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On Poincaré-Invariant Reduction and Exact Solutions of the Yang-Mills Equations

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Classical ideas and methods developed by Sophus Lie provide us with a powerful tool for constructing exact solutions of partial differential equations (PDE) (see, e.g., [1–4]). In the present paper we apply the above methods to obtain new explicit solutions of the nonlinear Yang-Mills equations (YME). By the classical YME, we mean the following nonlinear system of twelve second-order PDE:

$$\partial_\nu\partial^\nu\vec{A}_\mu - \partial^\mu\partial_\nu\vec{A}_\nu + e[(\partial_\nu\vec{A}_\nu)\times\vec{A}_\mu - 2(\partial_\nu\vec{A}_\mu)\times\vec{A}_\nu + (\partial^\mu\vec{A}_\nu)\times\vec{A}^\nu] + e^2\vec{A}_\nu\times(\vec{A}^\nu\times\vec{A}_\mu) = \vec{0}. \tag{1}$$

Here $\partial_\nu = \dfrac{\partial}{\partial x_\nu}$, $\mu = \overline{0,3}$, $e = \text{const}$, $\vec{A}_\mu = \vec{A}_\mu(x) = \vec{A}_\mu(x_0, x_1, x_2, x_3)$ is a three-component vector-potential of the Yang-Mills field. Hereafter, the summation over the repeated indices μ, ν from 0 to 3 is supposed. Raising and lowering the vector indices are peformed with the aid of the metric tensor $g_{\mu\nu}$, i.e., $\partial^\mu = g_{\mu\nu}\partial_\nu$ ($g_{\mu\nu} = 1$ if $\mu = \nu = 0$, $g_{\mu\nu} = -1$ if $\mu = \nu = 1, 2, 3$ and $g_{\mu\nu} = 0$ if $\mu \neq \nu$).

It should be said that there were several reviews devoted to classical solutions of YME. The solutions were obtained with the help of ad hoc substitutions suggested by Wu and Yang, Rosen, 't Hooft, Corrigan and Fairlie, Wilczek, Witten (for more detail see review [5] and references cited therein). But, in fact, symmetry properties of YME were not used.

It was known [6] that YME are invariant under the group $C(1,3)\bigotimes SU(2)$, where $C(1,3)$ is the 15-parameter conformal group and $SU(2)$ is the infinite-parameter special unitary group.

Symmetry properties of YME were used and some new exact solutions of equation(1) were obtained by W. Fushchych and W.Shtelen in [7] (see also [3]).

The present report is a continuation of the investigations which were realized by the author together with R. Zhdanov and W. Fushchych. With the aid of $P(1,3)$-inequivalent ansatzes for the Yang-Mills field, which are invariant under three-dimensional subgroups of the Poincaré group, reduction of YME to systems of ordinary differential equations is carried out and wide families of their exact solutions are constructed in [8, 9]. Here we carry out symmetry reduction of YME on four-dimensional subgroups of the Poincaré group to functional equations.

The symmetry group of YME contains as a subgroup the Poincaré group $P(1,3)$ having the following generators:

$$P_\mu = \partial_\mu, \quad J_{\alpha\beta} = x^\alpha\partial_\beta - x^\beta\partial_\alpha + A^{a\alpha}\partial_{A^a_\beta} - A^{a\beta}\partial_{A^a_\alpha}, \tag{2}$$

where $\partial_{A^a_\mu} = \dfrac{\partial}{\partial_{A^a_\mu}}$, $\mu, \alpha, \beta = \overline{0,3}$, $a = \overline{1,3}$.

The method of symmetry reduction is described in [1, 2]. A key idea of the symmetry approach to the problem of reduction of PDE is a special choice of the form of a solution. This choice is dictated by a structure of the symmetry group admitted by the equation under study.

In the case involved, to reduce YME by N variables, one has to construct ansatzes for the Yang-Mills field $A^a_\mu(2)$ invariant under N-dimensional subalgebras of the algebra with the basis elements (2) [1–4]. Hence, for Poincaré-invariant ansatzes reducing YME to systems of functional equations, N is equal to 4. Due to invariance of YME under the conformal group $C(1,3)$, it is sufficient to consider only subalgebras which can not be transformed one into another by group transformation, i.e., $C(1,3)$-inequivalent subalgebras. Complete description of the Poincaré algebra was obtained in [10] (see also [11]).

As basis elements of the Poincaré algebra $AP(1,3)$, $P_\mu, J_{\alpha\beta}$ from (2) have the form

$$X_a = \xi_{a\mu}(x)\partial_\mu + \rho^{bc}_{a\mu\nu}(x)A^c_\nu\partial_{A^b_\mu},$$

so the invariant ansatz for the field $A^a_\mu(x)$ is searched for in the form [3, 4]

$$A^a_\mu(x) = Q^{ab}_{\mu\nu}(x)B^b_\nu. \tag{3}$$

We make the symmetry reduction using 4-dimensional subalgebras of the Poincaré algebra $AP(1,3)$ and, consequently, in (3) B^b_ν are arbitrary constants, $Q^{ab}_{\mu\nu}(x)$ are particular solutions of the system of PDE $(\xi_{a\nu}\partial_\nu - \rho^{bc}_{a\mu\alpha})Q^{cd}_{\alpha\beta} = 0, \mu, \nu, \alpha, \beta = \overline{0,3},\ a = \overline{1,4},\ b, d = \overline{1,3}$, and the condition

$$\operatorname{rank} ||\xi_{a\mu}(x)||^{4}_{a=1}{}^{3}_{\mu=0} = 4 \tag{4}$$

holds (for the detail, see [1,4,9]).

General form of the ansatz invariant under a 4-dimensional subalgebra of the algebra $AP(1,3)$ with generators (2) is the following:

$$\vec{A}_\mu(x) = a_{\mu\nu}(x)\vec{B}^\nu, \tag{5}$$

where

$$\begin{aligned} a_{\mu\nu}(x) = {}& (a_\mu a_\nu - d_\mu d_\nu)ch\theta_0 + (d_\mu a_\nu - d_\nu a_\mu)sh\theta_0 + 2(a_\mu + d_\mu)[(\theta_1\cos\theta_3 + \\ & \theta_2\sin\theta_3)b_\nu + (\theta_2\cos\theta_3 - \theta_1\sin\theta_3)c_\nu + (\theta_1^2 + \theta_2^2)(a_\nu + d_\nu)e^{-\theta_0}] + \\ & (b_\mu c_\nu - b_\nu c_\mu)\sin\theta_3 - (c_\mu c_\nu + b_\mu b_\nu)\cos\theta_3 - 2(\theta_1 b_\mu + \theta_2 c_\mu)(a_\nu + d_\nu)e^{-\theta_0}, \end{aligned}$$

$a_\mu, b_\mu, c_\mu, d_\mu$ are components of the constant 4-vectors a, b, c, d which form orthonormal basis of the Minkowski space-time, i.e. $a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1$, $a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0$ and $\theta_0, \theta_1, \theta_2, \theta_3$ are smooth functions of x, whose explicit form is determined by the choice of the 4-dimensional subalgebras of the algebra $AP(1,3)$.

Below, we adduce a complete list of 4-dimensional $C(1,3)$-inequivalent subalgebras of the Poincaré algebra, which satisfy the condition (4).

$L_1 =< P_0, P_1, P_2, P_3 >,$ $L_2 =< J_{03}, M, P_1, P_2 >,$

$L_3 =< J_{12} + \alpha J_{03}, M, P_1, P_2 >,$ $L_4 =< G_1, J_{03}, M, P_2 >,$

$L_5 =< G_1, J_{03}, M, P_1 + \alpha P_2 >,$ $L_6 =< G_1, G_2, J_{03}, M >,$

$L_7 =< G_1, G_2, J_{12} + \alpha J_{03}, M >,$ $L_8 =< J_{12} + P_0, P_1, P_2, P_3 >,$

$L_9 =< J_{12} + P_3, P_0, P_1, P_2 >,$ $L_{10} =< J_{12} + P_0 - P_3, M, P_1, P_2 >,$

$L_{11} =< J_{03} + P_1, P_0, P_2, P_3 >,$ $L_{12} =< G_1 + P_2, P_0, P_1, P_3 >,$

$L_{13} =< G_1 + P_0 - P_3, M, P_1, P_2 >,$ $L_{14} =< G_1 + \alpha P_2, G_2 + P_0 - P_3, M, P_1 >,$

$L_{15} =< G_1, J_{03} + P_2, M, P_1 + \alpha P_2 >,$ $L_{16} =< G_1, J_{03} + P_1, M, P_2 >,$

$L_{17} =< G_1, G_2, J_{03} + P_1, M > .$

Here $M = P_0 + P_3,\ G_i = J_{0i} - J_{i3} (i = 1, 2),\ \alpha > 0.$

Ansatzes for the Yang-Mills field $A^a_\mu(x)$ are of the form (5), non-zero functions $\theta_\mu(x)$, $\mu = \overline{0,3}$ being determined by one of the following formulae:

$$
\begin{aligned}
&L_2 : \theta_0 = -\ln|kx|;\\
&L_3 : \theta_0 = -\ln|kx|,\ \theta_3 = \frac{1}{\alpha}\ln|kx|;\\
&L_4 : \theta_0 = -\ln|kx|, \theta_1 = \frac{1}{2}bx(kx)^{-1};\\
&L_5 : \theta_0 = -\ln|kx|,\ \theta_1 = \frac{1}{2}\Big(bx - \frac{1}{\alpha}cx\Big)(kx)^{-1};\\
&L_6 : \theta_0 = -\ln|kx|, \theta_1 = \frac{1}{2}bx(kx)^{-1}, \theta_2 = \frac{1}{2}cx(kx)^{-1},\ \theta_3 = \frac{1}{\alpha}\ln|kx|;\\
&L_8 : \theta_3 = -ax;\\
&L_9 : \theta_3 = dx;\\
&L_{10} : \theta_3 = -\frac{1}{2}kx;\\
&L_{11} : \theta_0 = -cx;\\
&L_{12} : \theta_1 = \frac{1}{2}cx;\\
&L_{13} : \theta_1 = -\frac{1}{4}kx;\\
&L_{14} : \theta_1 = \frac{1}{8\alpha}(4cx + (kx)^2),\ \theta_2 = -\frac{1}{4}kx;\\
&L_{15} : \theta_0 = -\ln|kx|,\ \theta_1 = \frac{1}{2}\Big[bx - \frac{1}{\alpha}cx + \frac{1}{\alpha}\ln|kx|\Big](kx)^{-1};\\
&L_{16} : \theta_0 = -\ln|kx|,\ \theta_1 = \frac{1}{2}[bx - \ln|kx|](kx)^{-1};\\
&L_{17} : \theta_0 = -\ln|kx|,\ \theta_1 = \frac{1}{2}[bx - \ln|kx|](kx)^{-1},\ \theta_2 = \frac{1}{2}cx(kx)^{-1}.
\end{aligned}
\tag{6}
$$

Here $ax = a_\mu x^\mu$, $bx = b_\mu x^\mu$, $cx = c_\mu x^\mu$, $dx = d_\mu x^\mu$, $\mu = \overline{0,3}$, $kx = ax + dx$.

In order to reduce YME to functional equations, it is necessary to substitute ansatz (5) into (1) and convolute the expression obtained with $a^\mu_\alpha(x)$. As a result, we get a system of twelve nonlinear equations for B^a_ν of the form

$$m_{\mu\gamma}\vec{B}^\gamma + eh_{\mu\nu\gamma}(\vec{B}^\nu \times \vec{B}^\gamma) + e^2(\vec{B}_\gamma \times (\vec{B}^\gamma \times \vec{B}_\mu)) = 0. \tag{7}$$

Coefficients of the reduced equations are given by the following formulae:

$$m_{\mu\gamma} = a^\alpha_\mu \Box a_{\alpha\gamma} - a_{\alpha\mu} a_{\beta\gamma x_\beta x_\alpha}, \tag{8}$$

where $h_{\mu\nu\gamma} = \frac{1}{2}(g_{\mu\gamma} a_{\alpha\nu x_\alpha} - g_{\mu\nu} a_{\alpha\gamma x_\alpha}) - a^\alpha_\mu a_{\alpha\nu x_\beta} a_{\beta\gamma} - a^\alpha_\nu a_{\alpha\gamma x_\beta} a_{\beta\mu} - a^\alpha_\gamma a_{\alpha\mu x_\beta} a_{\beta\nu}$, $g_{\mu\nu}$ is a metric tensor of the Minkowski space $R(1,3)$.

Substituting functions $a_{\mu\nu}(x)$ from (5), where $\theta_\mu(x)$ are determined by one of the formulae (6) into (8), we obtain coefficients of the corresponding systems of equations (7). Below we restrict ourselves only to non-zero coefficients:

$$\begin{aligned}
&L_2 : h_{\mu\nu\gamma} = \tfrac{\epsilon}{2}\Psi(k);\\
&L_3 : h_{\mu\nu\gamma} = \tfrac{\epsilon}{2}\Psi(k) + \tfrac{\epsilon}{\alpha}\Phi(b,c,k);\\
&L_4 : m_{\mu\gamma} = -k_\mu k_\gamma,\ h_{\mu\nu\gamma} = \epsilon\Psi(k);\\
&L_5 : m_{\mu\gamma} = -\Big(1 + \frac{1}{\alpha^2}\Big)k_\mu k_\gamma,\ h_{\mu\nu\gamma} = \epsilon\Psi(k) + \tfrac{\epsilon}{\alpha}\Phi(b,c,k);\\
&L_6 : m_{\mu\gamma} = -2k_\mu k_\gamma, h_{\mu\nu\gamma} = \tfrac{3\epsilon}{2}\Psi(k);\\
&L_7 : m_{\mu\gamma} = -2k_\mu k_\gamma,\ h_{\mu\nu\gamma} = \tfrac{3\epsilon}{2}\Psi(k) - \tfrac{\epsilon}{\alpha}\Phi(b,c,k);\\
&L_8 : m_{\mu\gamma} = b_\mu b_\gamma + c_\mu c_\gamma,\ h_{\mu\nu\gamma} = \Phi(b,c,a);\\
&L_9 : m_{\mu\gamma} = -(b_\mu b_\gamma + c_\mu c_\gamma),\ h_{\mu\nu\gamma} = -\Phi(b,c,d);\\
&L_{10} : h_{\mu\nu\gamma} = \frac{1}{2}\Phi(b,c,k);\\
&L_{11} : m_{\mu\gamma} = -(a_\mu a_\gamma - d_\mu d_\gamma),\ h_{\mu\nu\gamma} = -\Phi(a,d,c);\\
&L_{12} : m_{\mu\gamma} = k_\mu k_\gamma,\ h_{\mu\nu\gamma} = -\Phi(k,b,c);\\
&L_{14} : m_{\mu\gamma} = -\frac{1}{\alpha^2}k_\mu k_\gamma,\ h_{\mu\nu\gamma} = -\frac{1}{\alpha}\Phi(k,b,c);\\
&L_{15} : m_{\mu\gamma} = -\Big(1 + \frac{1}{\alpha^2}\Big)k_\mu k_\gamma,\ h_{\mu\nu\gamma} = \epsilon\Psi(k) + \tfrac{\epsilon}{\alpha}\Phi(k,b,c);\\
&L_{16} : m_{\mu\gamma} = -k_\mu k_\gamma,\ h_{\mu\nu\gamma} = \epsilon\Psi(k);\\
&L_{17} : m_{\mu\gamma} = -2k_\mu k_\gamma,\ h_{\mu\nu\gamma} = \tfrac{3\epsilon}{2}\Psi(k);
\end{aligned} \tag{9}$$

where $k_\mu = a_\mu + d_\mu$, $\Psi(k) = g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma$, $\Phi(a,b,c) = (a_\mu b_\nu - a_\nu b_\mu)c_\gamma + (a_\nu b_\gamma - a_\gamma b_\nu)c_\mu + (a_\gamma b_\mu - a_\mu b_\gamma)c_\nu$, $\epsilon = 1$ for $kx > 0$ and $\epsilon = -1$ for $kx < 0$, $\alpha > 0$.

The principal idea of our approach to solution of the systems of equations (8), (9) is rather simple and quite natural. It is a reduction of these systems by the number of components with the aid of ad hoc substitutions. Using this trick, we construct particular solutions of equations (8), (9). Below we adduce substitutions for $\vec{B}_\mu$ and corresponding equations:

L_{11} :

$$\vec{B}_\mu = a_\mu \lambda_1 \vec{e}_1 + c_\mu \lambda_2 \vec{e}_2 + d_\mu \lambda_3 \vec{e}_3, \tag{10}$$

$$\begin{aligned} &\lambda_1 - 2e\lambda_2\lambda_3 - e^2\lambda_1(\lambda_2^2 + \lambda_3^2) = 0, \\ &2\lambda_1\lambda_3 - e\lambda_2(\lambda_3^2 - \lambda_1^2) = 0, \\ &\lambda_3 + 2e\lambda_1\lambda_2 - e^2(\lambda_2^2 - \lambda_1^2)\lambda_3 = 0, \end{aligned} \tag{11}$$

where $\lambda_1, \lambda_2, \lambda_3$ are arbitrary real constants, $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, $\vec{e}_3 = (0,0,1)$.

The system (11) has only the following real solutions: $\lambda_1 = \pm e^{-1}\sqrt{2+2\sqrt{2}}$, $\lambda_3 = \pm e^{-1}\sqrt{2\sqrt{2}-2}$, $\lambda_2 = -e^{-1}$ if $\lambda_1\lambda_3 > 0$ and $\lambda_2 = e^{-1}$ if $\lambda_1\lambda_3 < 0$.

Substituting the results obtained into formulae (10), (5), we obtain exact solutions of the YME(1):

$$\vec{A}_\mu = e^{-1}\Big[\pm\sqrt{2+\sqrt{2}}(a_\mu ch(cx) - d_\mu sh(cx))\vec{e}_1 - c_\mu \vec{e}_2 \pm \sqrt{2\sqrt{2}-2}(d_\mu ch(cx) - a_\mu sh(cx))\vec{e}_3\Big];$$

$$\vec{A}_\mu = e^{-1}\Big[\pm\sqrt{2+\sqrt{2}}(a_\mu ch(cx) - d_\mu sh(cx))\vec{e}_1 + c_\mu \vec{e}_2 \mp \sqrt{2\sqrt{2}-2}(d_\mu ch(cx) - a_\mu sh(cx))\vec{e}_3\Big].$$

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