$$\begin{gathered}
[D, H] = -2H\,, \quad [H, C] = D\,, \quad [D, C] = 2C\,, \quad [H, B_j] = P_j\,, \\
[D, P_j] = -P_j\,, \quad [D, B_j] = B_j\,, \quad [C, P_j] = -B_j\,, \quad [P_j, B_k] = \delta_{jk}E\,, \\
[A, H] = H\,, \quad [A, C] = -C\,, \quad [A, E] = -E\,, \quad [A, B_j] = -B_j\,, \\
[L_{jk}, P_l] = \delta_{kl}P_j - \delta_{jl}P_k\,, \qquad [L_{jk}, B_l] = \delta_{kl}B_j - \delta_{jl}B_k\,, \\
[L_{jk}, L_{lm}] = \delta_{kl}L_{jm} + \delta_{jm}L_{kl} - \delta_{jl}L_{km} - \delta_{km}L_{jl}\,,
\end{gathered} \tag{20}$$

and the exceptional generator F of $sym_4(n)$ yields

$$[F,R] = (1-\iota_3)F\,, \qquad [F,R] = -\frac{\iota_3}{\iota_2}F\,. \tag{21}$$

In particular, the fundamental LIE symmetry algebra $sym_0(n)$ of (3), i.e., the symmetry of *all* equations, consists of the EUCLIDean algebra $e(n)$, dilations and time translations (spanning an affine sub-algebra $\mathit{aff}(1)$) and complex homogeneity ($t(2)$),

$$sym_0(n) = \Big(\mathit{aff}(1) \oplus\!\!\!\!\!\!\in e(n)\Big) \oplus t(2)\,, \tag{22}$$

and the LIE algebra $sym_1(n)$ is a direct sum of the centrally extended SCHRÖDINGER algebra $sch_e(n)$ and a one–dimensional algebra $t(1)$ (real homogeneity),

$$sym_1(n) = sch_e(n) \oplus t(1)\,. \tag{23}$$

These particular DG-equations thus fit into the classes of SCHRÖDINGER-invariant nonlinear evolution equations determined in [20, 21, 22, 23].

Furthermore we have three different infinite-dimensional LIE algebras, the additional infinite dimensional parts of which we denote by $a^\infty, b^\infty, c^\infty$, respectively.

The LIE algebra a^∞ is spanned by the generators

$$Y_f = f(\iota_2 r - s)\partial_r\,, \tag{24}$$

where $f = f(z)$ is a real valued function on $\mathbb{R}$. Their commutators are

$$\Big[Y_{f_1}, Y_{f_2}\Big] = \iota_2 Y_{(f_1 f_2' - f_2 f_1')}\,. \tag{25}$$

Thus, a^∞ is isomorphic either to the commutative algebra of smooth functions on $\mathbb{R}$ or to the LIE algebra of vector fields on the real line,

$$a^\infty \simeq \begin{cases} C^\infty(\mathbb{R}) & \text{for } \iota_2 = 0, \\ \mathrm{Vect}(\mathbb{R}) & \text{for } \iota_2 \neq 0. \end{cases} \tag{26}$$

In the commutative case, these additional generators add to the generators of $sym_2(n)$, whereas in the non-commutative case they add to the fundamental symmetry algebra $sym_0(n)$ to yield the infinite-dimensional LIE symmetry algebras

$$sym_0^a(n) = \langle H, D, L_{jk}, P_j, E, R, Y_f\rangle\,, \tag{27}$$

$$sym_2^a(n) = \langle H, D, L_{jk}, P_j, E, R, A, Y_f\rangle\,. \tag{28}$$

Note that $sym_4(n)$ is a sub-algebra of $sym_0^a(n)$. The nontrivial commutation relations of Y_f with the generators of $sym_0(n)$ and $sym_2(n)$ are

$$\Big[Y_f, R\Big] = -\mu_1 Y_{f'}\,, \qquad \Big[Y_f, E\Big] = \frac{1}{2}Y_{f'}\,, \qquad \Big[Y_f, A\Big] = Y_{zf'}\,. \tag{29}$$

The second infinite-dimensional LIE algebra b^∞ is spanned by the generators

$$\begin{aligned} Z_{\Phi_\pm} = \; & e^{-r}\Bigg(\Phi_+(\vec{x},t)e^{\frac{1}{\sqrt{-2\iota_1}}s}\left(\frac{1}{\sqrt{-2\iota_1}}\partial_r - \partial_s\right) + \\ & \Phi_-(\vec{x},t)e^{-\frac{1}{\sqrt{-2\iota_1}}s}\left(\frac{1}{\sqrt{-2\iota_1}}\partial_r + \partial_s\right)\Bigg)\,, \end{aligned} \tag{30}$$

where Φ_+ (Φ_-) is a smooth solution of the forward (backward) heat equation with a diffusion coefficient $\sqrt{-2\iota_1}$:

$$\partial_t \Phi_\pm \pm \sqrt{-2\iota_1}\,\Delta\Phi_\pm = 0\,. \tag{31}$$

The LIE algebra $b^\infty = \{Z_{\Phi_\pm} | \Phi_\pm \text{ solutions of (31)}\}$ is commutative, and together with the elements of $sym_1(n)$ it spans the infinite-dimensional LIE symmetry algebra

$$sym_1^b(n) = \langle H, D, L_{jk}, P_j, E, R, C, B_j, Z_{\Phi_\pm}\rangle\,. \tag{32}$$

By integration of the generators $Z_{\Phi_\pm}$ we find a transformation of the subfamily ($\iota_2 = \iota_3 = \iota_4 = \iota_5 = 0$, $\iota_1 < 0$) to the above pair of forward and backward heat equations, i.e., if $\Phi_\pm$ is a solution of (31), then

$$\psi(\vec{x},t) = \sqrt{\Phi_+(\vec{x},t)\Phi_-(\vec{x},t)}\exp\left(i\sqrt{-\frac{\iota_1}{2}}\ln\left(\frac{\Phi_-}{\Phi_+}\right)\right)\,, \tag{33}$$

is a solution of (3), a relation given for a particular subclass of (3) already in [15].

Finally, there is a third infinite-dimensional LIE algebra involved, spanned by the generators

$$\begin{aligned} Z_\Psi \;=\; & e^{-r}|\Psi(\vec{x},t)|\Bigg(\sin\left(\frac{1}{\sqrt{2\iota_1}}s - \arg\Psi(\vec{x},t)\right)\partial_r + \\ & \sqrt{2\iota_1}\cos\left(\frac{1}{\sqrt{2\iota_1}}s - \arg\Psi(\vec{x},t)\right)\partial_s\Bigg)\,, \end{aligned} \tag{34}$$

where Ψ is a solution of the free linear SCHRÖDINGER equation

$$i\partial_t\Psi = -\sqrt{2\iota_1}\,\Delta\Psi\,. \tag{35}$$

Again the LIE algebra $c^\infty = \{Z_\Psi | \Psi \text{ solution of (35)}\}$ is commutative, and generates together with the elements of $sym_1(n)$ the infinite-dimensional LIE symmetry algebra

$$sym_1^c(n) = \langle H, D, L_{jk}, P_j, E, R, C, B_j, Z_\Psi\rangle\,. \tag{36}$$

Integrating the vector-field Z_Ψ, it turns out that this particular symmetry corresponds to the nonlinear gauge transformation $N_{(\sqrt{2\iota_1},0)}$!

4 Integrable sub-families

We have noted in the previous section that two of the sub-families of (3) with infinite-dimensional symmetries are linearizable by a local transformation of the dependent variables for any space dimension. Therefore, it seems worthwhile to examine the integrability of the sub-families with the other infinite dimensional symmetry algebras $sym_0^a(1)$ and $sym_2^a(1)$. Indeed, it turns out that in one space dimension an integration of the sub-families

$$sym_2(1): \quad \iota_2 = \iota_3 = \iota_4 = \iota_5 = 0\,, \tag{37}$$

$$sym_0^a(1): \quad \iota_1 = \iota_5 = 0,\ \iota_3 = -1,\ \iota_2 = \iota_4 \neq 0 \tag{38}$$

can be carried out by solving a set of implicit equations and quadratures [13].

Integrating the first sub-family $sym_2(1)$, we have to distinguish the case $\iota_3 = 0$ admitting the larger symmetry algebra $sym_3(1)$. In both cases we obtain either travelling wave solutions or solutions involving the (local) solution of the implicit equation for an arbitrary smooth function f

$$2(1-\iota_3)tz - x + f'(z) = 0\,. \tag{39}$$

The general solution of the particular case $\iota_3 = 0$ of the sub-family (37) reads

$$\psi(x,t) = g(x-2C_1t)e^{i(C_1x - C_1^2t + C_2)}\,, \tag{40}$$

$$\psi(x,t) = \Big(f''(z) + 2t\Big)^{-\frac{1}{2}} g(z) e^{-i(tz^2 - xz + f(z))}\,, \tag{41}$$

whereas the general solution for $\iota_3 \neq 0$ has the form

$$\psi(x,t) = g(x-2C_1t)e^{i(C_1x - \mu_3 C_1^2t + C_2)}\,, \tag{42}$$

$$\psi(x,t) = z^{-\frac{1}{2\iota_3}} g\left(-2\iota_3 t z^{\frac{\iota_3-1}{\iota_3}} + \int^z \zeta^{-\frac{1}{\iota_3}} f''(\zeta)d\zeta\right) e^{-i((1-\iota_3)tz^2 - xz + f(z))}\,. \tag{43}$$

In both cases f, g are arbitrary sufficiently smooth real-valued functions on $\mathbb{R}$, and C_j arbitrary real parameters.

The general solution of the second sub-family (38) involves an arbitrary solution of the heat equation

$$u_t + \iota_2 u_{xx} = 0\,, \tag{44}$$

and an arbitrary smooth real-valued function f on $\mathbb{R}$:

$$\psi(x,t) = (u(x,t))^{\frac{1}{2}} f\Big(\int_0^x u(\xi,t)d\xi - \mu_1 \int_0^t u_x(0,\tau)d\tau\Big) e^{\frac{i\mu_1}{2}\ln u(x,t)}\,. \tag{45}$$

5 Conclusions

We have summarized the results of [11, 12, 13] on the DG-equation (3). First we have seen that the family (3) admits a nonlinear gauge description as it is invariant under the nonlinear gauge transformation (9). Using this fact, we reduced the number of parameters of the family by two and proposed a set of parameters ι to describe the gauge invariant sub-families.

Since the nonlinear gauge transformations are *local* transformations, we were able to use a particular gauge to determine the Lie symmetries of the free DG-equations. Besides the linearization of a certain sub-family by a nonlinear gauge transformation, this examination led to a linearization of another sub-family to a forward/backward heat equation. Using Lie symmetries as an indicator for integrability of PDEs, we were able to integrate free DG-equations in one space dimension by quadratures and implicit equations. But as the implicit equation can in general only be solved locally for a certain space-time region, the solutions obtained in this way are only local solutions in general. On the other hand, the solutions of the sub-families indicate an infinite-dimensional generalized symmetry of the DG-equation. For instance, the infinite-dimensional Lie symmetry of the heat equation (44) induces an infinite-dimensional *nonlocal* symmetry of the DG-equation (38).

Though we summarized the methods of integration only for the free DG-equations, they can be extended to DG-equations with potentials V. Labelling sub-families by their symmetries, the DG-equations $sym_1^b(n)$ and $sym_1^c(n)$ can be integrated in arbitrary space dimension n, and $sym_2(1)$ and $sym_0^a(1)$ in one space dimension $n = 1$. Thus, at least in one space dimension we can solve the bottom line of sub-families in Fig.1. It remains in a first step to extend these methods to arbitrary space dimensions n and to search for an integration of the other, larger sub-families.

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