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Symmetry Group of Vlasov-Maxwell Equations in Plasma Theory

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1 Introduction

The classical group analysis developed by S. Lie at the end of the past centure (see also ([1]–[2]) deals with systems of differential equations. For systems of integro-differential equations an algoritm suggested by Lie should be modified. Difficulties arise when one tries to construct and what is more important to solve *nonlocal determining equations (NDE)*, which result from invariance conditions for a basic system of nonlocal equations under the group of transformations. The problem of obtaining NDE was repeatedly discussed (see examples in [3], [4]). Following [5]–[9], in this paper the solution of NDE is used to obtain a Lie point-symmetry group for Vlasov-Maxwell equations and to extend this group upon nonlocal variables.

2 Vlasov-Maxwell equations

A hot rarefied plasma is a quasineutral gas of charged particles with a negligibly low rate of Coulomb collisions. The macroscopic state of plasma particles is described by distribution functions f (specific for different plasma particles species) which depend on time t, radius-vector $\mathbf{r}$ of a particle in the coordinate space and particle's velocity $\mathbf{v}$. The evolution of distribution functions is governed by kinetic equations:

$$f^\alpha_t + \mathbf{v} f^\alpha_{\mathbf{r}} + e_\alpha \Big\{ \mathbf{E} + \frac{1}{c}[\mathbf{v}\mathbf{B}] \Big\} f^\alpha_{\mathbf{p}_\alpha} = 0 \,; \quad \mathbf{p}_\alpha = \frac{m_\alpha \mathbf{v}}{\sqrt{1-(\mathbf{v}/c)^2}} \,. \tag{1}$$

The index α indicates a sort of plasma particles (with the charge e_α and mass m_α), c is the free-space light velocity. The fields $\mathbf{E}$ and $\mathbf{B}$ in (1) obey the Maxwell equations

$$\mathbf{B}_t + c \,\mathrm{rot}\, \mathbf{E} = 0 \,; \quad \mathrm{div}\, \mathbf{E} = 4\pi\rho \,; \quad \mathbf{E}_t - c\,\mathrm{rot}\, \mathbf{B} + 4\pi \mathbf{j} = 0 \,; \quad \mathrm{div}\, \mathbf{B} = 0 \,, \tag{2}$$

where the charge density ρ and current density $\mathbf{j}$ are in turn governed by motion of particles:

$$\rho = \sum_\alpha e_\alpha \int d\mathbf{p}_\alpha \, f^\alpha \,, \quad \mathbf{j} = \sum_\alpha e_\alpha \int d\mathbf{p}_\alpha \, f^\alpha \mathbf{v} \,. \tag{3}$$

Equations (1)–(3) are known as Vlasov-Maxwell equations [10].

3 Group analysis of Vlasov-Maxwell equations. First stage: constructing of local determining equations

The group admitted by Vlasov-Maxwell equations for homogeneous electron plasma in the one-dimensional nonrelativistic limit was first found in [11]. By introducing moments of a distribution function, the author transformed the basic integro-differential manifold to an infinite set of differential equations which was analyzed using the standard Lie approach. A similar method was used in [12], where group analysis was done for one-dimensional kinetic Benney equations (Vlasov-type equations) and some of their dissipative analogues. For the one-dimensional nonrelativistic model of Vlasov-Maxwell equations, the direct construction of NDE was fulfilled in [13] with the help of finite transformations generated by the Lie point group.

An approach that enables to construct NDE for a wide class of integro-differential systems was suggested in [14]. In particular, the result of [11] was reproduced in [14] by a direct solution of NDE. Based on [14] we presented a general concept in group analysis of Vlasov-Maxwell equations [6]. Its main idea combines a local group analysis of a differential part [1]–[2] of Vlasov-Maxwell equations with a nonlocal group analysis of material relationships (3). The construction of a NDE is formalized thanks to the special representation (see the formula (7) below). Employment of a solution of local determinig equations in order to simplify a NDE and a subsequent splitting of NDE by means of variational differentiation with respect to the distribution function[†] make closed the whole procedure of group analysis of Vlasov-Maxwell equations [1]–[3]. The latter operation is not described in [14].

Local determinig equations (LDE) yield invariance criteria of equations (1)–(2) under the point transformation group with the canonical *infinitesimal operator (IO)* [2]

$$Y = \sum_{\alpha} \chi^{1\,\alpha}\partial_{f^{\alpha}} + \overrightarrow{\chi^2}\partial_{\mathbf{E}} + \overrightarrow{\chi^3}\partial_{\mathbf{B}} + \overrightarrow{\chi^4}\partial_{\mathbf{j}} + \chi^5\partial_{\rho} \tag{4}$$

and are obtained in a standard way [1]–[2], namely applying the operator Y to basic equations (1)–(2). The result is as follows:

$$\begin{gathered}
D_t\left(\chi^{1\,\alpha}\right) + \mathbf{v}D_{\mathbf{r}}\left(\chi^{1\,\alpha}\right) + \frac{e_\alpha}{m_\alpha}\sqrt{1-(\mathbf{v}/c)^2}\left\{\mathbf{E} + \frac{1}{c}\left[\mathbf{v},\,\mathbf{B}\right] - \mathbf{v}\frac{(\mathbf{v},\,\mathbf{E})}{c^2}\right\} D_{\mathbf{v}}\left(\chi^{1\,\alpha}\right) + \\
\frac{e_\alpha}{m_\alpha}\sqrt{1-(\mathbf{v}/c)^2}\left\{\overrightarrow{\chi^2} + \frac{1}{c}\left[\mathbf{v},\,\overrightarrow{\chi^3}\right] - \frac{\mathbf{v}}{\mathrm{c}^2}\left(\mathbf{v},\,\overrightarrow{\chi^2}\right)\right\} f^{\alpha}{}_{\mathbf{v}} = 0\,; \\
c\left[D_{\mathbf{r}},\,\overrightarrow{\chi^2}\right] + D_t\left(\overrightarrow{\chi^3}\right) = 0\,; \quad c\left[D_{\mathbf{r}},\,\overrightarrow{\chi^3}\right] - D_t\left(\overrightarrow{\chi^2}\right) = 4\pi\overrightarrow{\chi^4}\,; \\
\left(D_{\mathbf{r}},\overrightarrow{\chi^2}\right) = 4\pi\chi^5\,; \quad \left(D_{\mathbf{r}},\,\overrightarrow{\chi^3}\right) = 0\,; \quad D_{\mathbf{v}}\left(\overrightarrow{\chi^2}\right) = 0\,, \quad D_{\mathbf{v}}\left(\overrightarrow{\chi^3}\right) = 0\,.
\end{gathered} \tag{5}$$

[†]See below section 4.

The last two vector LDE in (5) follow from the equations $\mathbf{E}_{\mathbf{v}} = 0$ and $\mathbf{B}_{\mathbf{v}} = 0$, which expresses the "trivial" fact of independence of the fields $\mathbf{E}$ and $\mathbf{B}$ upon the velocity $\mathbf{v}$ of plasma particles; D_s is the total derivative with respect to s.

The solution of LDE (5) is found in a usual way [1]–[2] and is presented by the following equalities which define *the group of an intermediate symmetry* (see [5]–[9]):

$$\begin{aligned}
&\chi^{1\,\alpha} = \eta^{1\,\alpha}(f^\alpha) - \xi^1 f^\alpha_t - \overrightarrow{\xi^2} f^\alpha_{\mathbf{r}} - \overrightarrow{\xi^3} f^\alpha_{\mathbf{v}}\,; \quad \overrightarrow{\chi^2} = -A_4\mathbf{E} + [\mathbf{g},\,\mathbf{E}] - c\,[\mathbf{b},\,\mathbf{B}] - \\
&\quad \xi^1\mathbf{E}_t - \left(\overrightarrow{\xi^2},\,\nabla\right)\mathbf{E}\,; \quad \overrightarrow{\chi^3} = -A_4\mathbf{B} + [\mathbf{g},\,\mathbf{B}] + c\,[\mathbf{b},\,\mathbf{E}] - \xi^1\mathbf{B}_t - \left(\overrightarrow{\xi^2},\,\nabla\right)\mathbf{B}\,; \\
&\overrightarrow{\chi^4} = -2A_4\mathbf{j} + [\mathbf{g},\,\mathbf{j}] + c^2\mathbf{b}\,\rho - \xi^1\mathbf{j}_t - \left(\overrightarrow{\xi^2},\,\nabla\right)\mathbf{j}\,; \quad \chi^5 = -2A_4\rho + (\mathbf{b},\,\mathbf{j}) - \\
&\quad \xi^1\rho_t - \overrightarrow{\xi^2}\rho_{\mathbf{r}}\,; \quad \xi^1 = A_0 + (\mathbf{b},\,\mathbf{r}) + A_4 t\,; \\
&\overrightarrow{\xi^2} = \mathbf{A} + c^2\mathbf{b}t + [\mathbf{g},\,\mathbf{r}] + A_4\mathbf{r}\,; \quad \overrightarrow{\xi^3} = c^2\mathbf{b} - (\mathbf{b},\,\mathbf{v})\,\mathbf{v} + [\mathbf{g},\,\mathbf{v}]\,.
\end{aligned} \tag{6}$$

Here A_0, $\mathbf{A}$, A_4, $\mathbf{b}$ and $\mathbf{g}$ are arbitrary constants, $\eta^{1\,\alpha}(f^\alpha)$ are unknown functions of a single group variable, namely the distribution function f^α.

4 Second stage: construction of nonlocal determining equations

As material relationships (3) are nonlocal, we use the special representation [5] for derivatives with respect to f^α in the canonical IO (4):

$$\chi^{1\,\alpha}\partial_{f^\alpha} \equiv \int d\mathbf{v}\,\chi^{1\,\alpha}(\mathbf{v})\frac{\delta}{\delta f^\alpha(\mathbf{v})}\,. \tag{7}$$

In view of the formula (7) the direct action of the canonical IO (4) upon equalities (3) gives the desired NDE:

$$\chi^5 - \sum_\alpha e_\alpha \int d\mathbf{p}\,\chi^{1\,\alpha} = 0\,, \quad \chi^4 - \sum_\alpha e_\alpha \int d\mathbf{p}\,\chi^{1\,\alpha}\,\mathbf{v} = 0\,. \tag{8}$$

Substituting explicit expressions for coordinates of the canonical IO (6) in the first of equations (8) for the particular case of electron-ion plasma ($\alpha = e, i$) †:

$$\int d\mathbf{v}\left(1 - (\mathbf{v}/c)^2\right)^{-\frac{5}{2}}\left[em^3\left(\eta^1(f) + 2A_4 f\right) + \bar{e}(\bar{m})^3\left(\overline{\eta^1}(\bar{f}) + 2A_4\bar{f}\right)\right] = 0\,. \tag{9}$$

As any other determining equation, the NDE (9) is an identity with respect to the group variables involved, namely distribution functions f and $\bar{f}$. In contrast to local DE, where splitting is achieved by differentiating with respect to group variables, in NDE one should use variational differentiation. Hence, applying variational derivatives $\delta/\delta f(\mathbf{v}')$ and $\delta/\delta\bar{f}(\mathbf{v}')$ to NDE (9), we obtain the following two ordinary differential equations:

$$\eta^1_f + 2A_4 = 0\,; \quad \overline{\eta^1}_{\bar{f}} + 2A_4 = 0\,.$$

†In what follows e, m, f denote the charge, mass and distribution function of electrons; the variables which refer to ions are marked with the bar (e.g., $\bar{e}$ is the charge of ions with the mass $\bar{m}$).

Integrating these relashionships gives solution of the first NDE (8):

$$\eta^1 = -2A_4 f + \frac{1}{em^3} A_5 \,; \quad \overline{\eta^1} = -2A_4 \bar{f} - \frac{1}{\bar{e}(\bar{m})^3} A_5 \,. \tag{10}$$

Solving the second NDE (8) yields the same results (10).

The obtained formulas (6) and (10) completely describe the continuous Lie point group of Vlasov-Maxwell equations [5]. It appears more convenient to present the operators of this group in a non-canonical form:

$$\begin{aligned}
&X_0 = \partial_t \,; \quad \mathbf{X} = \partial_{\mathbf{r}} \,; \\
&\mathbf{Y} = \mathbf{r}\partial_t + c^2 t \partial_{\mathbf{r}} + c^2 \partial_{\mathbf{v}} - \mathbf{v}\,(\mathbf{v},\, \partial_{\mathbf{v}}) - c\,[\mathbf{B},\, \partial_{\mathbf{E}}] + c\,[\mathbf{E},\, \partial_{\mathbf{B}}] + c^2 \rho \partial_{\mathbf{j}} + \mathbf{j}\partial_\rho \,; \\
&\mathbf{Z} = [\mathbf{r},\, \partial_{\mathbf{r}}] + [\mathbf{v},\, \partial_{\mathbf{v}}] + [\mathbf{E},\, \partial_{\mathbf{E}}] + [\mathbf{B},\, \partial_{\mathbf{B}}] + [\mathbf{j},\, \partial_{\mathbf{j}}] \,; \\
&X_4 = t\partial_t + \mathbf{r}\partial_{\mathbf{r}} - 2f\partial_f - 2\bar{f}\partial_{\bar{f}} - \mathbf{E}\partial_{\mathbf{E}} - \mathbf{B}\partial_{\mathbf{B}} - 2\mathbf{j}\partial_{\mathbf{j}} - 2\rho\partial_\rho \,; \\
&X_5 = \frac{1}{em^3}\partial_f - \frac{1}{\bar{e}(\bar{m})^3}\partial_{\bar{f}} \,.
\end{aligned} \tag{11}$$

The Poincaré group, which is obvious from physical considerations, corresponds to a subgroup represented here by ten (scalar) IO:

$$L_{10} =< X_0,\, \mathbf{X},\, \mathbf{Y},\, \mathbf{Z} > .$$

This subgroup is supplemented with IO corresponding to dilatations X_4 and translations X_5. If parameters of the theory (namely, e, $\bar{e}$, m, $\bar{m}$, c) are invariant, the continuous Lie group admitted by Vlasov-Maxwell equations (1)–(3) is a finite 12-parameter group. In general, the set of IO (11) is supplemented by four more dilatation operators [5], [7].

5 Prolongation of group generators on nonlocal variables

The use of the symmetry group obtained (11) and the special representation of the derivative in the canonical IO (7) enable to prolong the group operator on nonlocal variables [8] and to find transformation laws for various nonlocal variables [9] such as entropy, plasma temperature, etc. To illustrate the possibility of such prolongation, we rewrite the symmetry group IO of Vlasov-Maxwell equations in Fourier variables (ω, $\mathbf{k}$ representation).

The Fourier-transform of the distribution function of electrons is defined by a standard formula:

$$\hat{f}(\omega,\, \mathbf{k},\, \mathbf{v}) = \int dt\, d\mathbf{r} f(t,\, \mathbf{r},\, \mathbf{v}) \exp(i\omega t - i\mathbf{k}\mathbf{r}) \,. \tag{12}$$

Prolonging the canonical IO (4) on the Fourier-transform $\hat{f}$

$$\hat{Y} = Y + \hat{\chi}^1 \partial_{\hat{f}} \tag{13}$$

and applying the operator (13) to the formula (12), we obtain the following integral relationship between coordinates $\hat{\chi}^1$ and χ^1:

$$\hat{\chi}^1 = \int dt\, d\mathbf{r} \chi^1 \exp(i\omega t - i\mathbf{kr})\,. \qquad (14)$$

Substituting in (14) the coordinate of IO χ^1 given by (6), (10) and integrating by parts, we obtain the final result for the coordinate of IO (13) which defines the transformation of the Fourier-transform of the distribution function of electrons. In a similar way one can calculate coordinates of the canonical IO that define transformations of remaining functions in Vlasov-Maxwell equations. The point group thus obtained (in ω, $\mathbf{k}$ representation) is expressed as follows:

$$\begin{aligned}
&\hat{X}_0 = i\omega \left(\hat{f}\partial_{\hat{f}} + \hat{\bar{f}}\partial_{\hat{\bar{f}}} + \hat{\mathbf{E}}\partial_{\hat{\mathbf{E}}} + \hat{\mathbf{B}}\partial_{\hat{\mathbf{B}}} + \hat{\mathbf{j}}\partial_{\hat{\mathbf{j}}} + \hat{\rho}\partial_{\hat{\rho}} \right) ;\\
&\hat{\mathbf{X}} = -i\mathbf{k} \left(\hat{f}\partial_{\hat{f}} + \hat{\bar{f}}\partial_{\hat{\bar{f}}} + \hat{\mathbf{E}}\partial_{\hat{\mathbf{E}}} + \hat{\mathbf{B}}\partial_{\hat{\mathbf{B}}} + \hat{\mathbf{j}}\partial_{\hat{\mathbf{j}}} + \hat{\rho}\partial_{\hat{\rho}} \right) ;\\
&\hat{\mathbf{Y}} = c^2\mathbf{k}\partial_\omega + \omega\partial_{\mathbf{k}} + c^2\partial_{\mathbf{v}} - \mathbf{v}\,(\mathbf{v},\, \partial_{\mathbf{v}}) - c\left[\hat{\mathbf{B}},\, \partial_{\hat{\mathbf{E}}}\right] + c\left[\hat{\mathbf{E}},\, \partial_{\hat{\mathbf{B}}}\right] + c^2\hat{\rho}\partial_{\hat{\mathbf{j}}} + \hat{\mathbf{j}}\partial_{\hat{\rho}} \,;\\
&\hat{\mathbf{Z}} = [\mathbf{k},\, \partial_{\mathbf{k}}] + [\mathbf{v},\, \partial_{\mathbf{v}}] + \left[\hat{\mathbf{E}},\, \partial_{\hat{\mathbf{E}}}\right] + \left[\hat{\mathbf{B}},\, \partial_{\hat{\mathbf{B}}}\right] + \left[\hat{\mathbf{j}},\, \partial_{\hat{\mathbf{j}}}\right] ;\\
&\hat{X}_4 = -\omega\partial_\omega - \mathbf{k}\partial_{\mathbf{k}} + 2\hat{f}\partial_{\hat{f}} + 2\hat{\bar{f}}\partial_{\hat{\bar{f}}} + 3\hat{\mathbf{E}}\partial_{\hat{\mathbf{E}}} + 3\hat{\mathbf{B}}\partial_{\hat{\mathbf{B}}} + 2\hat{\mathbf{j}}\partial_{\hat{\mathbf{j}}} + 2\hat{\rho}\partial_{\hat{\rho}} \,;\\
&\hat{X}_5 = \delta(\omega)\,\delta(\mathbf{k}) \left[\frac{1}{em^3}\,\partial_{\hat{f}} - \frac{1}{\bar{e}(\bar{m})^3}\,\partial_{\hat{\bar{f}}} \right] .
\end{aligned} \qquad (15)$$

It should be noticed that the commutation table for IO (15) coincides with that for (11).

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