

Some Properties of Regular Crypto $H^{(l)}$ -abundant Semigroups

Lili Wang

School of Mathematics and Statistics, Chongqing
University of Technology
Chongqing, P. R. China
e-mail: wllaf@163.com

Aifa Wang

(Corresponding author)
School of Mathematics and Statistics,
Chongqing University of Technology
Chongqing, P. R. China
e-mail: wangaf1980@163.com

Abstract—A regular crypto $H^{(l)}$ -abundant semigroup is a generalized regular cryptogroup in the range of right principal projective semigroups. In our paper, we describe the property of regular crypto $H^{(l)}$ -abundant semigroups by the (l) -Green's relations.

Keywords- Abundant semigroups; $H^{(l)}$ -abundant semigroups; Completely regular semigroups

I. INTRODUCTION

We call a completely regular semigroup S a normal (regular) cryptogroup if H is a congruence and S/H is a normal (regular) band. The structure of normal cryptogroup has been described by Petrich and Reilly [7] in 1999. They proved that a completely regular semigroup is a normal cryptogroup if and only if it is a strong semilattice of completely simple semigroups.

In [4], Kong and Shum characterize the theorem of regular cyber groups, which generalize the characterization theorems on the completely regular semigroups to the super abundant semigroups. In our paper we further generalize the characterization theorems on the superabundant semigroups to the $H^{(l)}$ -abundant semigroups.

In our paper, we will define regular crypto $H^{(l)}$ -abundant semigroups and give some characteristics of regular crypto $H^{(l)}$ -abundant semigroups. In section 2, we give some concepts related with $H^{(l)}$ -abundant semigroups and (l) -Green's relations. In section 3, we will give the main result on $H^{(l)}$ -abundant semigroups.

II. PRELIMINARIES

Suppose that $T(S)$ is the set of all transformations on S . For every $f \in T(S)$, $\text{Ker}f$ denote the kernel of f and $\text{Im}f$ denote the image of f :

$$\text{Ker}f = \{(x, y) \in S^1 \times S^1 \mid f(x) = f(y)\}$$

$$\text{Im}f = \{f(x) \mid x \in S^1\}.$$

For any $a \in S$, put

$$a_r : x \rightarrow xa \quad (a_l : x \rightarrow ax).$$

In [6], the Green relations on S

$$aLb \Leftrightarrow \text{Im}a_r = \text{Im}b_r,$$

$$aRb \Leftrightarrow \text{Im}a_l = \text{Im}b_l,$$

$$D \Leftrightarrow L \cup R = L \circ R,$$

$$H \Leftrightarrow L \cap R,$$

$$\begin{aligned} aJb &\Leftrightarrow S^1 a S^1 = S^1 b S^1, \\ \text{be extended to } * \text{-Green's relations on } S \\ aL^*b &\Leftrightarrow \text{Ker}a_l = \text{Ker}b_l, \\ aR^*b &\Leftrightarrow \text{Ker}a_r = \text{Ker}b_r, \\ D^* &\Leftrightarrow L^* \cup R^* = L^* \circ R^*, \\ H^* &\Leftrightarrow L^* \cap R^*, \\ aJ^*b &\Leftrightarrow J^*(a) = J^*(b). \end{aligned}$$

where $J^*(a)$ saturated by L^* and R^* . It is easy to see that L^* is a right congruence on S and R^* is a left congruence on S .

It is easy to see that, $L^* \cup R^* \neq L^* \circ R^*$, in [6], Pastijn give the Green relations $D^{(l)}$; $L^{(l)}$; $R^{(l)}$ and $H^{(l)}$. They together with $J^{(l)}$ form a new Green's relations on S , we call them (l) -Green's relations

$$\begin{aligned} L^{(l)} &= L^*, \\ R^{(l)} &= R, \\ D^{(l)} &\Leftrightarrow L^{(l)} \cup R^{(l)}, \\ H^{(l)} &\Leftrightarrow L^{(l)} \cap R^{(l)}, \\ aJ^{(l)}b &\Leftrightarrow J^{(l)}(a) = J^{(l)}(b). \end{aligned}$$

Where $J^{(l)}(a)$ saturated by $L^{(l)}(a)$. Since $J^{(l)}(a)$ clearly is also saturated by R , we have $D^{(l)}(a)$ refines $J^{(l)}(a)$.

A semigroup S is an right principal projective semigroup, if all of its principal right ideals aS^1 are projective (see [2] and [3]). It was prove in [3] that a semigroup S is right principal projective, if and only if, for any $a \in S$, the set

$$M_a = \{e \in E(S) \mid S^1 \subseteq Se \text{ \& } (\forall x, y \in S^1) ax = ay \Rightarrow ex = ey\}$$

is nonempty. The concept of left principal projective semigroups is the left-right dual of right principal projective semigroups. A semigroup is said to be abundant [2], if it is both right principal projective and leftprincipal projective.

Obviously, $L \leq L^*$ [$R \leq R^*$]. It is obviously that for any $a, b \in \text{Re}g(S)$, if aL^*b [aR^*b], then we have aLb [aRb]. And all abundant semigroups are right principal projective semigroups.

Using the (l) -Green's relations, we can describe a completely regular semigroup S as a $H^{(l)} (= L^* \cap R)$ -abundant semigroup. And so $D^{(l)} = D$ is a semilattice

congruence, meanwhile, every $H^{(l)}$ -class is a group. S is called superabundant, if S is H^* -abundant [2]. Obviously, superabundant semigroups [$H^{(l)}$ -abundant semigroups] play a impotent role in the class of abundant semigroups.

By $*$ -Green's relations, we can study abundant semigroups, [super-abundant semigroups] such as [2] and [3]. A series of studies have shown that $*$ -Green relations are very important in the abundant semigroups. This paper will apply (l) -Green relations to study rpp semigroups.

Lemma 2.1.[8]

- (i) $L \subseteq L^{(l)}, D \subseteq D^{(l)}, J \subseteq J^{(l)}, H \subseteq H^{(l)}$.
- (ii) $L^{(l)} \subseteq L^*, D^{(l)} \subseteq D^*, J^{(l)} \subseteq J^*, H^{(l)} \subseteq H^*$.

Lemma 2.2. [8] If S be a semigroup. Then we have:

- (a). $D^{(l)} = L^* \circ R = R \circ L^{(l)}$.
- (b). S is a right principal projective semigroup if and only if its $D^{(l)}$ -class includes a regular D -class.
- (c). Each $D^{(l)}$ -class contains at most one regular D -class.

Lemma 2.3. [8] If a, b are two elements of semigroup S . Then b is in $J^{(l)}$ if and only if we have $a_0, a_1, \dots, a_n \in S$ with $a_0 = a$ and $a_n = b$, and $x_1, x_2, \dots, x_n; y_1, y_2, \dots, y_n \in S^1$, such that $a_i L^{(l)}(x_i a_{i-1} y_i), i = 1, 2, \dots, n$.

Lemma 2.4. [8] If S be a semigroup. Suppose, for any $a \in S, aJ^{(l)}a^2$, then $J^{(l)}$ is a semilattice congruence on S .

III. ON $H^{(l)}$ -ABUNDANT SEMIGROUPS

Theorem 3.1. Let S be an $H^{(l)}$ -abundant semigroup. Then

- (i) $H^{(l)}$ is a congruence if and only if for any $a, b \in S, (ab)^0 = (a^0 b^0)^0$.
- (ii) If $e, f \in S$ and $e D^{(l)} f$, then $e D f$.
- (iii) If e, f in S , e, f are idempotents and $e J f$, then $e D f$.

Proof: (i) (\Rightarrow). For every $a, b \in S$, it is obviously, $aH^{(l)}a^0$ and $bH^{(l)}b^0$. From $H^{(l)}$ is a congruence, thus $abH^{(l)}a^0 b^0$. Again $abH^{(l)}(ab)^0$, and so we have $(ab)^0 = (a^0 b^0)^0$.

(\Leftarrow) Let $(a, b) \in H^{(l)}$ and $c \in S$. It is obviously that $(ca)^0 = (c^0 a^0)^0 = (c^0 b^0)^0 = (cb)^0$. Thus $H^{(l)}$ is left compatible congruence on S . Similarly, $H^{(l)}$ is right compatible and thus $H^{(l)}$ is a congruence.

(ii) Since $e D^{(l)} f$, we have $a_1, \dots, a_k \in S$ such that $e L^{(l)} a_1 R a_2 \dots a_k L^{(l)} f$. From S is an $H^{(l)}$ -abundant semigroup, we have $e L a_1^0 R a_2^0 \dots a_k^0 L f$.

(iii) By $SeS = SfS$, we have $x, y, s, t \in S$ and $e = xfy$. Put $h = (fy)^0, k = (se)^0$. It follows that $fy = ffy$ and so $h = h^2 = fh$, and $sek = see$, we can see $k = k^2 = ke$. Thus hf, ek are the idempotents, $hfRh$ and $ekLk$. It follows that

$ehf = Re h$ and $ekfLk$. By $eh = xfyh = xfy = e$ and $kf = kset = set = f$, we get $eRe = fLf$. Thus eDf .

Definition 3.2. An $H^{(l)}$ -abundant semigroup S is crypto, if $H^{(l)} \in C(S)$, where we use $C(S)$ to denote all congruences on S .

Theorem 3.3. Let S be a crypto $H^{(l)}$ -abundant semigroup. Then $J^{(l)}$ is a semilattice congruence on S .

Proof: Since S be a crypto $H^{(l)}$ -abundant semigroup, for any $a \in S; aJ^{(l)}a^2$. Thus, from Lemma 2.4, $J^{(l)}$ is a semilattice congruence on S .

From the above Theorem, we can denote the $H^{(l)}$ -abundant semigroup $S = (Y; S_\alpha)$, where S_α is a $J^{(l)}$ -class of S .

Definition 3.4. If $J^{(l)}$ is the universal relation on S , then we call the $H^{(l)}$ -abundant semigroup S is completely $J^{(l)}$ -simple. An $H^{(l)}$ -abundant semigroup S is called a regular crypto $H^{(l)}$ -abundant semigroup, if S/H is a regular band.

Theorem 3.5. If S be a regular crypto $H^{(l)}$ -abundant semigroup. Then

- (i) For any $a \in S, J^{(l)}(a) = Sa^0S$.
- (ii) $J^{(l)} = D^{(l)}$.
- (iii) If the regular crypto $H^{(l)}$ -abundant semigroup S is completely $J^{(l)}$ -simple, then the idempotents of S are primitive.
- (iv) If the a regular crypto $H^{(l)}$ -abundant semigroup S is completely $J^{(l)}$ -simple, then the $Reg(S)$ is a regular subsemigroup.

Proof. (i) we can easy to see $a^0 \in J^{(l)}(a)$ and so $Sa^0S \subseteq J^{(l)}(a)$. Let $b = xa^0y \subseteq Sa^0S(x, y \in S)$ and $k = (a^0y)^0$. Then $a^0a^0y = a^0y$ and so $a^0(a^0y)^0 = k^2 = k$. Again $H^{(l)}$ is a congruence, we have $xa^0yH^{(l)}xk$. Put $h = (xk)^0 = (xa^0y)^0$. Thus $xkh = xk = xkk$ so that $h = h^2 = hk = ha^0k \in Sa^0S$. So we have $c \in L_b^*, d \in R_b^*$, then $c = ch, d = hd \in Sa^0S$ and so, Sa^0S saturated by L^* and R^* . Since $a = aa^0 \in Sa^0S$, we have $J^{(l)}(a) \subseteq Sa^0S$.

(ii) If $(a, b) \in S$ and $aJ^{(l)}b$. Then by (i), we have $Sa^0S = Sb^0S$. Now, by Theorem 3.1 (iii), a^0Db^0 and thus $aH^{(l)}a^0Db^0H^{(l)}b$. It follows that $aD^l b$ and hence $J^l \subseteq D^l$. Conversely, suppose that $a, b \in S$ with aDb . Thus by Lemma 2.2 (i), we have for some $c \in S, aL^*cRb$. This implies that $a^0Lc^0Rb^0$ and so $Sa^0S = Sc^0S = Sb^0S$. Hence, by (i), we have $(a, b) \in J^{(l)}$ and thus $D^{(l)} \subseteq J^{(l)}$. It follows that $D^{(l)} = J^{(l)}$.

(iii) If e, f be idempotents and $e \leq f$. From S is completely J^l -simple, $f \in SeS$. Hence by the Exercise 3 in [1][Sec.8.4], we have an idempotent g of S such that

$g \leq e$ and fDg . If $a \in S$ and $fLaRg$. We have fLa^0Rg and since gLf and $g \leq e$, then $f = fg = g$. However, from $g \leq e$, we have $e = f$ and thus, all idempotents of S are primitive.

(iv) Let a, b be regular elements of S . From S consists of a single $D^{(i)}$ -class, we have for some $c \in S$, aL^*cRb . So aL^*c^0Rb . This implies that aLc^0 and $c^0b = b$ by a is regular. Thus $abLb$ and hence ab is regular. Since b is regular.

REFERENCES

[1] Clifford A. H. and Preston, G. B., The algebraic theory of semigroups, Vol I & Vol II, Math. Surveys of the American Math. Soc,

Providence R.I., (1961) & (1967)

- [2] Fountain, J. B., Right PP monoid with central idempotents, Semigroup Forum, 13 (1977), 229-237.
- [3] Fountain, J. B., Abundant semigroups, Proc. London Math. Soc., V. 44, N. 3 (1982), 103-129.
- [4] Kong, X. Z. and Shum, K. P., Semilattice structure of regular cyber groups, Pragmatic Algebra, SAS Int. Publ., Delhi, 2006, 1-99.
- [5] Lawson, Mark V., Rees matrix semigroups, Proc. of the Edinburgh Math. Soc., 33 (1990), 23-39.
- [6] Pastijn, F., A representation of a semigroup by a semigroup of matrices over a group with zero, Semigroup Forum, V. 10 (1975), 238-249.
- [7] Petrich, M. and Reilly, N. R., Completely Regular Semigroups, New York: Wiley-Interscience Publication (1999).
- [8] Guo, X. J. Guo, Y. Q. and Shum, K. P., Super rpp semigroups. Indian J. Pure Appl. Math, 41(3) (2010), 505-533.