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Quasiclassical Solutions of the Schrödinger Equations as a Consequense of the Nonlinear Problem

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Abstract

Quasiclassic method of solving of the Schrödinger equation with quadratic Hamiltonian is used to derive solutions of Klein-Fock equation for the particle in the constant magnetic field and the jumping magnetic field.

Let us consider the relativistic wave equation

$$\left(D_\lambda^2 - k_0^2\right)\psi = 0,$$
$$k_0 = \frac{mc}{\hbar}, \tag{1}$$
$$D_\mu = \frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}A_\mu$$

with the electromagnetic field of the form

$$A_\mu = A_\mu^T = \frac{\mathbf{H}}{2}(-x_2, x_1, 0, 0),$$
$$\mathbf{H} = \text{const}. \tag{2}$$

It should be noted that the Maxwell equations let to use the magnetic field $\mathbf{H}$ depending on the variable $\xi = ct - z \quad (or\ \eta = ct + z)$. We provide the following investigation of Eq.(1) in the characteristics representation

$$\xi = ct - z,$$
$$\eta = ct + z, \tag{3}$$
$$\vec{x_\perp} = (x_1, x_2, 0)$$

which may be introduced in covariant form. We introduce the axial and transverse coordinates

$$x = x_1 + ix_2,$$
$$x^+ = x_1 - ix_2. \tag{4}$$

Thus the equation (1) has the form

$$\left(D_\perp^2 - 4\frac{\partial^2}{\partial\xi\partial\eta} - k_0^2\right)\psi = 0 \tag{5}$$

where

$$D_\perp^2 = 4\frac{\partial^2}{\partial x\partial x^+} + \frac{e\mathbf{H}}{2\hbar c}\left(x^+\frac{\partial}{\partial x^+} - x\frac{\partial}{\partial x}\right) - \frac{e^2\mathbf{H}^2}{16\hbar^2c^2}x^+x. \tag{6}$$

If the operator $D_\perp^2$ does not depend on the variables ξ, η we can apply to Eq.(5) the Fourier-Bessel transformation

$$\psi(x^T, \xi, \eta) = \int J_0(\sqrt{\xi\eta Q^2})\varphi(x, x^+, Q)QdQ \tag{7}$$

and to consider the Gorsat problem solutions [1]. If the operator $D_\perp^2$ depends on the characteristics variable (that is $\mathbf{H} = \mathbf{H}(\xi)$) then we apply to Eq.(5) the Fourier transformation in respect to the advanced relativistic time η [2]

$$\psi(x^T, \xi, \eta) = \frac{1}{2\pi\hbar}\int d\beta\ \exp\left(\frac{-i\beta}{\hbar}\eta\right)\varphi(x, x^+, \beta) \tag{8}$$

and then we obtain "nonrelativistic" equation for function φ

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial\xi} + \mathbf{H}\right)\varphi = 0,$$

$$\mathbf{H} = -\frac{\hbar^2}{4\beta}\left(D_\perp^2 - k_0^2\right). \tag{9}$$

To find the quasiclassical solutions of Eq.(8) we introduce

$$\varphi(x, x^+, \beta, \xi) = \phi(\xi, \beta)\exp\left(\frac{i}{\hbar}S(x, x^+, \xi, \beta)\right) \tag{10}$$

with the complex amplitude ϕ and action S. Two equations arise

$$\frac{\partial\phi}{\partial\xi} + \frac{1}{\beta}\cdot\frac{\partial^2 S}{\partial x\partial x^+}\phi = 0,$$

$$\frac{\partial S}{\partial\xi} + \frac{1}{\beta}\frac{\partial S}{\partial x}\frac{\partial S}{\partial x^+} - \frac{i\Omega}{2\beta}\left(x^+\frac{\partial S}{\partial x^+} - x\frac{\partial S}{\partial x}\right) \tag{11}$$

where $\Omega = \dfrac{eH}{\hbar c}$. The Hamiltonian (8) is quadratic and so it can be supposed that the action S has the form

$$S = A(\xi, \beta) + B(\xi, \beta)x + B^+(\xi, \beta)x^+ + C(\xi, \beta)x^+x \tag{12}$$

and then the problem reduces to the solution of the system of ordinary differential equations of first order

$$\frac{dA}{d\xi} + \frac{1}{\beta}B^+B + \frac{m^2c^2}{4\beta} = 0, \tag{13}$$

$$\frac{dB}{d\xi}+\frac{1}{\beta}\left(C+\frac{i\Omega}{2}\right)B=0, \qquad \frac{dB^{+}}{d\xi}+\frac{1}{\beta}\left(C-\frac{i\Omega}{2}B^{+}\right)=0, \tag{14}$$

$$\frac{dC}{d\xi}+\frac{1}{\beta}\left(C^{2}+\frac{\Omega^{2}}{4}\right)=0. \tag{15}$$

The nonlinear equation (15) is principal and we call the function C key-function. It may be shown that all other coefficients in the action S and amplitude ϕ can be expressed via the key-function C.

The Eq.(15) and its solutions have the following properties (Ω is a constant):

1. If C ia a solution then $\widetilde{C}=-\frac{\Omega^{2}}{4}\frac{1}{C}$ is a solution also.
2. A fuction of C and C^{+} is a solution $C(1+\Omega)=C(-\Omega)$.
3. There is pure imaginary solution if $C=C_{0}=\pm\frac{i\Omega}{2}$.

The solution of Eq.(15) can be found by linearization of Eq.(15) by means of applying Caley–Klein transformation of the complex function C

$$F=\frac{C-i\Omega/2}{C+i\Omega/2},$$

$$C=\frac{i\Omega}{2}\cdot\frac{1+F}{1-F} \tag{16}$$

where F must be the solution of the linear equation

$$\frac{dF}{d\xi}+\frac{i\Omega}{\beta}F=0 \tag{17}$$

with the periodical solutions. That gives the key-function

$$C(\xi)=\frac{i\Omega}{2}\cdot\frac{1+b\cdot\exp\left(\frac{-i\Omega}{\beta\xi}\right)}{1-b\cdot\exp\left(\frac{-i\Omega}{\beta\xi}\right)} \tag{18}$$

where b is a complex parameter of the constant of integration. It follows from Eq.(18) that the imaginary part of the key-function has a form

$$\operatorname{Im}C=\frac{\Omega(1-\rho^{2})}{2\left(1+\rho^{2}-2\rho\cos\left(\frac{\Omega\xi}{2\beta}-\vartheta\right)\right)} \tag{19}$$

where $\rho=b^{*}b$ and $\vartheta=\arctan\left(\frac{\operatorname{Im}\beta}{\operatorname{Re}\beta}\right)$. The importance of the parameter b is seen from the investigation of the behaviour of the function ImC. If $\rho^{2}<1$ and $\operatorname{Im}C>0$ then the states are normalizable. Passing to the region where $\rho^{2}>1$ it is seen that function $\operatorname{Im}C$ disappears if $\rho^{2}=1$ and then the sign changes of the Eq.(19).

There is the periodicity of key-function C in relativistic retarded time due to its interesting property $\left(T=\frac{2\pi\beta}{\Omega}\right)$

$$\langle C(\xi)\rangle=\frac{1}{T}\int_{\xi_{0}}^{\xi_{0}+T}d\xi^{'}C(\xi^{'})=\frac{i\Omega}{2}. \tag{20}$$

The value $C_0 = \dfrac{i\Omega}{2}$ corresponds to the stable coherent state we obtain some singular state when $b = 1$ and $\operatorname{Im} C = 0$, $\operatorname{Re} C = \dfrac{\Omega}{2} \cot \dfrac{\Omega\xi}{2\beta}$. This solution represents the fundamental solution of the problem and it may be used as the Green function. The region $\rho^2 < 1$ represents the pulsating states unknown in traditional theory. The problem to be developed in this approach is the behaviour of the system by the jumps of magnetic field H and the jumps of the value Ω. The continuity of solutions (φ or ψ) is guaranted by the continuity of the key-function C, and when the frequency Ω_1 changes at $\xi = \xi_0$ into Ω_2 the condition of continuity is

$$C(\Omega_1, \xi_0) = C(\Omega_2, \xi_0).$$

We remind that if $\Omega = 0$ then the solution for a free particle is

$$C(\xi) = \frac{C_0}{1 + C_0\xi/\beta}. \tag{21}$$

It two particular cases can be considered:

1. the case of the plane wave if $C_0 = 0$.
2. the case of the free Green function of the Schrödinger equation if $C_0 \to \infty$, then the key-function C is $C = \dfrac{\beta}{\xi}$ and

$$\varphi = \frac{\phi_0}{\xi} \exp\left(\frac{i\beta x_\perp^2}{\hbar\xi} - \frac{im^2c^2\xi}{4\hbar\beta}\right). \tag{22}$$

We emphasize that if the magnetic field $H \to 0$ then the state with $b = 1$ and $C = \dfrac{\beta}{\xi}$ is the Green function and there is a stable coherent state if $b = 0$. If the parameter $0 < b < 1$ then the stable coherent state with $C = C_0 = \dfrac{i\Omega}{2}$ become pulsatily.

Let before the point ξ_0 the frequency was Ω_0 and after ξ_0 become it Ω_1. Then the condition of continuity gives ($b_0 = 0$)

$$b_1 = \frac{\Omega_0 - \Omega_1}{\Omega_0 + \Omega_1} \exp\left(\frac{i\Omega_1\xi_0}{\beta}\right), \tag{23}$$

and

$$b_1^* b_1 = \left(\frac{\Omega_0 - \Omega_1}{\Omega_0 + \Omega_1}\right)^2 < 1.$$

The inverse process of transfer from pulsating state with given b_1 into stable coherent state gives

$$b_1 = \frac{\Omega_1 - \Omega_0}{\Omega_0 + \Omega_1} \exp\left(\frac{i\Omega_0\xi_0}{\beta}\right), \tag{24}$$

and $b_1^* b_1 < 1$ once again.

Making two such the jumps ($\xi_1 < \xi_0$) we obtain the condition of conservation of the stable coherent state

$$\exp\left(\frac{i\Omega_1(\xi_1 - \xi_0)}{\beta}\right) = \frac{\Omega_0 - \Omega_1}{\Omega_0 + \Omega_1} \cdot \frac{\Omega_1 + \Omega_2}{\Omega_2 - \Omega_1} \tag{25}$$

that gives the restriction on the time interval $\Delta\xi = \xi_1 - \xi_0$. This is possible at $\Omega_1 = \Omega_0$ only. In this case Eq.(25) has the solution

$$\Delta\xi_n = \frac{\pi\beta}{2\Omega_1}(2n+1). \tag{26}$$

After the rectangle impulse of amplitude of $\Omega_1 - \Omega_0$ and the duration of $\Delta\xi_n$ the system returns into the stable coherent state.

References

[1] Courant R., Partial Differential Equations, New York, London, 1962.

[2] Borghardt A., Karpenko D., On the class of the integral transformations in relativistic quantum theory, Kiev, 1975, Preprint /ITP-75-19E, 16p.