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On the Poincaré-Invariant Second-Order Partial Equations for a Spinor Field

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Abstract

The different second-order nonlinear partial equations are found that are invariant under the representation $D(\frac{1}{2},0)\bigoplus D(0,\frac{1}{2})$ of the Poincaré group $P(1,3)$ and also under conformal group $C(1,3)$. The some exact solutions are constructed for the one of these equations.

1 Introduction

It is well-known that the Dirac equation (DE)

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{1}$$

(where $\psi=\psi(x)$ is a four-component complex function (column), $\mu=\overline{0,3}$, $\partial_0\equiv\dfrac{\partial}{\partial x_0}$, $\partial_i\equiv -\dfrac{\partial}{\partial x_i}$, $i=\overline{1,3}$ and γ^μ are 4×4 Dirac matrices) describes a spinor field, since DE is invariant under the representation $D\Big(\frac{1}{2},0\Big)\bigoplus D\Big(0,\frac{1}{2}\Big)$ of the Poincaré algebra $AP(1,3)$ with the generators

$$P_\mu=\frac{\partial}{\partial x^\mu},\qquad J_{\mu\nu}=x_\nu\partial_\mu - x_\mu\partial_\nu - \frac{1}{4}[\gamma^\mu,\gamma^\nu]. \tag{2}$$

Any component of a solution of the DE (1) satisfies the D'Alembert equation

$$(\partial_\mu\partial^\mu + m^2)\psi = 0. \tag{3}$$

System (3) may describe fields with different spins (for example, $s=0$, when $\gamma^\mu=0$ in operators (2)). Then system (3) cannot be considered as describing a spinor field.

2 The system (3) with additional conditions

For system (3) to describe a spinor field it is necessary to put an additional condition on the function $\psi(x)$. A simplest Poincare–invariant condition has one of the forms [1]

$$\lambda_1\partial_\mu(\overline{\psi}\gamma^\mu\psi) + \lambda_2\partial_\mu(\overline{\psi}\gamma^4\gamma^\mu\psi) = G(\overline{\psi}\psi,\overline{\psi}\gamma^4\psi), \tag{4}$$

$$\lambda_1\overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \lambda_2\overline{\psi}\gamma^4(i\gamma^\mu\partial_\mu - m)\psi = H(\overline{\psi}\psi,\overline{\psi}\gamma^4\psi), \tag{5}$$

$$\lambda_1\psi^\top\gamma^0\gamma^2\gamma^\mu\partial_\mu\psi+\lambda_2\psi^\top\gamma^3\gamma^1\gamma^\mu\partial_\mu\psi=R(\overline{\psi}\psi,\overline{\psi}\gamma^4\psi), \tag{6}$$

where $\gamma^4=\gamma^0\gamma^1\gamma^2\gamma^3$, and so on.

Let us consider system (3), (5), when $\lambda_2=0$, $H\equiv 0$. It is evident that any solution of DE (1) satisfies that system but the reverse is not true. The function $\psi=(\gamma^0+\gamma^1)\{\exp(-im\gamma^3x_3)\chi+\exp(im\gamma^3x_3)\eta\}$, where χ , η are constant spinors, is the solution of system (3), (5), but doesn't of DE (1). Then it is possible to pick out among solutions of the D'Alembert equations (3) such a Poincaré-invariant set, corresponding to a field with $s=\frac{1}{2}$, that is more wide than the set of solutions of DE.

It is interesting to note that system (3) together with the additional condition

$$i\overline{\psi}\gamma^\mu\partial_\mu\psi-i(\partial_\mu\overline{\psi})\gamma^\mu\psi=2\lambda\overline{\psi}\psi,$$

where $\lambda=\sqrt{m^2+1}$ is invariant under the Poincaré algebra with the second-order (non-local) differential operators with matrix coeficients

$$\begin{aligned}
&P_\mu=\partial_\mu,\\
&\breve{J}_{01}=J_{01}+\frac{1}{2}i\gamma^0\gamma^2\tilde{\psi}\partial_\psi-\frac{1}{2}i\gamma^0\gamma^2\psi\partial_{\tilde{\psi}},\\
&\breve{J}_{02}=J_{02}+\frac{1}{2}\gamma^0\gamma^2\tilde{\psi}\partial_\psi+\frac{1}{2}\gamma^0\gamma^2\psi\partial_{\tilde{\psi}},\\
&\breve{J}_{03}=J_{03}+\frac{1}{2}\gamma^5(i\gamma^\mu\partial_\mu-\lambda)\psi\partial_\psi+\frac{1}{2}\gamma^5(i(\gamma^\mu)^\top\partial_\mu+\lambda)\tilde{\psi}\partial_{\tilde{\psi}},\\
&\breve{J}_{12}=J_{12}-\frac{1}{2}i\psi\partial_\psi+\frac{1}{2}i\tilde{\psi}\partial_{\tilde{\psi}},\\
&\breve{J}_{13}=J_{13}+\frac{1}{2}\gamma^3\gamma^1(i(\gamma^\mu)^\top\partial_\mu+\lambda)\tilde{\psi}\partial_\psi+\frac{1}{2}\gamma^1\gamma^3(i\gamma^\mu\partial_\mu-\lambda)\psi\partial_{\tilde{\psi}},\\
&\breve{J}_{23}=J_{23}+\frac{1}{2}i\gamma^1\gamma^3(i(\gamma^\mu)^\top\partial_\mu+\lambda)\tilde{\psi}\partial_\psi+\frac{1}{2}i\gamma^1\gamma^3(i\gamma^\mu\partial_\mu-\lambda)\psi\partial_{\tilde{\psi}},
\end{aligned} \tag{7}$$

where $\tilde{\psi}=\gamma^0\psi^*$, $J_{\mu\nu}$ are the operators (2).

It is important to note that the operators (7) satisfy commutation relations for the Poincaré algebras on the set of solutions of system (3) only, in contrast to operators (2).

3 The generalization of system (3)

Another way for describing of fields with $s=\frac{1}{2}$ is to generalize system (3):

$$\partial_\mu\partial^\mu\psi+F(\overline{\psi},\psi,\partial_\alpha\overline{\psi},\partial_\beta\psi)\psi=0, \tag{8}$$

where F is a matrix function of ψ and its derivations. Evidently, for equation (8) to be invariant under the representation (2) only, it is necessary that F depend on derivatives of ψ. Here we have considered the restricted class of the equations (8) [2,3] (although the more wide class has been found)

$$\partial_\mu\partial^\mu\psi+F^1(u,v,j^\mu,\tilde{j^\mu})\gamma^\mu\partial_\mu\psi+F^2(u,v,j^\mu,\tilde{j^\mu})\psi=0, \tag{9}$$

where $u=\overline{\psi}\psi$, $v=\overline{\psi}\gamma^4\psi$, $j^\mu=\partial_\mu(\overline{\psi}\psi)$, $\tilde{j^\mu}=\partial_\mu(\overline{\psi}\gamma^4\psi)$.

Theorem 1 *The system of equations (9) is invariant under the Poincaré group with the generators (2) if and only if*

$$\begin{aligned}
&F^1=g_1+g_2\gamma^4+\gamma^\mu j^\mu(g_3+g_4\gamma^4)+\gamma^\mu\tilde{j^\mu}(g_5+g_6\gamma^4),\\
&F^2=f_1+f_2\gamma^4+\gamma^\mu j^\mu(f_3+f_4\gamma^4)+\gamma^\mu\tilde{j^\mu}(f_5+f_6\gamma^4),
\end{aligned} \tag{10}$$

where $g_i,\ f_i,\ i=\overline{1,6}$ *are arbitrary functions of the Poincaré-group invariants* $u,\ v,\ j^\mu j_\mu$, $\tilde{j^\mu}\tilde{j_\mu},\ \tilde{j^\mu}j_\mu$.

It is known that among nonlinear Dirac equations there are conformally-invariant ones. The generators of the conformal group $C(1,3)$ are (2) and such ones

$$
\begin{aligned}
&D = x^\mu\partial_\mu + k, \quad \text{(the dilatation)}\\
&K_\mu = 2x_\mu D - (x^\nu x_\nu)\partial_\mu - \tfrac{1}{2}[\gamma^\mu,\gamma^\nu]x^\nu, \quad \mu,\nu=\overline{0,3} \quad \text{(the conformals).}
\end{aligned} \tag{11}
$$

Similarly, there is the class of the conformally–invariant equations (9), (10).

Theorem 2 *The system of equations (9), (10) is invariant under the conformal group* $C(1,3)$ *with the generators (2), (11) if and only if*

$$
\begin{aligned}
F^1 &= -\tfrac{1}{3}(\overline{\psi}\psi)^{-1}\gamma^\mu j^\mu + (h_1+h_2\gamma^4)\left(\frac{\gamma^\mu j^\mu}{\overline{\psi}\psi} - \frac{\gamma^\mu\tilde{j^\mu}}{\overline{\psi}\gamma^4\psi}\right) + (\overline{\psi}\psi)^{\frac{1}{3}}(h_3+h_4\gamma^4),\\
F^2 &= (\overline{\psi}\psi)^{\frac{1}{3}}(q_1+q_2\gamma^4)\left(\frac{\gamma^\mu j^\mu}{\overline{\psi}\psi} - \frac{\gamma^\mu\tilde{j^\mu}}{\overline{\psi}\gamma^4\psi}\right) + (\overline{\psi}\psi)^{\frac{2}{3}}(q_3+q_4\gamma^4),
\end{aligned} \tag{12}
$$

where $h_i,\ q_i,\ i=\overline{1,4}$ *are smooth arbitrary functions of the conformal invariants* $(\overline{\psi}\psi)(\overline{\psi}\gamma^4\psi)^{-1}$ *and* $\{(\overline{\psi}\psi)^2\tilde{j^\mu}\tilde{j_\mu} + (\overline{\psi}\gamma^4\psi)^2 j^\mu j_\mu - 2(\overline{\psi}\psi)(\overline{\psi}\gamma^4\psi)\tilde{j^\mu}j_\mu\}(\overline{\psi}\psi)^{-\frac{14}{3}}$, *and the conformal power* $k=\frac{3}{2}$.

Making the substitution [2, 4]

$$
\psi = (\bar{\xi}\xi)^s, \quad s = \frac{1}{2k}(\frac{3}{2}-k)
$$

in equations (9), (12), we obtain the $C(1,3)$-invariant ones with an arbitrary conformal power k, for example

$$
D_\mu D^\mu\xi - \frac{(2s+1)}{3}\frac{\gamma^\mu j^\mu}{\bar{\xi}\xi}\gamma^\mu D_\mu\xi = 0,
$$

where $D_\mu = \partial_\mu + s\partial_\mu(\ln\bar{\xi}\xi),\quad j^\mu = \partial_\mu(\bar{\xi}\xi)$.

4 The some exact solutions

Let we consider the simplest conformally-invariant equation (9), (12)

$$
\partial_\mu\partial^\mu\psi - \frac{1}{3}\left\{(\overline{\psi}\psi)^{-1}\gamma^\mu\partial_\mu(\overline{\psi}\psi)\right\}\gamma^\mu\partial_\mu\psi = 0 \tag{13}
$$

Take the ansatz [4] that is invariant under 3-dimensional subalgebras $< J_{01}+J_{31}, J_{03}, P_2 >$ of the Poincaré algebra

$$
\begin{aligned}
&\psi(x) = \exp\left\{\frac{x_1}{2(x_0+x_3)}(\gamma^0+\gamma^3)\gamma^1\right\}\exp\left\{\tfrac{1}{2}\gamma^0\gamma^3\ln(x_0+x_3)\right\}\varphi(\omega),\\
&\omega = {x_0}^2 - {x_1}^2 - {x_3}^2.
\end{aligned} \tag{14}
$$

Substituting this ansatz into (13) we obtain the reduced equation

$$4\omega\varphi'' + 2(3+\gamma^0\gamma^3)\varphi' - \frac{1}{3}\frac{(\overline{\varphi}\varphi)'}{\overline{\varphi}\varphi}[2(1+\gamma^0\gamma^3)\varphi + 4\omega\varphi'] = 0. \quad (15)$$

For example,

$$\varphi = \omega^{-\frac{3}{4}} \exp\left\{-\frac{1}{4}\gamma^0\gamma^3 \ln\omega\right\} \quad (16)$$

is the solution of equation (15). Evidently any solution of the massless DE is a solution of (13). But the solution (14), (16) does not satisfy the DE.

Take the substitution

$$\varphi(\omega) = \exp\{-\frac{1}{2}(1+\gamma^0\gamma^3)\ln\omega\}\xi(\omega). \quad (17)$$

Substituting (17) into (15), we obtain the equation for ξ which can be integrated once:

$$\xi' = \frac{1}{2}\{\omega^{-\frac{1}{3}}(\gamma^0-\gamma^3) + \omega^{-\frac{4}{3}}(\gamma^0+\gamma^3)\}(\bar{\xi}\xi)^{\frac{1}{3}}\chi,$$

where χ is a constant spinor. Looking for solutions in the form

$$\xi = [(\gamma^0-\gamma^3)\varphi_1(\omega) + (\gamma^0+\gamma^3)\varphi_2(\omega)]\chi, \quad (18)$$

and substituting this expression into the last equation, we obtain the system of ODEs for the scalar functions φ_1, φ_2

$${\varphi_1}' = \omega^{-\frac{1}{3}}(\varphi_1\varphi_2)^{\frac{1}{3}}c, \quad {\varphi_2}' = \omega^{-\frac{4}{3}}(\varphi_1\varphi_2)^{\frac{1}{3}}c, \quad c = \frac{\sqrt[3]{(4)}(\overline{\chi}\chi)^{\frac{1}{3}}}{2}.$$

This system is invariant under the operator $Q = \omega\partial_\omega + \varphi_1\partial_{\varphi_1}$. Then we obtain the substitution $\varphi_1 = \omega f(\varphi_2)$ as the ansatz that is invariant under Q. This substitution gives

$$c(f'(\varphi_2)-1)\varphi_2^{\frac{1}{3}} + f^{\frac{2}{3}} = 0, \quad {\varphi_2}' = \omega^{-1}[f(\varphi_2)\varphi_2]^{\frac{1}{3}}c.$$

So, solving the first–order ODE for $f(\varphi_2)$ and then for $\varphi_2(\omega)$, we obtain the solution (14), (17), (18) of equation (13).

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