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Differential Operators, Symmetries and the Inverse Problem for Second-Order Differential Equations

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Abstract

With each second-order differential equation $\mathbf{Z}$ in the evolution space $J^1(M_{n+1})$ we associate, using the natural $f(3,-1)$-structure $\dot{S}$ and the $f(3,1)$-structure K, a group $\mathfrak{G}$ of automorphisms of the tangent bundle $T(J^1(M_{n+1}))$, with $\mathfrak{G}$ isomorphic to a dihedral group of order 8. Using the elements of $\mathfrak{G}$ and the Lie derivative, we introduce new differential operators on $J^1(M_{n+1})$ and new types of symmetries of $\mathbf{Z}$. We analyze the relations between the operators and the "dynamical" connection induced by $\mathbf{Z}$. Moreover, we analyze the relations between the various symmetries, also in connection with the inverse problem for $\mathbf{Z}$. Both the approach based on the Poincaré–Cartan two forms and the one relying on the introduction of the so-called metrics compatible with $\mathbf{Z}$ are explicitly worked out.

1 Introduction

The lagrangian approach to Analytical Mechanics and the geometric study of systems of second-order ordinary differential equations (briefly SODEs) were the objects of a renewed interest in the last 15 years, principally due to a deeper understanding of the geometry of the tangent bundle and to the introduction of jet-bundle structures.

The inverse problem for a given SODE (or equivalently for a mechanical system with a finite number of degrees of freedom), that is, the problem of finding when the SODE admits a (regular) lagrangian function L, is one of the most important problems in this wide subject, and it was recently studied, among the others, by Cantrijn, Carinena, Crampin, Hojman, Martinez, Marmo, Massa and Pagani, Santilli, Sarlet and many others. The principal result however was given by Helmholtz, and it is the so-called "Helmholtz conditions", that are sufficient conditions for a SODE to be lagrangian.

The presence of relations between the existence of symmetries of the SODE, the first integrals of the SODE and the inverse problem is well known and many authors, especially using the techniques of modern Differential Geometry, gave a contribution to this subject and to the evolution of the concept of symmetry.

This paper goes exactly in this direction. Using some special type (1,1) tensor fields defined over the first jet-bundle $J^1(M_{n+1})$ of a suitable fibre bundle $M_{n+1} \to I\!R$, tensor fields naturally determined by the SODE, we can introduce some new differential operators acting on the tensor algebra of $J^1(M_{n+1})$, and new kinds of symmetries for the SODE. The symmetries turn out to be distinguished in two different types ("vector type" symmetries and "form type" symmetries), and the whole context frames in a natural way some well known concepts, such as dynamical and adjoint symmetries. Moreover, we introduce four different subspaces of the module of vector fields over $J^1(M_{n+1})$ ("vector type" spaces) and four subspaces of the module of 1-forms over $J^1(M_{n+1})$ ("form type" spaces), that are the natural places where one have to seek symmetries.

We show that there are bijections between every pair of "vector type" spaces and between every pair of "form type" spaces, and moreover, that these bijections map also symmetries in symmetries, but there are no natural relations between any "vector type" space and any "form type" space. Moreover, we show that the existence of such a relation is strictly related to the two Helmholtz conditions that involve explicitly the SODE. More precisely, we prove that we have a bijection between "vector type" spaces and "form type" spaces if and only if one of the Helmholtz conditions involving the SODE hold. Moreover, we show that we have a correspondence between "vector type" spaces and "form type" spaces and between "vector type" symmetries and "form type" symmetries if and only if both the Helmholtz conditions involving the SODE hold.

The paper is divided into three parts.

In Sec.2, for the convenience of the reader and in order to fix notations, we introduce the geometric context and we present the principal results we shall use later, without any intention to be exhaustive.

In Sec.3 we introduce the differential operators $G_{\mathbf{Z}}$, the concepts of $G_{\mathbf{Z}}$ and $G^*_{\mathbf{Z}}$-symmetry, and the "vector type" and "form type" spaces where one can find the $G_{\mathbf{Z}}$-symmetries and the $G^*_{\mathbf{Z}}$-symmetry, respectively. We analyze the relations between spaces and symmetries of the same type and we present sufficient conditions for an element in these spaces to be a symmetry.

In Sec.4 we present the Helmholtz conditions for the SODE, and we analyze the relations between different types of spaces and symmetries in connection with the two Helmholtz conditions that involve explicitly the SODE. To this aim, we introduce two classes of important geometric objects: the Poincaré–Cartan–like 2-forms and the metrics compatible with the almost contact structure on $J^1(M_{n+1})$ given by the SODE. These two objects allow the explicit construction of the bijections between spaces and symmetries.

2 Preliminaries

In this section we recall the principal definitions and properties of a SODE and of the geometric context where the SODEs can be studied. For a more exhaustive discussion, see, e.g., [9], [4], [14], [3], [2], [13], [6], [7].

The natural geometric environment to study the SODEs, or to study the evolution of a mechanical system with a finite number of degrees of freedom, is the first jet-extension $J^1(M_{n+1})$ of an $(n+1)$-dimensional differentiable manifold M_{n+1} fibred over the real line

$\mathbb{R}$ by the "absolute time" function t, or, more precisely, the diagram

$$J^1(M_{n+1}) \xrightarrow{\pi} M_{n+1} \xrightarrow{t} \mathbb{R}.$$

This structure is rich of geometric objects powerful for the study of SODEs, such as the form dt, the contact forms $\omega^i \quad i = 1, \ldots, n$, the vertical subbundle $V(J^1(M_{n+1})) \subset T(J^1(M_{n+1}))$ of the vectors vertical with respect to the projection $\pi : J^2(M_{n+1}) \to J^1(M_{n+1})$, the "horizontal" subbundle $H^*(J^1(M_{n+1})) \subset T^*(J^1(M_{n+1}))$ of the virtual 1-forms over $J^1(M_{n+1})$ and the type $(1,1)$ tensor field S over $J^1(M_{n+1})$ that gives the vertical endomorphism of $J^1(M_{n+1})$.

We recall that, using fibred local coordinates $(t, q, \dot{q})$ on $J^1(M_{n+1})$, we have the representations

$$\omega^i \;=\; dq^i - \dot{q}^i dt, \quad i = 1, \ldots, n,$$

$$V \in V(J^1(M_{n+1})) \;\Leftrightarrow\; V = v^i \,\frac{\partial}{\partial \dot{q}^i},$$

$$\alpha \in H^*(J^1(M_{n+1})) \;\Leftrightarrow \alpha = a_i\, \omega^i,$$

$$S \;=\; \frac{\partial}{\partial \dot{q}^i} \otimes \omega^i.$$

In this context, a SODE can be represented by a vector field $\mathbf{Z}$ defined over $J^1(M_{n+1})$ obeying the conditions

$$\langle \mathbf{Z}, dt \rangle \;=\; 1; \qquad \langle \mathbf{Z}, \omega^i \rangle \;=\; 0, \quad i \;=\; 1, \ldots, n.$$

Using fibred coordinates, the vector field $\mathbf{Z}$ takes the form

$$\mathbf{Z} \;=\; \frac{\partial}{\partial t} \;+\; \dot{q}^i \frac{\partial}{\partial q^i} \;+\; Z^i \frac{\partial}{\partial \dot{q}^i},$$

and its integral curves are the first prolongations of the solutions of the system of differential equations

$$\ddot{q}^i \;=\; Z^i(t, q, \dot{q}).$$

The presence of a SODE over $J^1(M_{n+1})$, together with the vertical endomorphism S, allows the construction of a $f(3,-1)$-structure $\dot{S} = \mathcal{L}_{\mathbf{Z}} S$, i.e., a type $(1,1)$ tensor field over $J^1(M_{n+1})$ of rank $2n$ that satisfies the condition $\dot{S}^3 - \dot{S} = 0$ (see, e.g., [4]). Using local coordinates, the tensor $\dot{S}$ takes the form

$$\dot{S} \;=\; -\frac{\partial}{\partial q^i} \otimes \omega^i \;+\; \frac{\partial}{\partial \dot{q}^i} \otimes \left(dq^i - Z^i dt \;-\; \frac{\partial Z^i}{\partial \dot{q}^k} \omega^k \right).$$

Moreover it is easy to show that $\dot{S}$ is such that

$$\dot{S}^2 \;=\; I \;-\; \mathbf{Z} \otimes dt.$$

It is well known (see, e.g., [4], [3], [7]) that $\dot{S}$ induces a decomposition

$$T(J^1(M_{n+1})) = \mathcal{M}(J^1(M_{n+1})) \oplus \mathcal{H}(J^1(M_{n+1})) \oplus \mathcal{V}(J^1(M_{n+1})) \tag{1}$$

of the tangent bundle $T(J^1(M_{n+1}))$ given by the three eigenspaces $\mathcal{M}, \mathcal{V}, \mathcal{H}$ associated to the eigenvalues $0, \pm 1$ of the endomorphism $\dot{S}$. Since, for every point $p \in J^1(M_{n+1})$ we have that

$$\mathcal{M}_p = span\{\mathbf{Z}_p\};$$

$$\mathcal{V}_p = V_p(J^1(M_{n+1})),$$

the $f(3,-1)$–structure $\dot{S}$ induces in particular a "horizontal" distribution $\mathcal{H}(J^1(M_{n+1})) \subset T(J^1(M_{n+1}))$. In the same way $\dot{S}$ induces a decomposition of the cotangent bundle

$$T^*(J^1(M_{n+1})) = \mathcal{M}^*(J^1(M_{n+1})) \oplus \mathcal{H}^*(J^1(M_{n+1})) \oplus \mathcal{V}^*(J^1(M_{n+1})). \tag{2}$$

Similarly to the case of the tangent bundle, we have that

$$\mathcal{M}^*(J^1(M_{n+1})) = span\{dt\},$$

$$\mathcal{H}^*(J^1(M_{n+1})) = H^*(J^1(M_{n+1})).$$

It is then possible (see, e.g. [7]) to introduce two suitable local bases $\{\mathbf{Z}, D_i, V_i\}_{i=1,\ldots,n}$ of the module $\mathcal{X}(J^1(M_{n+1}))$ of vector fields over $J^1(M_{n+1})$ and $\{dt, \omega^i, \Omega^i\}_{i=1,\ldots,n}$ of the module $\mathcal{X}^*(J^1(M_{n+1}))$ of 1–forms over $J^1(M_{n+1})$, adapted to decompositions, and dual one of the other. In local coordinates they are given by

$$D_i = \frac{\partial}{\partial q^i} + \frac{1}{2}\frac{\partial Z^k}{\partial \dot{q}^i}\frac{\partial}{\partial \dot{q}^k}; \quad V_i = \frac{\partial}{\partial \dot{q}^i};$$

$$\Omega^i = d\dot{q}^i - Z^i dt - \frac{1}{2}\frac{\partial Z^i}{\partial \dot{q}^k}\omega^k.$$

In particular we have that

$$\begin{aligned}
&\mathcal{H}(J^1(M_{n+1})) = span\{D_i\};\\
&\mathcal{V}(J^1(M_{n+1})) = span\{V_i\} = V(J^1(M_{n+1}));\\
&\mathcal{H}^*(J^1(M_{n+1})) = span\{\omega^i\} = H^*(J^1(M_{n+1}));\\
&\mathcal{V}^*(J^1(M_{n+1})) = span\{\Omega^i\}.
\end{aligned}$$

For later use, we introduce also the "weakly" horizontal submodules

$$\begin{aligned}
&\mathcal{H}'(J^1(M_{n+1})) = span\{\mathbf{Z}, D_i\};\\
&\mathcal{H}'^*(J^1(M_{n+1})) = span\{dt, \omega^i\};
\end{aligned}$$

and the "weakly" vertical submodules

$$\begin{aligned}
&\mathcal{V}'(J^1(M_{n+1})) = span\{\mathbf{Z}, V_i\};\\
&\mathcal{V}'^*(J^1(M_{n+1})) = span\{dt, \Omega^i\}.
\end{aligned}$$

Note that, using these bases, the $f(3,-1)$-structure $\dot{S}$ assumes the very simple expression

$$\dot{S} = -D_i \otimes \omega^i + V_i \otimes \Omega^i.$$

It is also well known (see, e.g. [4], [1]) that a SODE $\mathbf{Z}$ determines an almost contact structure $(K, \mathbf{Z}, dt)$ over $J^1(M_{n+1})$ and, in particular, an $f(3,1)$-structure K, i.e., a type (1,1) tensor field over $J^1(M_{n+1})$ of rank $2n$ obeying the condition $K^3 + K = 0$. Using the natural bases introduced above, K takes the form

$$K = -D_i \otimes \Omega^i + V_i \otimes \omega^i$$

and it is then straightforward to verify that $K^2 = -I + \mathbf{Z} \otimes dt$.

The SODE $\mathbf{Z}$ determines, together with the decompositions (1), (2), a "dynamical" connection ∇ (see, e.g., [7]). For our purposes, the only relevant operator determined by the connection ∇ is the operator $\nabla_{\mathbf{Z}}$, whose action on the elements of the local bases $\{\mathbf{Z}, D_i, V_i\}$ and $\{dt, \omega^i, \Omega^i\}$ is the following:

$$\begin{array}{rclrcl} \nabla_{\mathbf{Z}}\mathbf{Z} &=& 0; & \nabla_{\mathbf{Z}}dt &=& 0; \\ \nabla_{\mathbf{Z}}D_i &=& -\tau_i^k D_k; & \nabla_{\mathbf{Z}}\omega^i &=& \tau_k^i \omega^k; \\ \nabla_{\mathbf{Z}}V_i &=& -\tau_i^k V_k; & \nabla_{\mathbf{Z}}\Omega^i &=& \tau_k^i \Omega^k \end{array} \tag{3}$$

where the coefficients

$$\tau_k^i = \frac{1}{2}\frac{\partial Z^i}{\partial \dot{q}^k} \tag{4}$$

are (part of) the connection coefficients of the dynamical connection.

To conclude the section, we recall (see, e.g. [5], [7]) that to each type (1,1) tensor field W we can associate a derivation ψ_W such that

- $\psi_W(f) = 0 \quad \forall f \in C^\infty(J^1(M_{n+1}))$;
- $\psi_W(X) = W(X) \quad \forall X \in \mathcal{X}(J^1(M_{n+1}))$;
- ψ_W commutes with contractions.

3 The symmetries and their spaces

The tensor fields $\dot{S}, K$ do not determine automorphisms of the tangent (or cotangent) bundle, since they are not of maximum rank. However it is possible to modify the two tensor fields, obtaining the tensor fields

$$A = \dot{S} + \mathbf{Z} \otimes dt = \mathbf{Z} \otimes dt - D_i \otimes \omega^i + V_i \otimes \Omega^i, \tag{5}$$

$$B = K + \mathbf{Z} \otimes dt = \mathbf{Z} \otimes dt - D_i \otimes \Omega^i + V_i \otimes \omega^i. \tag{6}$$

A straightforward computation shows that $A^2 = I = B^4$, so that both tensors are of maximum rank. Then they define automorphisms of the tangent bundle $T(J^1(M_{n+1}))$ and the cotangent bundle $T^*(J^1(M_{n+1}))$.

Later on, we often consider the type (1,1) tensor fields as operators acting on the tangent bundle $T(J^1(M_{n+1}))$ and on the cotangent bundle $T^*(J^1(M_{n+1}))$.

The operators A, B can be taken as the generators of a finite subgroup $\mathfrak{G}$ of the group $Aut(T(J^1(M_{n+1})))$ with the operation of composition. A straightforward calculation shows that $\mathfrak{G}$ has order 8, and that the elements of $\mathfrak{G}$, using the suitable bases introduced above, can be written as

$$\begin{array}{ll} I = \mathbf{Z} \otimes dt + D_i \otimes \omega^i + V_i \otimes \Omega^i, & E = \mathbf{Z} \otimes dt - D_i \otimes \omega^i - V_i \otimes \Omega^i, \\ A = \mathbf{Z} \otimes dt - D_i \otimes \omega^i + V_i \otimes \Omega^i, & F = \mathbf{Z} \otimes dt + D_i \otimes \omega^i - V_i \otimes \Omega^i, \\ B = \mathbf{Z} \otimes dt - D_i \otimes \Omega^i + V_i \otimes \omega^i, & M = \mathbf{Z} \otimes dt + D_i \otimes \Omega^i - V_i \otimes \omega^i, \\ C = \mathbf{Z} \otimes dt + D_i \otimes \Omega^i + V_i \otimes \omega^i, & N = \mathbf{Z} \otimes dt - D_i \otimes \Omega^i - V_i \otimes \omega^i. \end{array} \tag{7}$$

Later on we shall use the notation G to indicate the generic element of the group $\mathfrak{G}$. For later use, we present explicitly the composition table of the group $\mathfrak{G}$.

$2^{nd}\backslash 1^{st}$	I	A	B	C	E	F	M	N
A	A	I	N	M	F	E	C	B
B	B	C	E	F	M	N	I	A
C	C	B	A	I	N	M	F	E
E	E	F	M	N	I	A	B	C
F	F	E	C	B	A	I	N	M
M	M	N	I	A	B	C	E	F
N	N	M	F	E	C	B	A	I

(8)

Remark. The group $\mathfrak{G}$ is isomorphic to the diedral group D_8 of the rigid movement of the square.

Note that, if we focus our attention on the action of the elements of $\mathcal{G}$ on the subspaces $\mathcal{H}, \mathcal{V}, \mathcal{H}', \mathcal{V}'$ of $T(J^1(M_{n+1}))$ and on the subspaces $\mathcal{H}^*, \mathcal{V}^*, \mathcal{H}'^*, \mathcal{V}'^*$ of $T^*(J^1(M_{n+1}))$, then $\mathfrak{G}$ can be divided into two parts: the subgroup $\{I, A, E, F\}$ that maps horizontal (resp., vertical) objects into horizontal (resp., vertical) ones, and the subset $\{B, C, M, N\}$ whose elements change the "character" of the objects.

To each element $G \in \mathfrak{G}$ we can associate a differential operator $G_{\mathbf{Z}}$ acting on the tensor fields over $J^1(M_{n+1})$ determined by the conditions:

- $G_{\mathbf{Z}}(f) \;=\; \mathbf{Z}(f) \qquad \forall f \in C^\infty(J^1(M_{n+1}));$ (9)
- $G_{\mathbf{Z}}(X) \;=\; G^{-1}\left[\mathcal{L}_{\mathbf{Z}}\left(G(X)\right)\right] \qquad \forall X \in \mathcal{X}(J^1(M_{n+1}));$ (10)
- $G_{\mathbf{Z}}$ commutes with contraction, i.e. (11)
  $\forall X \in \mathcal{X}(J^1(M_{n+1})),\ \eta \in \mathcal{X}^*(J^1(M_{n+1}))$ we have
  $G_{\mathbf{Z}}\left(X \rfloor \eta\right) \;=\; \left(G_{\mathbf{Z}} X\right) \rfloor \eta \;+\; X \rfloor \left(G_{\mathbf{Z}} \eta\right).$

An easy calculation shows that $G_{\mathbf{Z}}$ is a derivation of degree 0 of the tensor algebra of $J^1(M_{n+1})$. Moreover, the explicit calculation of the differential operators $G_{\mathbf{Z}}$, when G varies in $\mathfrak{G}$, shows that to different elements of $\mathfrak{G}$ is associated the same differential operator. In particular, we have the following

Proposition 3.1 *For the operators $G_{\mathbf{Z}}$, $G \in \mathfrak{G}$, the following identities hold:*

$$I_{\mathbf{Z}} \;=\; E_{\mathbf{Z}} \;=\; \mathcal{L}_{\mathbf{Z}};$$

$$A_{\mathbf{Z}} \;=\; F_{\mathbf{Z}};$$

$$B_{\mathbf{Z}} \;=\; M_{\mathbf{Z}};$$

$$C_{\mathbf{Z}} \;=\; N_{\mathbf{Z}}.$$

Proof. Since the operators $I_{\mathbf{Z}}, E_{\mathbf{Z}}$ are both derivations, it is sufficient to show that the two actions are the same on functions and on vector fields. Of course, we have $I_{\mathbf{Z}}(f) = E_{\mathbf{Z}}(f) = \mathbf{Z}(f)\ \forall f \in C^\infty(J^1(M_{n+1}))$.

Let X be a vector field, $X = x^0\mathbf{Z} + x^i D_i + y^i V_i$. We have:

$$\mathcal{L}_{\mathbf{Z}}(X) \;=\; \mathbf{Z}\left(x^0\right)\mathbf{Z} + \left[\mathbf{Z}\left(x^i\right) - x^k\tau^i_k - y^i\right] D_i + \left[\mathbf{Z}\left(y^i\right) - y^k\tau^i_k + x^k Q^i_k\right] V_i, \qquad (12)$$

where the coefficients Q^i_k are such that:

$$Q^i_k = \mathbf{Z}\left(\tau^i_k\right) - \frac{\partial Z^i}{\partial q^k} - \tau^i_r \tau^r_k. \tag{13}$$

Then we have, using equations (7),

$$E(X) = x^0 \mathbf{Z} - x^i D_i - y^i V_i,$$

$$\mathcal{L}_{\mathbf{Z}}\left(E(X)\right) = \mathbf{Z}\left(x^0\right)\mathbf{Z} - \left[\mathbf{Z}\left(x^i\right) - x^k \tau^i_k - y^i\right] D_i - \left[\mathbf{Z}\left(y^i\right) - y^k \tau^i_k + x^k Q^i_k\right] V_i,$$

$$E^{-1}\left(\mathcal{L}_{\mathbf{Z}}\left(E(X)\right)\right) = \mathcal{L}_{\mathbf{Z}}(X).$$

The other identities can be proved in the same way. □

Definition 3.2 *Let* $\mathbf{Z}$ *be a SODE,* G *be an element of* $\mathfrak{G}$*,* $G_{\mathbf{Z}}$ *be the differential operator associated to* G*,* X *be a vector field and* α *be a 1-form over* $J^1(M_{n+1})$*. Then:*

- X *is a* $G_{\mathbf{Z}}$*-symmetry for the SODE* $\mathbf{Z}$ *iff* $G_{\mathbf{Z}}(X) = h\mathbf{Z}$, $h \in C^\infty(J^1(M_{n+1}))$*;*
- α *is a* $G^*_{\mathbf{Z}}$*-symmetry for the SODE* $\mathbf{Z}$ *iff* $G_{\mathbf{Z}}(\alpha) = hdt$, $h \in C^\infty(J^1(M_{n+1}))$*.*

Note that the definition (3.2) includes some well-known types of symmetries of a SODE: for example, the $I_{\mathbf{Z}}$-symmetries are the so-called dynamical symmetries, and the $A^*_{\mathbf{Z}}$-symmetries are the so-called adjoint symmetries (see, e.g. [8]).

For later use, we present explicitly the actions of the four operators on elements of the bases. We have the following expressions:

$$\begin{array}{lcllcl}
I_{\mathbf{Z}}(\mathbf{Z}) & = & 0; & I_{\mathbf{Z}}(dt) & = & 0; \\
I_{\mathbf{Z}}(D_i) & = & -\tau^k_i D_k + Q^k_i V_k; & I_{\mathbf{Z}}(\omega^i) & = & \tau^i_k \omega^k + \Omega^i; \\
I_{\mathbf{Z}}(V_i) & = & -D_i - \tau^k_i V_k; & I_{\mathbf{Z}}(\Omega^i) & = & -Q^i_k \omega^k + \tau^i_k \Omega^k; \\
 & & & & & \\
A_{\mathbf{Z}}(\mathbf{Z}) & = & 0; & A_{\mathbf{Z}}(dt) & = & 0; \\
A_{\mathbf{Z}}(D_i) & = & -\tau^k_i D_k - Q^k_i V_k; & A_{\mathbf{Z}}(\omega^i) & = & \tau^i_k \omega^k - \Omega^i; \\
A_{\mathbf{Z}}(V_i) & = & D_i - \tau^k_i V_k; & A_{\mathbf{Z}}(\Omega^i) & = & +Q^i_k \omega^k + \tau^i_k \Omega^k; \\
 & & & & & \\
B_{\mathbf{Z}}(\mathbf{Z}) & = & 0; & B_{\mathbf{Z}}(dt) & = & 0; \\
B_{\mathbf{Z}}(D_i) & = & -\tau^k_i D_k + V_i; & B_{\mathbf{Z}}(\omega^i) & = & \tau^i_k \omega^k + Q^i_k \Omega^k; \\
B_{\mathbf{Z}}(V_i) & = & -Q^k_i D_k - \tau^k_i V_k; & B_{\mathbf{Z}}(\Omega^i) & = & -\omega^i + \tau^i_k \Omega^k; \\
 & & & & & \\
C_{\mathbf{Z}}(\mathbf{Z}) & = & 0; & C_{\mathbf{Z}}(dt) & = & 0; \\
C_{\mathbf{Z}}(D_i) & = & -\tau^k_i D_k - V_i; & C_{\mathbf{Z}}(\omega^i) & = & \tau^i_k \omega^k - Q^i_k \Omega^k; \\
C_{\mathbf{Z}}(V_i) & = & +Q^k_i D_k - \tau^k_i V_k; & C_{\mathbf{Z}}(\Omega^i) & = & \omega^i + \tau^i_k \Omega^k.
\end{array} \tag{14}$$

The first result about the $G_{\mathbf{Z}}$-symmetries is the following:

Theorem 3.3 *Let* $\mathbf{Z}$ *be a SODE,* X *be a vector field,* α *be a 1-form. Then:*

- X *is a* $I_{\mathbf{Z}}$*-symmetry* $\Longrightarrow$ $X = A_{\mathbf{Z}}(V)$ *for some* $V \in \mathcal{V}'$*;*
- X *is a* $A_{\mathbf{Z}}$*-symmetry* $\Longrightarrow$ $X = I_{\mathbf{Z}}(V)$ *for some* $V \in \mathcal{V}'$*;*

- *X is a $B_{\mathbf{Z}}$-symmetry* $\Longrightarrow$ $X = C_{\mathbf{Z}}(H)$ *for some* $H \in \mathcal{H}'$*;*
- *X is a $C_{\mathbf{Z}}$-symmetry* $\Longrightarrow$ $X = B_{\mathbf{Z}}(H)$ *for some* $H \in \mathcal{H}'$*;*
- *α is a $I^*_{\mathbf{Z}}$-symmetry* $\Longrightarrow$ $\alpha = A_{\mathbf{Z}}(\beta)$ *for some* $\beta \in (\mathcal{H}')^*$*;*
- *α is a $A^*_{\mathbf{Z}}$-symmetry* $\Longrightarrow$ $\alpha = I_{\mathbf{Z}}(\beta)$ *for some* $\beta \in (\mathcal{H}')^*$*;*
- *α is a $B^*_{\mathbf{Z}}$-symmetry* $\Longrightarrow$ $\alpha = C_{\mathbf{Z}}(\gamma)$ *for some* $\gamma \in (\mathcal{V}')^*$*;*
- *α is a $C^*_{\mathbf{Z}}$-symmetry* $\Longrightarrow$ $\alpha = B_{\mathbf{Z}}(\gamma)$ *for some* $\gamma \in (\mathcal{V}')^*$*.*

Proof. If $V = x^0\mathbf{Z} + x^i V_i$ is a vertical vector field, we have, using equations (14):

$$A_{\mathbf{Z}}(V) \;=\; \mathbf{Z}(x^0)\mathbf{Z} + x^i D_i + \left[\mathbf{Z}(x^i) - \tau^i_k x^k\right] V_k.$$

Comparing with equation (12), we have the first statement. The other statements can be obtained in the same way. □

Taking into account the result of the theorem, it is very natural to introduce the spaces of vector fields and 1–forms that obey the necessary condition to be $G_{\mathbf{Z}}$ or $G^*_{\mathbf{Z}}$–symmetries. Then we have the following:

Definition 3.4 *Let* $\mathbf{Z}$ *be a SODE,* $G_{\mathbf{Z}}$ *be the operators defined above. Then we define the subspaces "of vector type" of the module* $\mathcal{X}(J^1(M_{n+1}))$ *of vector fields over* $J^1(M_{n+1})$

$$\begin{aligned}
\mathcal{I}_{\mathbf{Z}}(J^1(M_{n+1})) &= \{X | X = A_{\mathbf{Z}}(V),\, V \in \mathcal{V}'\},\\
\mathcal{A}_{\mathbf{Z}}(J^1(M_{n+1})) &= \{X | X = I_{\mathbf{Z}}(V),\, V \in \mathcal{V}'\},\\
\mathcal{B}_{\mathbf{Z}}(J^1(M_{n+1})) &= \{X | X = C_{\mathbf{Z}}(H),\, H \in \mathcal{H}'\},\\
\mathcal{C}_{\mathbf{Z}}(J^1(M_{n+1})) &= \{X | X = B_{\mathbf{Z}}(H),\, H \in \mathcal{H}'\}.
\end{aligned}$$

Analogously, we define the subspaces "of form type" of the module $\mathcal{X}^*(J^1(M_{n+1}))$ *of 1-forms over* $J^1(M_{n+1})$

$$\begin{aligned}
\mathcal{I}^*_{\mathbf{Z}}(J^1(M_{n+1})) &= \{\alpha | \alpha = A_{\mathbf{Z}}(\beta),\, \beta \in (\mathcal{H}')^*\},\\
\mathcal{A}^*_{\mathbf{Z}}(J^1(M_{n+1})) &= \{\alpha | \alpha = I_{\mathbf{Z}}(\beta),\, \beta \in (\mathcal{H}')^*\},\\
\mathcal{B}^*_{\mathbf{Z}}(J^1(M_{n+1})) &= \{\alpha | \alpha = C_{\mathbf{Z}}(\gamma),\, \gamma \in (\mathcal{V}')^*\},\\
\mathcal{C}^*_{\mathbf{Z}}(J^1(M_{n+1})) &= \{\alpha | \alpha = B_{\mathbf{Z}}(\gamma),\, \gamma \in (\mathcal{V}')^*\}.
\end{aligned}$$

Later on we shall use the notation $\mathcal{G}_{\mathbf{Z}}(J^1(M_{n+1}))$ for the generic "vector type" subspace and the notation $\mathcal{G}^*_{\mathbf{Z}}(J^1(M_{n+1}))$ for the generic "form type" subspace. For later use, we present explicitly the conditions for a vector field X to be an element of the spaces $\mathcal{G}_{\mathbf{Z}}(J^1(M_{n+1}))$. If $X = x^0\mathbf{Z} + x^i D_i + y^i V_i$, then we have:

$$\begin{array}{lcccl} X \in \mathcal{I}_{\mathbf{Z}}(J^1(M_{n+1})) & \Longleftrightarrow & y^i & = & \mathbf{Z}(x^i) - \tau^i_k x^k, \\ X \in \mathcal{A}_{\mathbf{Z}}(J^1(M_{n+1})) & \Longleftrightarrow & y^i & = & -\mathbf{Z}(x^i) + \tau^i_k x^k, \\ X \in \mathcal{B}_{\mathbf{Z}}(J^1(M_{n+1})) & \Longleftrightarrow & x^i & = & -\mathbf{Z}(y^i) + \tau^i_k y^k, \\ X \in \mathcal{C}_{\mathbf{Z}}(J^1(M_{n+1})) & \Longleftrightarrow & x^i & = & \mathbf{Z}(y^i) - \tau^i_k y^k. \end{array} \tag{15}$$

Moreover, if $\alpha = a_0 dt + a_i\omega^i + b_i\Omega^i \in \mathcal{G}^*_{\mathbf{Z}}(J^1(M_{n+1}))$, then α obeys the following conditions:

$$\begin{array}{lcccl} \alpha \in \mathcal{I}^*_{\mathbf{Z}}(J^1(M_{n+1})) & \Longleftrightarrow & a_i & = & -\mathbf{Z}(b_i) - \tau^k_i b_k, \\ \alpha \in \mathcal{A}^*_{\mathbf{Z}}(J^1(M_{n+1})) & \Longleftrightarrow & a_i & = & \mathbf{Z}(b_i) + \tau^k_i b_k, \\ \alpha \in \mathcal{B}^*_{\mathbf{Z}}(J^1(M_{n+1})) & \Longleftrightarrow & b_i & = & \mathbf{Z}(a_i) + \tau^k_i a_k, \\ \alpha \in \mathcal{C}^*_{\mathbf{Z}}(J^1(M_{n+1})) & \Longleftrightarrow & b_i & = & -\mathbf{Z}(a_i) - \tau^k_i a_k. \end{array} \tag{16}$$

It is important to note that the conditions (15), (16) express the n vertical components of the object as functions of the horizontal ones, or vice versa. Then every vector field in the space $\mathcal{G}_{\mathbf{Z}}(J^1(M_{n+1}))$ and every 1-form of $\mathcal{G}^*_{\mathbf{Z}}(J^1(M_{n+1}))$ have only $n+1$ free components. Moreover, from the condition $\mathcal{L}_{\mathbf{Z}}\mathbf{Z} = \mathcal{L}_{\mathbf{Z}}dt = 0$, it is easy to see that the component along $\mathbf{Z}$ or, respectively, along dt takes no role in the definitions (3.2) and (3.4) (we shall return on this fact in section 4). Then the conditions (15), (16) leave to the object only n meaningful components.

The first result about the spaces $\mathcal{G}_{\mathbf{Z}}(J^1(M_{n+1}))$ and $\mathcal{G}^*_{\mathbf{Z}}(J^1(M_{n+1}))$ is the following:

Theorem 3.5 *For every pair of subspaces "of vector type", there exists an element of the group $\mathfrak{G}$ that gives a bijection between the subspaces.*

Analogously, for every pair of subspaces "of form type", there exists a element of the group $\mathfrak{G}$ that gives a bijection between the subspaces.

Proof. Taking into account equations (15), (16), let $x^i, i = 0, \ldots, n$ be $n+1$ functions defined over $J^1(M_{n+1})$. We can construct, for example,

$$X = x^0\mathbf{Z} + x^i D_i + \left[-\mathbf{Z}(x^i) + \tau^i_k x^k\right] V_i \in A_{\mathbf{Z}};$$

$$Y = x^0\mathbf{Z} + \left[\mathbf{Z}(x^i) - \tau^i_k x^k\right] D_i + x^i V_i \in C_{\mathbf{Z}}$$

that are the generic elements of $A_{\mathbf{Z}}$ and $C_{\mathbf{Z}}$, respectively. It is easy to see that, to map X in Y, we need an element of $\mathfrak{G}$ such that

$$D_i \rightsquigarrow V_i \qquad V_i \rightsquigarrow -D_i.$$

A straightforward comparision with the table (8) shows that B is the suitable element of $\mathfrak{G}$. Every other bijection can be found with the same procedure. $\square$

The result of the theorem (3.5) allows the construction of the following two diagrams:

$$\begin{array}{ccc} \mathcal{I}_{\mathbf{Z}}(J^1(M_{n+1})) & & \mathcal{I}^*_{\mathbf{Z}}(J^1(M_{n+1})) \\ \updownarrow & & \updownarrow \\ \mathcal{A}_{\mathbf{Z}}(J^1(M_{n+1})) & & \mathcal{A}^*_{\mathbf{Z}}(J^1(M_{n+1})) \\ \updownarrow & & \updownarrow \\ \mathcal{B}_{\mathbf{Z}}(J^1(M_{n+1})) & & \mathcal{B}^*_{\mathbf{Z}}(J^1(M_{n+1})) \\ \updownarrow & & \updownarrow \\ \mathcal{C}_{\mathbf{Z}}(J^1(M_{n+1})) & & \mathcal{C}^*_{\mathbf{Z}}(J^1(M_{n+1})) \end{array} \tag{17}$$

It is well-known (see, e.g., [10], [11], [8]) that the existence of a "horizontal" relation between the diagrams (or, more generally, the existence of arrows between the left and right diagrams) is strictly related to the inverse problem for the SODE **Z**. We shall return on the relations between the diagrams of symmetries of a SODE and the inverse problem for the SODE in the next section.

The second result of this section is the following:

Theorem 3.6 *A vector field $X \in \mathcal{G}_{\mathbf{Z}}(J^1(M_{n+1}))$ is a $G_{\mathbf{Z}}$-symmetry for the SODE* **Z** *if and only if the "free" components $u^i, i = 1, \ldots, n$ of X obey the following second-order equation:*

$$\mathbf{Z}\left(\mathbf{Z}\left(u^i\right)\right) - 2\,\tau^i_k\,\mathbf{Z}\left(u^k\right) - u^k\,\frac{\partial Z^i}{\partial q^k} = 0. \tag{18}$$

A 1-form $\alpha \in \mathcal{G}^*_{\mathbf{Z}}(J^1(M_{n+1}))$ is a $G^*_{\mathbf{Z}}$-symmetry for the SODE **Z** *if and only if the "free" components $c_i, i = 1, \ldots, n$, of α obey the following second-order equation:*

$$\mathbf{Z}\left(\mathbf{Z}\left(c_i\right)\right) + 2\,\mathbf{Z}\left(\tau^k_i c_k\right) - c_k\,\frac{\partial Z^k}{\partial q^i} = 0. \tag{19}$$

Proof. Let $X = x^0\mathbf{Z} + x^i D_i + y^i V_i$ be an $I_{\mathbf{Z}}$–symmetry for **Z**. Then, using equation (12), we have the conditions:

$$\left\{\begin{array}{l} y^i - \mathbf{Z}(x^i) + \tau^i_k x^k = 0, \\ \left[\mathbf{Z}\left(y^i\right) - y^k\tau^i_k + x^k Q^i_k\right] = 0. \end{array}\right. \tag{20}$$

Taking into account equation (13), a straightforward calculation shows that the system is equivalent to one given by the first equation of (20) and the equation (18). So we have the first statement for the operator $I_{\mathbf{Z}}$ acting on vector fields. For the other operators acting on vector fields, the proofs are analogous and are left to the reader.

For the operators acting on 1-forms, let us consider, for example,

$$\begin{aligned} B_{\mathbf{Z}}(\alpha) = B_{\mathbf{Z}}\left(a_o dt + a_i\omega^i b_i + \Omega^i\right) = \\ \mathbf{Z}(a_0)dt + \left[\mathbf{Z}(a_i) + a_k\tau^k_i - b_i\right]\omega^i + \left[\mathbf{Z}(b_i) + b_k\tau^k_i + a_k Q^k_i\right]\Omega^i. \end{aligned}$$

Then the condition $B_{\mathbf{Z}}(\alpha) = hdt$ is equivalent to the system

$$\left\{\begin{array}{l} b_i - \mathbf{Z}(a_i) - \tau^k_i a_k = 0, \\ \left[\mathbf{Z}\left(b_i\right) + a_k\tau^k_i + b_k Q^k_i\right] = 0. \end{array}\right. \tag{21}$$

Once again, taking into account equation (13), a straightforward calculation shows that the system is equivalent to one given by the first equation of (21) and the equation (19). For the other operators acting on 1-forms, the proofs are analogous, and are left to the reader. □

Remark. Equations (18), (19) are well–known in literature. Moreover, equation (19) is known as the "adjoint equation" of equation (18) (see, e.g. [10], [12], [8]).

Corollary 3.7 *The bijections of theorem (3.5) map also symmetries in symmetries.*

Proof. The proof is straightforward and is left to the reader. □

As a final comment, we point out that, in some sense, theorem (3.6) is a negative result. In fact, if, on the one side, the result has an intrinsic elegance, since we have that the condition to obtain a "vector type" symmetry from an element of the space $\mathcal{G}_{\mathbf{Z}}(J^1(M_{n+1}))$ is independent of the space (and analogously for "form type" symmetries), on the other side, the theorem says also that, if we want to **construct explicitly** a "vector type" symmetry, we have **in any case** to solve equation (18) (or, respectively, equation (19)), independently of the space where we decide to work.

4 The symmetries and the inverse problem

We have already recalled that the existence of relations connecting the two diagrams of (17) is strictly related to the inverse problem for the SODE $\mathbf{Z}$, i.e., to the problem of estabilishing when a given SODE can be obtained by a Lagrangian function.

The most classical result about the inverse problem was given by Helmholtz, with the so-called Helmholtz conditions:

Theorem 4.1 A SODE $\mathbf{Z}$ is lagrangian if there exists a matrix g_{ij} of functions defined over $J^1(M_{n+1})$ such that:

A1) g_{ij} is nonsingular;

A2) g_{ij} is symmetric;

D1) $\mathbf{Z}\,(g_{ij}) + g_{ik}\tau^k_j + g_{jk}\tau^k_i \;=\; 0;$

D2) $g_{ik}Q^k_j = g_{jk}Q^k_i;$

C1) $\dfrac{\partial g_{ik}}{\partial \dot{q}^j} \;=\; \dfrac{\partial g_{ij}}{\partial \dot{q}^k}.$

We point out that the Helmholtz conditions can be naturally divided into three different types: the algebraic conditions (conditions A1 and A2), the differential conditions (conditions D1 and D2), that are the conditions that explicitly involve the SODE $\mathbf{Z}$, and a "closure" condition (C1).

Several geometric formulations of the Helmoltz conditions can be found in the literature (see, among the others, [3], [7]). For our purposes it is remarkable that these geometric formulations are based essentially on the existence of a 2-form ω of the form

$$\omega \;=\; g_{ij}\,\omega^i \wedge \Omega^j \tag{22}$$

with the properties g_{ij} non–singular and symmetric. A 2–form of type (22) will be called a Poincaré–Cartan–like form, since, as it is immediate to see, when a regular Lagrangian function L for $\mathbf{Z}$ is known, the Poincaré–Cartan form determined by L is exactly of type (22), with $g_{ij} = \dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$.

A Poincaré–Cartan-like form gives a natural correspondence between vector fields and 1-form, simply considering the interior product

$$X \ \rightsquigarrow X \lrcorner \omega. \tag{23}$$

Note that, since ω has rank $2n$, the map (23) is not a bijection. To avoid this inconvenience, it is a standard procedure to introduce the following equivalence relation:

Definition 4.2 *Let X, Y be vector fields defined over $J^1(M_{n+1})$, α, β be 1–forms defined over $J^1(M_{n+1})$. We say that:*

- *X is equivalent to Y iff $X - Y = h\mathbf{Z}$ for some $h \in C^\infty(J^1(M_{n+1}))$;*
- *α is equivalent to β iff $\alpha - \beta = hdt$ for some $h \in C^\infty(J^1(M_{n+1}))$.*

It is immediate to see that, since the kernel of the map (23) is exactly the space $span\{\mathbf{Z}\}$, then (23) gives a bijection up to equivalence classes.

Moreover, following [1], we can consider, together with the Poincaré–Cartan–like form ω, a Riemannian metric Φ defined over $J^1(M_{n+1})$ and given by:

$$\Phi \ = \ dt \otimes dt + g_{ij}\omega^i \otimes \omega^j + g_{ij}\Omega^i \otimes \Omega^j. \tag{24}$$

The metric Φ turns out to be compatible with the almost contact structure over $J^1(M_{n+1})$ given by $(K, \mathbf{Z}, dt)$, and Φ gives also another natural bijection between vector fields and 1-forms

$$X \ \rightsquigarrow \ \Phi(X). \tag{25}$$

Using the Poincaré–Cartan-like form ω and the compatible metric Φ, we can state the first result about the diagram (17).

Theorem 4.3 *Let g_{ij} be a matrix of functions defined over $J^1(M_{n+1})$ obeying the algebraic Helmholtz conditions A1 and A2, let ω be the Poincaré–Cartan-like form determined by g_{ij} and let Φ be the compatible metric determined by g_{ij}. Then the following conditions are equivalent:*

1) g_{ij} obeys the first differential Helmholtz condition D1;

2) $\nabla_{\mathbf{Z}}\omega \ = \ 0$;

3) $\nabla_{\mathbf{Z}}\Phi \ = \ 0$;

4) $\lrcorner\, \omega : \mathcal{I}_{\mathbf{Z}}(J^1(M_{n+1})) \rightarrow \mathcal{I}^*_{\mathbf{Z}}(J^1(M_{n+1}))$ is a bijection up to equivalence classes;

5) $\Phi : \mathcal{I}_{\mathbf{Z}}(J^1(M_{n+1})) \rightarrow \mathcal{B}^*_{\mathbf{Z}}(J^1(M_{n+1}))$ is a bijection.

Proof. Taking into account equations (3), we have that:

$$\nabla_{\mathbf{Z}}\omega = \nabla_{\mathbf{Z}}\left(g_{ij}\omega^i\wedge\Omega^j\right) = \mathbf{Z}(g_{ij})\omega^i\wedge\Omega^j + g_{ij}\left(\tau^i_k\omega^k\right)\wedge\Omega^j +$$
$$g_{ij}\omega^i\wedge\left(\tau^j_k\Omega^k\right) = \left(\mathbf{Z}(g_{ij}) + g_{kj}\tau^k_i + g_{ik}\tau^k_j\right)\omega^i\wedge\Omega^j.$$

Moreover we have:

$$\begin{aligned}\nabla_{\mathbf{Z}}\Phi &= \nabla_{\mathbf{Z}}\left(dt\otimes dt + g_{ij}\omega^i\otimes\omega^j + g_{ij}\Omega^i\otimes\Omega^j\right) = \\ &\mathbf{Z}(g_{ij})\omega^i\otimes\omega^j + g_{ij}\left(\tau^i_k\omega^k\right)\otimes\omega^j + g_{ij}\omega^i\otimes\left(\tau^j_k\omega^k\right) + \\ &g_{ij}\left(\tau^i_k\Omega^k\right)\otimes\Omega^j + g_{ij}\Omega^i\otimes\left(\tau^j_k\Omega^k\right) = \\ &\left(\mathbf{Z}(g_{ij}) + g_{kj}\tau^k_i + g_{ik}\tau^k_j\right)\omega^i\otimes\omega^j + \left(\mathbf{Z}(g_{ij}) + g_{kj}\tau^k_i + g_{ik}\tau^k_j\right)\Omega^i\otimes\Omega^j.\end{aligned}$$

The equivalence of conditions 1), 2), 3) is now evident. Moreover, let X be an element of $\mathcal{I}_{\mathbf{Z}}(J^1(M_{n+1}))$, $X = x^0\mathbf{Z} + x^iD_i + \left(\mathbf{Z}(x^i) - \tau^i_kx^k\right)V_i$. Then we have:

$$X\rfloor\omega = -g_{ij}\left(\mathbf{Z}(x^j) - \tau^j_kx^k\right)\omega^i + g_{ij}x^j\Omega^i. \tag{26}$$

Then, taking into account equations (16), we have that:

$$\begin{aligned}X\rfloor\omega\in\mathcal{I}^*_{\mathbf{Z}}(J^1(M_{n+1})) \quad&\Leftrightarrow\quad -g_{ij}\left(\mathbf{Z}(x^i) - \tau^i_kx^k\right) = -\mathbf{Z}\left(g_{ij}x^j\right) - \tau^k_ig_{kj}x^j \\ &\Leftrightarrow\quad -x^j\left(\mathbf{Z}\left(g_{ij}\right) + g_{kj}\tau^k_i + g_{ik}\tau^k_j\right) = 0.\end{aligned}$$

For the converse, we have to consider a 1-form $\alpha = \left(-\mathbf{Z}(b_i) - \tau^k_ib_k\right)\omega^i + b_i\Omega^i \in \mathcal{I}^*_{\mathbf{Z}}(J^1(M_{n+1}))$, and the unique vector field $X = x^iD_i + y^iV_i$ such that $X\rfloor\omega = \alpha$. Then a straightforward calculation shows that $X\in\mathcal{I}_{\mathbf{Z}}(J^1(M_{n+1}))$ if and only if the condition D1) holds.

Then we have the equivalence of condition 4) with the previous 1), 2), 3).

Moreover, taking $X = x^0\mathbf{Z} + x^iD_i + \left(\mathbf{Z}(x^i) - \tau^i_kx^k\right)V_i$, we have that:

$$\Phi(X) = x_0dt + g_{ij}x^j\omega^i + g_{ij}\left(\mathbf{Z}(x^j) - \tau^j_kx^k\right)\Omega^i. \tag{27}$$

Then, taking once again into account equations (16), we obtain:

$$\begin{aligned}\Phi(X)\in\mathcal{B}^*_{\mathbf{Z}}(J^1(M_{n+1})) \quad&\Leftrightarrow\quad g_{ij}\left(\mathbf{Z}(x^i) - \tau^i_kx^k\right) = \mathbf{Z}\left(g_{ij}x^j\right) + \tau^k_ig_{kj}x^j \\ &\Leftrightarrow\quad x^j\left(\mathbf{Z}\left(g_{ij}\right) + g_{kj}\tau^k_i + g_{ik}\tau^k_j\right) = 0.\end{aligned}$$

The converse can be obtained in the same way, using $\Phi^{-1}(\alpha)$. Then we have the last equivalence. □

Corollary 4.4 *In the hypotheses of theorem (4.3), we have that:*

- *the map* $\rfloor\,\omega$ *gives bijections also between the spaces:*

  $\mathcal{A}_{\mathbf{Z}}(J^1(M_{n+1}))$ *and* $\mathcal{A}^*_{\mathbf{Z}}(J^1(M_{n+1}))$*;*

$\mathcal{B}_{\mathbf{Z}}(J^1(M_{n+1}))$ *and* $\mathcal{B}^*_{\mathbf{Z}}(J^1(M_{n+1}))$*;*
$\mathcal{C}_{\mathbf{Z}}(J^1(M_{n+1}))$ *and* $\mathcal{C}^*_{\mathbf{Z}}(J^1(M_{n+1}))$

(up to equivalence classes);

- *the map* Φ *gives bijections also between the spaces:*

$\mathcal{A}_{\mathbf{Z}}(J^1(M_{n+1}))$ *and* $\mathcal{C}^*_{\mathbf{Z}}(J^1(M_{n+1}))$*;*
$\mathcal{B}_{\mathbf{Z}}(J^1(M_{n+1}))$ *and* $\mathcal{I}^*_{\mathbf{Z}}(J^1(M_{n+1}))$*;*
$\mathcal{C}_{\mathbf{Z}}(J^1(M_{n+1}))$ *and* $\mathcal{A}^*_{\mathbf{Z}}(J^1(M_{n+1}))$*.*

Proof. The proof follows easily from theorem (3.5) and straightforward computations. □

Then the situation, when a matrix g_{ij} obeying the Helmholtz condition D1 is known, can be summarized by the following diagrams:

$$\begin{array}{ccc} & \boxed{\omega} & \\ \mathcal{I}_{\mathbf{Z}}(J^1(M_{n+1})) & \leftrightarrow & \mathcal{I}^*_{\mathbf{Z}}(J^1(M_{n+1})) \\ \updownarrow & & \updownarrow \\ \mathcal{A}_{\mathbf{Z}}(J^1(M_{n+1})) & \leftrightarrow & \mathcal{A}^*_{\mathbf{Z}}(J^1(M_{n+1})) \\ \updownarrow & & \updownarrow \\ \mathcal{B}_{\mathbf{Z}}(J^1(M_{n+1})) & \leftrightarrow & \mathcal{B}^*_{\mathbf{Z}}(J^1(M_{n+1})) \\ \updownarrow & & \updownarrow \\ \mathcal{C}_{\mathbf{Z}}(J^1(M_{n+1})) & \leftrightarrow & \mathcal{C}^*_{\mathbf{Z}}(J^1(M_{n+1})) \end{array} \qquad \begin{array}{ccc} & \boxed{\Phi} & \\ \mathcal{I}_{\mathbf{Z}}(J^1(M_{n+1})) & \leftrightarrow & \mathcal{B}^*_{\mathbf{Z}}(J^1(M_{n+1})) \\ \updownarrow & & \updownarrow \\ \mathcal{A}_{\mathbf{Z}}(J^1(M_{n+1})) & \leftrightarrow & \mathcal{C}^*_{\mathbf{Z}}(J^1(M_{n+1})) \\ \updownarrow & & \updownarrow \\ \mathcal{B}_{\mathbf{Z}}(J^1(M_{n+1})) & \leftrightarrow & \mathcal{I}^*_{\mathbf{Z}}(J^1(M_{n+1})) \\ \updownarrow & & \updownarrow \\ \mathcal{C}_{\mathbf{Z}}(J^1(M_{n+1})) & \leftrightarrow & \mathcal{A}^*_{\mathbf{Z}}(J^1(M_{n+1})) \end{array} \tag{28}$$

To obtain a deeper insight of the diagrams (28), we need to describe the relations between the operators $G_{\mathbf{Z}}$ and the connection ∇ induced by the SODE $\mathbf{Z}$. Since $\nabla_{\mathbf{Z}}(f) = G_{\mathbf{Z}}(f) = \mathbf{Z}(f)$, $\forall G \in \mathfrak{G}$, $\forall f \in C^\infty(J^1(M_{n+1}))$, a standard argument (see, e.g., [5]) shows that we can decompose the action of $G_{\mathbf{Z}}$ as:

$$G_{\mathbf{Z}} \;=\; \nabla_{\mathbf{Z}} + \psi_O,$$

where ψ_O is the derivation of the tensor algebra of $J^1(M_{n+1})$ induced by a suitable type (1,1) tensor field O. We have the following:

Theorem 4.5 *Let* Q^i_k *be as in (13), and define the two type (1,1) tensor fields*

$$W \;=\; -D_i \otimes \Omega^i + Q^i_k V_i \otimes \omega^k,$$

$$U \;=\; K(W) \;=\; -V_i \otimes \omega^i + Q^i_k D_i \otimes \Omega^k.$$

Then we have the following identities:

$$I_{\mathbf{Z}} \;=\; \nabla_{\mathbf{Z}} + \psi_W; \qquad B_{\mathbf{Z}} \;=\; \nabla_{\mathbf{Z}} + \psi_U;$$

$$A_{\mathbf{Z}} \;=\; \nabla_{\mathbf{Z}} - \psi_W; \qquad C_{\mathbf{Z}} \;=\; \nabla_{\mathbf{Z}} - \psi_U.$$

Proof. It is sufficient to show that the derivations give the same results acting on functions of $J^1(M_{n+1})$ and on vector fields. Since $\psi_U(f) = \psi_W(f) = 0$, all the derivations have the same action on functions. Moreover, taking into account equations (3, 14), we have:

$$I_{\mathbf{Z}}(\mathbf{Z}) = \nabla_{\mathbf{Z}}(\mathbf{Z}) + \psi_W(\mathbf{Z}) = 0,$$

$$I_{\mathbf{Z}}(D_i) = -\tau_i^k D_k + q_i^k V_k = \nabla_{\mathbf{Z}}(D_i) + W(D_i) = \nabla_{\mathbf{Z}}(D_i) + \psi_W(D_i),$$

$$I_{\mathbf{Z}}(V_i) = -D_i - \tau_i^k V_k = \nabla_{\mathbf{Z}}(V_i) + W(V_i) = \nabla_{\mathbf{Z}}(V_i) + \psi_W(V_i).$$

The other statements can be proved in the same way. □

Then we can state the second result about the diagrams (28):

Theorem 4.6 *Let g_{ij} be a matrix of functions defined over $J^1(M_{n+1})$ obeying the algebraic Helmholtz conditions A1 and A2, let ω be the Poincaré–Cartan-like form determined by g_{ij} and let Φ be the compatible metric determined by g_{ij}. Then the following conditions are equivalent:*

1) g_{ij} obeys the first and the second differential Helmholtz condition D1 and D2;

2) $\begin{cases} \nabla_{\mathbf{Z}}\omega = 0; \\ \psi_U(\omega) = \psi_W(\omega); \end{cases}$

3) $\begin{cases} \nabla_{\mathbf{Z}}\Phi = 0; \\ \psi_U(\Phi) = \psi_W(\Phi); \end{cases}$

4) $\lrcorner\, \omega : \mathcal{I}_{\mathbf{Z}}(J^1(M_{n+1})) \to \mathcal{I}^*_{\mathbf{Z}}(J^1(M_{n+1}))$ is a bijection between the spaces and between $I_{\mathbf{Z}}$-symmetries and $I^*_{\mathbf{Z}}$-symmetries (up to equivalence classes);

5) $\Phi : \mathcal{I}_{\mathbf{Z}}(J^1(M_{n+1})) \to \mathcal{B}^*_{\mathbf{Z}}(J^1(M_{n+1}))$ is a bijection between spaces and between $I_{\mathbf{Z}}$-symmetries and $B^*_{\mathbf{Z}}$-symmetries.

Proof. We have that:

$$\psi_U(\omega) = g_{ij}U(\omega^i) \wedge \Omega^j + g_{ij}\omega^i \wedge U(\Omega^j) = g_{ij}Q^i_k\Omega^k \wedge \Omega^j - g_{ij}\omega^i \wedge \omega^j = g_{ij}Q^i_k\Omega^k \wedge \Omega^j,$$

$$\psi_W(\omega) = g_{ij}W(\omega^i) \wedge \Omega^j + g_{ij}\omega^i \wedge W(\Omega^j) = -g_{ij}\Omega^i \wedge \Omega^j + g_{ij}Q^j_k\omega^i \wedge \omega^k = g_{ij}Q^j_k\omega^i \wedge \omega^k.$$

Moreover we have:

$$\psi_U(\Phi) = g_{ij}U(\omega^i) \otimes \omega^j + g_{ij}\omega^i \otimes U(\omega^j) + g_{ij}U(\Omega^i) \otimes \Omega^j + g_{ij}\Omega^i \otimes U(\Omega^j) =$$

$$\left(-g_{ij} + g_{ik}Q^k_j\right)\omega^i \otimes \Omega^j + \left(-g_{ij} + g_{jk}Q^k_i\right)\Omega^i \otimes \omega^j;$$

$$\psi_W(\Phi) = g_{ij}W(\omega^i) \otimes \omega^j + g_{ij}\omega^i \otimes W(\omega^j) + g_{ij}W(\Omega^i) \otimes \Omega^j + g_{ij}\Omega^i \otimes W(\Omega^j) =$$

$$\left(-g_{ij} + g_{jk}Q^k_i\right)\omega^i \otimes \Omega^j + \left(-g_{ij} + g_{ik}Q^k_j\right)\Omega^i \otimes \omega^j.$$

The equivalence between the Helmholtz conditions D1 and D2 and the conditions 2) and 3) is now evident.

About conditions 4), given an $I_{\mathbf{Z}}$–symmetry $X = x^0\mathbf{Z} + x^i D_i + \left(\mathbf{Z}(x^i) - \tau^i_k x^k\right)$ with the functions $x^i, i = 1, \ldots, n$ obeying the equation (18) and recalling equation (26), we have to show that the equation

$$\mathbf{Z}\left(\mathbf{Z}(g_{ij}x^j)\right) + 2\mathbf{Z}\left(\tau^k_i g_{kj}x^j\right) - g_{kj}x^j \frac{\partial Z^k}{\partial q^i} \;=\; 0$$

holds if and only if the Helmholtz conditions D1 and D2 hold. The statement follows from a very long and tedious, but straightforward, calculation. In fact we have:

$$\begin{aligned}\mathbf{Z}\left(\mathbf{Z}(g_{ij}x^j)\right) + 2\mathbf{Z}\left(\tau^k_i g_{kj}x^j\right) - g_{kj}x^j \frac{\partial Z^k}{\partial q^i} &= \mathbf{Z}\left(\mathbf{Z}(g_{ij})x^j + g_{ij}\mathbf{Z}(x^j) + \right.\\ \left. 2\tau^k_i g_{kj}x^j\right) - g_{kj}x^j \frac{\partial Z^k}{\partial q^i} &= g_{kj}x^j Q^k_i + g_{kj}x^j\tau^k_r\tau^r_i - g_{ik}\tau^k_j\mathbf{Z}(x^j) + \\ g_{ij}\mathbf{Z}(\mathbf{Z}(x^j)) - \mathbf{Z}\left(g_{ki}\tau^k_j x^j\right) &= g_{ij}\mathbf{Z}(\mathbf{Z}(x^j)) - 2g_{ik}\tau^k_j\mathbf{Z}(x^j) - \\ g_{kr}\tau^k_i\tau^r_j x^j - \mathbf{Z}(g_{ki})x^j\tau^k_j + g_{ij}&\mathbf{Z}(\mathbf{Z}(x^j)) - g_{kj}x^j Q^k_i - g_{ki}x^j\mathbf{Z}(\tau^k_j)\end{aligned}$$

that is equivalent, using the condition D2, to the expression

$$g_{ij}\left(\mathbf{Z}\left(\mathbf{Z}(x^j)\right) - 2\tau^j_k\mathbf{Z}(x^k) - x^k\frac{\partial Z^j}{\partial q^k}\right)$$

from which we have the relation 1) $\Rightarrow$ 4). The other implications can be proved in the same way. □

Corollary 4.7 *In the hypotheses of theorem (4.6), we have that:*

- *the map* $\lrcorner\, \omega$ *gives bijections also between the spaces:*

  $\mathcal{A}_{\mathbf{Z}}(J^1(M_{n+1}))$ *and* $\mathcal{A}^*_{\mathbf{Z}}(J^1(M_{n+1}))$;
  $\mathcal{B}_{\mathbf{Z}}(J^1(M_{n+1}))$ *and* $\mathcal{B}^*_{\mathbf{Z}}(J^1(M_{n+1}))$;
  $\mathcal{C}_{\mathbf{Z}}(J^1(M_{n+1}))$ *and* $\mathcal{C}^*_{\mathbf{Z}}(J^1(M_{n+1}))$

  and between the relative $G_{\mathbf{Z}}$ *and* $G^*_{\mathbf{Z}}$*-symmetries (up to equivalence classes);*

- *the map* Φ *gives bijections also between the spaces:*

  $\mathcal{A}_{\mathbf{Z}}(J^1(M_{n+1}))$ *and* $\mathcal{C}^*_{\mathbf{Z}}(J^1(M_{n+1}))$;
  $\mathcal{B}_{\mathbf{Z}}(J^1(M_{n+1}))$ *and* $\mathcal{I}^*_{\mathbf{Z}}(J^1(M_{n+1}))$;
  $\mathcal{C}_{\mathbf{Z}}(J^1(M_{n+1}))$ *and* $\mathcal{A}^*_{\mathbf{Z}}(J^1(M_{n+1}))$

  and between the relative $G_{\mathbf{Z}}$ *and* $G^*_{\mathbf{Z}}$*-symmetries.*

Proof. The proof follows easily from theorems (3.5, 4.3), corollary (4.4) and straightforward computations. □

As a final remark, we point out that an easy calculation shows that the condition 3) of theorem (4.6) is equivalent to the condition $\mathcal{L}_{\mathbf{Z}}(\omega) = 0$. Then the theorem (4.6) includes naturally a classical result (see, e.g., [3]).

References

[1] Blair D. E., Contact Manifolds in Riemannian Geometry, Springer Verlag, Berlin, 1976.

[2] Cariñena J.F. and Martinez E., Symmetry theory and lagrangian inverse problem for time-dependent second-order differential equations, *J.Phys. A: Math. Gen.,* 1989, V.22, 2659–2665.

[3] Crampin M., Prince G.E., and Thompson G., A geometrical version of the Helmholtz conditions in time-dependent lagrangian dynamics, *J. Phys. A: Math. Gen.,* 1984, V.17, 1437–1447.

[4] de Leon M. and Rodrigues P.R., Methods of Differential Geometry in Analytical Mechanics, North Holland, Amsterdam, 1990.

[5] Kobayashi S. and Nomizu K., Foundations of Differential Geometry, Willey Intersc., New York, 1963.

[6] Massa E. and Pagani E., Classical dynamics of non-holonomic systems: a geometric approach, *Ann. Inst. H. Poincarè,* 1991, V.55, 511–544.

[7] Massa E. and Pagani E., Jet-bundle geometry, dynamical connections, and the inverse problem of lagrangian mechanics, *Ann. Inst. H. Poincarè,* 1994, V.61, 17–62.

[8] Morando P. and Pasquero S., The symmetry in the structure of dynamical and adjoint symmetries of second-order differential equations, *J. Phys. A:Math. Gen.,* 1995, V.28, 1943–1955.

[9] Pommaret J.F., Systems of Partial Differential Equations and Lie Pseudogroups, Gordon & Breach, New York, 1978.

[10] Santilli R.M., Foundations of Theoretical Mechanics, Springer, Berlin, 1978.

[11] Sarlet W., Symmetries, first integrals and the inverse problem of lagrangian mechanics, *J. Phys. A: Math. Gen.,* 1981, V.14, 2227–2238.

[12] Sarlet W., Cantrijn F., and Crampin M., Pseudo-symmetries, Noether's theorem and the adjoint equation, *J.Phys A: Math. Gen.,* 1987, V.20, 1365–1376.

[13] Sarlet W., Prince G.E., and Crampin M., Adjoint symmetries for time-dependent second order equations, *J. Phys A: Math. Gen.,* 1990, V.23, 1335–1347.

[14] Saunders D.J., The Geometry of the Jet Bundles, Cambridge University Press, Cambridge, 1989.