Topology structures of the families of gray images

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Abstract. Let L be a subspace of Euclidean Space E^1. $\downarrow$ USC(X,L) denote all regions below of upper semi-continuous maps from X to L and $\downarrow$ C(X,L) denote all regions below of continuous maps from X to L. For an infinite compact metric space X, $\downarrow$ USC(X,I) with Vietoris topology is homeomorphic to Hilbert cube Q and $\downarrow$ C(X,I) is its subspace, where Q=[-1, 1]∞. $\downarrow$ USC(X, I) could be regarded as a mathematical model of all gray images. In the present paper, the following result is proved: $\downarrow$ USC(X, [0,1)) is homeomorphic to Q\{(0)}. Therefore the topological structure of $\downarrow$ C(X, [0,1)) is also clear.

Keywords: Upper semi-continuous maps. Continuous maps. Vietoris topology. Regions below of maps. Hilbert cube.

1.1 Introduction

For a Tychonoff space X, the hyperspace Cld(X) is the set consisting of all non-empty closed subsets of X endowed with the Vietoris topology which is generated by the subbase $\{U^-, U^+ : U \subset X$ is open $\}$, where $U^- = \{A \in Cld(X) \mid A \cap U \neq \emptyset\}$ and $U^+ = \{A \in Cld(X) \mid A \subset U\}$, where U is open in X. Thus $\{< U_1, U_2, …, U_n >:$ Ui is open in X} is a base for this topology, where $< U_1, U_2, … , U_n >=\{A \in$

$$Cld(X): A \subset \bigcup_{i=1}^{n} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i = 1, 2, …, n\}.$$

It is well-known that Cld(X) with this topology is metrizable if and only if X is compact and metrizable [1, Theorem I.3.4]. For a compact space X and for any admissible metric d on X, the Vietoris topology of Cld(X) is induced by the Hausdorff metric d_H defined as follows:

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$$d_H(A,B) = Max\{Sup_{a\in A}inf_{b\in B}\, d(a,b), Sup_{b\in B}inf_{a\in A}\, d(a,b)\}$$
for any A, B ∈ Cld (X).

Let $\Lambda L = (L, \lambda L)$ be a Lawson semilattice L with the Lawson topology λL. Let ↓USC(X,L) denote all lower closed sets including X×{0} in the product space X×ΛL (a such set is called an upper-semi-continuous set). ↓C(X,L) denotes all regions below of continuous maps from X to ΛL. In what follows, ↓USC(X, L) and ↓C(X,L) are considered as subspaces of Cld (X×ΛL).

Pattern analysis and image management is one of practical undergrounds of the research on ↓USC(X, L). In mathematical morphology which is one of the most important mathematical theories of pattern analysis and image management, 2-value image is thought to be a closed set of a fixed compact space which usually is a compact subspace of 3-dimensional Euclidean space and interval-value image a lower closed set of the product of the compact space and the unit interval. Hence ↓USC(X, L) could be regarded as a mathematical model of all colored images. For example, ↓USC(X, I^3) stands for the family of all three primary colored images. Moreover, the approach of images is just the Vietoris convergence of their mathematical Models (see [2], [3]). In fact, the topological structure of the family of all colored image is Hilbert cube (see [4, Theorem 1]).

A (single-valued) function f : X → R is called upper semi-continuous if $f^{-1}(-\infty, t)$ is open in X for every t ∈ R. For a Tychonoff space X and L⊂R, let USC(X,L) and C(X,L) denote the family of all upper semi-continuous maps from X to L and the family of all continuous maps form X to L respectively. For every f ∈USC (X, I), let ↓f be the region below of f, that is, ↓f ={(x, λ) ∈X×I: λ≤f (x) } then ↓f ∈Cld (X×I). Hence ↓USC (X, I) = { ↓f: f ∈USC (X, I) } and ↓C (X, I)= { ↓f: f ∈ C(X, I) }. They are all subspaces of the hyperspace Cld(X × I). For convenience , let USC(X)=USC(X,I), C(X)=C(X,I). For any A⊂USC(X), let↓A={ ↓f: f∈A}. Let USC₁(X)=USC(X)\USC(X,[0, 1)). In this paper, we mainly research the topological structures of ↓USC(X, [0, 1)).

For two pairs of spaces (X_1, Y_1) and (X_2, Y_2) with $Y_1 \subset X_1$ and $Y_2 \subset X_2$, the symbol $(X_1, Y_1) \approx (X_2, Y_2)$ means that there exists a homeomorphism h: $X_1 \rightarrow X_2$ such that $h(Y_1)=Y_2$.

For a metric space X, we use X_0 and $cl_X(\cdot)$ to denote the set of all isolated points of X and the closure-operator in X, respectively.

In [5], there was the following result.

Theorem 1. For a Tychonoff space X, the following conditions are equivalent:

(a) X is a compactum and $cl_X(X_0) \neq X$;

(b) $\downarrow C(X, I) \approx c_0$;

(c) $(\downarrow USC(X, I), \downarrow C(X, I)) \approx (Q, c_0)$

where Q=[-1, 1]∞ is Hilbert cube and $c_0 = \{ x \in (-1,1)^\infty : \lim_{n\to\infty} x_n = 0 \}$ is a subspace of Q.

In the present paper we mainly prove the following results.

Theorem 2. If X is an infinite compact metric space, then

$$(\downarrow USC(X), \downarrow USC_1(X)) \approx (I \times Q, \{1\} \times Q).$$

It is well-known that $[0,1) \times Q \approx Q\backslash\{(0)\}$ (where $(0)=(0,\ldots,0,\ldots)\in Q$), hence we have the following corollary.

Corollary 1. For a compact metric space X,

$$\downarrow USC(X, [0, 1)) \approx \begin{cases} [0,1)^{|X|}, & X \text{ is finite} \\ Q \backslash \{(0)\}, & \text{others} \end{cases},$$

where |X| denotes the cardinal of X.

A closed subset A of X is said to be a *Z-set* of X if the identity id_X can be approximated by continuous maps from X to X\A.

The following theorem is due to [6, Theorem 5.4.12 , Lemma 5.5.4, Lemma 5.5.12].

Theorem 3. If A is a Z-set in Q, then $(Q, c_0\backslash A) \approx (Q, c_0)$.

Note that $\{1\} \times Q$ is a Z-set of $I \times Q$. Hence by Theorems 1, 2 and 3, there is the following Corollary.

Corollary 2. If X is a compact metric space with $cl_X(X_0) \neq X$, then

$$\downarrow C(X,[0,1)) \approx c_0$$

1.2 Preliminaries

All spaces under discussion are assumed to be separable and metrizable spaces.

If X is a metric space, we can give another equivalent definition for upper semi-continuous map.

Definition 1. A map f: $X \rightarrow R$ is called a upper semi-continuous, if for every $x_0 \in X$ and $\varepsilon > 0$, there exists $\delta > 0$, such that for every $x \in U(x_0, \delta)$, $f(x_0) > f(x) - \varepsilon$.

Definition 2. A space X is called an absolute retract, abbreviated AR, provided that for every space Y containing X as a closed subspace, X is a retract of Y, that is, there exists a continuous map r:$Y \rightarrow X$ such that $r|_X = id_X$.

Definition 3. A subset A of a space Y is called *homotopy dense* in Y if there exists a homotopy h:$Y \times I \rightarrow Y$ such that $h_0 = id_Y$ and $h_t(Y) \subset A$ for every t > 0.

Definition 4. A topological semilattice is a topological space S equipped with a continuous operator $\vee : S \times S \rightarrow S$ which is idempotent, commutative and associative (i.e., $x \vee x = x$, $x \vee y = y \vee x$; $(x \vee y) \vee z = x \vee (y \vee z)$ for any x, y, $z \in S$). A topological semilattice S is called a Lawson semilattice if S admits an open basis consisting of subsemilattices.

Definition 5. Let X be a space. A subset $Y \subset X$ is relatively LC^0 in X if for every $x \in X$, each neighborhood U of x in X contains a smaller neighborhood V of x such that every two points of $V \cap Y$ can be joined by a path in $U \cap Y$.

Lemma 1. ([7, Theorem 5.1]) Let X be a metrizable Lawson semilattice with $Y \subset X$ a dense subsemilattice. If Y is relatively LC^0 in X and path-connected, then X is an AR and Y is homotopy dense in X, hence Y is also an AR.

Definition 6. We say a space (X, d) has the disjoint-cells property, if for any $\varepsilon > 0$, $k \in N$ and continuous functions f, g: $I^k \to X$, there exist continuous functions f_1, g_1 such that $d(f_1,f) < \varepsilon$, $d(g_1,g) < \varepsilon$ and $f_1 (I^k) \cap g_1(I^k) = \varnothing$.

Lemma 2. (The Toru´nczyk's Characterization Theorem) [8] (cf. [9, Corollary 7.8.4]) : A space X is homeomorphic to the Hilbert cube Q if and only if it is a compact AR with the disjoint-cells property.

The above lemma is a key to prove Theorem 2.

Lemma 3. [9, Theorem 6.4.6]Let E, F be Z-sets of Q, $\varepsilon > 0$, and h: $E \to F$ be a homeomorphism such that $\hat{\rho}$ (x, h(x)) $< \varepsilon$ for every $x \in E$. Then there exists a homeomorphism H:$Q \to Q$, such that H$|_E$=h and $\hat{\rho}$ (x, H(x)) $< \varepsilon$ for every $x \in Q$.

Let ϕ : $A \to B$ be a map from a set A to a set B. If $A \subset USC$ (X) and/or $B \subset USC$ (Y) for spaces X and Y, we may define a corresponding map $\downarrow \phi: \downarrow A \to \downarrow B$ or $\downarrow \phi: A \to \downarrow B$ or $\downarrow \phi: \downarrow A \to B$ as $\downarrow \phi (\downarrow f) = \downarrow (\phi (f))$ or $\downarrow \phi (f) = \downarrow (\phi (f))$ or $\downarrow \phi (\downarrow f) = \phi (f)$, respectively.

1.3 Proof of theorem 2

In follows, we always consider that $\hat{\rho}$ is the metric of the metric space $I \times Q$, where $\hat{\rho} (x,y) = \sqrt{\sum_{i=0}^{\infty} \frac{|x_i - y_i|^2}{2^i}}$ for every x= $x_i)_{i \in N}$, y= $y_i)_{i \in N} \in I \times Q$.

Lemma 4. $\{1\} \times Q$ is a Z-set in $I \times Q$.

Proof. It is easy to check.

Lemma 5. Let A be a space of $I \times Q$, then $(I \times Q, A) \approx (I \times Q, \{1\} \times Q)$ if and only if A is a copy of Q and a Z-set of $I \times Q$.

Proof. The essentiality is trivial. The sufficiency follows form lemmas 3 and 4.

In this section we always assume that X is an infinite compact space and ρ is an admissible metric of X. d is the metric on $X \times I$ defined by
$$d((x_1,t_1),(x_2,t_2)) = \max\{ \rho (x_1, x_2), |t_1 - t_2|\}.$$

Lemma 6. $\downarrow USC_1(X)$ is closed in $\downarrow USC$ (X).

Proof. Let $\{f_n\}$ be a sequence in $USC_1(X)$ and $\downarrow f_n \to \downarrow f$ $(n \to \infty$). It suffices to prove that $f \in USC_1(X)$. For every $n \in N$, there exists $x_n \in X$ such that

$f_n(x_n)=1$ since $f_n \in USC_1(X)$. By the compactness of X, $\{x_n\}$ has a convergence subsequence $\{x_{nk}\}$. Assume $x_{nk} \to y$, $k \to \infty$, then $(x_{nk}, 1) \to (y, 1)(k \to \infty)$. It follows from $(x_{nk}, 1) \in \downarrow f_{nk}$ and $\downarrow f_{nk} \to \downarrow f$ $(k \to \infty)$ that $(y, 1) \in \downarrow f$. That is $f(y)=1$. We are done.

Lemma 7. $\downarrow USC_1(X)$ is a Z-set of $\downarrow USC(X)$.

Proof. For every $\varepsilon >0$, pick $y \in (1-\varepsilon, 1)$. Define $\alpha: I \to [0, y]$ by $\alpha(t)=yt$ for every $t \in I$. Define $H: USC(X) \to USC(X, [0, y])$ by $H(f(x))= \alpha f(x)$ for every $f \in USC(X)$ and $x \in X$. It is trivial that $\downarrow H$ is continuous and $d_H(\downarrow H(f), \downarrow f)<\varepsilon$. Hence Lemma 7 holds by Lemma 6.

Lemma 8. $\downarrow C_1(X)$ is dense in $\downarrow USC_1(X)$.

Proof. For every $\varepsilon >0$ and $f \in \downarrow USC_1(X)$, it suffices to prove that there exists $g \in C_1(X)$ such that $d_H(\downarrow g, \downarrow f)<\varepsilon$. Since X is compact, there exists $\{x_1, x_2, \cdots, x_n\} \subset X$ such that $\{U(x_i, \frac{\varepsilon}{2})|i=1,2, \dots, n\}$ covers X and none of $U(x_i, \frac{\varepsilon}{2})$ could be covered by other balls.

Let $\delta = 0.5 \times \min\{\varepsilon, \min\{d(x_i, x_j)| i \neq j, i, j \in \{1,2, \dots, n\}\}\}$, then $\delta >0$. Let $g(x_i)=\text{Sup}\{f(x)|x \in U(x_i, \varepsilon/2)\}$, $g(x)=0$ for every $x \in X\backslash \bigcup_{i=1}^{n} U(x_i, \delta)$. If $x \in \bigcup_{i=1}^{n} U(x_i, \delta)$, it follows from the definition of δ that there exists only one $i \in \{1, 2, \dots, n\}$ such that $x \in U(x_i, \delta)$. Thus we can define $g(x)=(1-d(x, x_i)/\delta)g(x_i)$. It is trivial that $g(x) \in C_1(X)$ and we only need to show the following fact.

Fact. $d_H(\downarrow f, \downarrow g)<\varepsilon$.

On one hand, for every $(x, t) \in \downarrow f$. There exists $i \in \{1, 2, \dots, n\}$ such that $x \in U(x_i, \varepsilon/2)$. Since $g(x_i)=\sup\{f(x)|x \in U(x_i, \varepsilon/2)\}$, $(x_i, t) \in \downarrow g$. Noting that $d((x, t), (x_i, t))= \rho(x, x_i)<\varepsilon/2$. We conclude that $\downarrow f \in B_d(\downarrow g, \varepsilon/2)$. On the other hand, for every $(x, \lambda) \in \downarrow g$, there exists $i \in \{1, 2, \dots, n\}$ such that $x \in U(x_i, \varepsilon/2)$ and then $(x_i, \lambda) \in \downarrow g$ by the definition of g. Since $g(x_i)=\text{Sup}\{f(x)|x \in U(x_i, \varepsilon/2)\}$, there exists $y \in U(x_i, \varepsilon/2)$ such that $g(x_i)-f(y)<\varepsilon/2$. If $g(x_i) \geq \lambda >f(y)$, then $\lambda -f(y)<\varepsilon/2$, and hence $d((x, \lambda),(y, f(y))) \leq d((x, \lambda), (x_i, \lambda))+d((x_i, \lambda), (y, f(y)))= \rho(x, x_i)+\min\{\rho(x_i, y), \lambda -f(y)\}<\varepsilon/2+\varepsilon/2=\varepsilon$. If $\lambda <f(y)$, then $(y, \lambda) \in \downarrow f$ and $d((x, \lambda), (y, \lambda))= \rho(x, y) \leq \rho(x, x_i)+ \rho(y, x_i)<\varepsilon/2+\varepsilon/2=\varepsilon$. Thus $\downarrow g \in B_d(\downarrow f, \varepsilon)$. That is $d_H(\downarrow f, \downarrow g)<\varepsilon$.

Lemma 9. $\downarrow USC_1(X)$ is AR and $\downarrow C_1(X)$ is homotopy dense in $\downarrow USC_1(X)$.

Proof. Define $\vee: \downarrow USC_1(X) \times \downarrow USC_1(X) \to \downarrow USC_1(X)$ by $\downarrow f \vee \downarrow g =\downarrow \max\{f(x), g(x)\}=\downarrow f \vee \downarrow g$, then $\vee$ is continuous.

Claim 1. $(\downarrow USC_1(X), \vee)$ is a semi-lattice.

It suffices to prove that $\downarrow g \vee \downarrow h \in B_{d_H}(\downarrow f, \varepsilon)$ for any $\downarrow g$, $\downarrow h \in B_{d_H}(\downarrow f, \varepsilon)$, where $B_{d_H}(\downarrow f, \varepsilon)=\{\downarrow f \in \downarrow USC(X) \mid d_H(\downarrow f, \downarrow g) < \varepsilon\}$. On one hand, for any point $p \in \downarrow f$, since $d_H(\downarrow f, \downarrow g) < \varepsilon$, there exists $p_0 \in \downarrow g \subset \downarrow g \vee \downarrow h$ such that $d(p, p_0) < \varepsilon$. On the other hand, if the point $q \in \downarrow g \vee \downarrow h$, then $q \in \downarrow g$ or $q \in \downarrow h$. Since $\downarrow g$, $\downarrow h \in B_{d_H}(\downarrow f, \varepsilon)$, there exists $q_0 \in \downarrow f$ such that $d(q, q_0) < \varepsilon$. Thus $\downarrow g \vee \downarrow h \in B_{d_H}(\downarrow f, \varepsilon)$.

Claim 2. $\downarrow C_1(X)$ is a sub semi-lattice of $\downarrow USC_1(X)$.

For any $\downarrow g_1$, $\downarrow g_2 \in \downarrow C_1(X)$, if $\downarrow g = \downarrow g_1 \vee \downarrow g_2$, it is easy to check that $g \in C_1(X)$. Thus by Claim 1, $\downarrow C_1(X)$ is a sub semi-lattice of $\downarrow USC_1(X)$.

Claim 3. $\downarrow C_1(X)$ is relatively LC^0 in $\downarrow USC_1(X)$ and path connected.

For any $\downarrow f \in \downarrow USC_1(X)$ and $\varepsilon > 0$, we shall prove that $\downarrow U = B_{d_H}(\downarrow f, \varepsilon) \cap \downarrow C_1(X)$ is path connected. For any $\downarrow g_1$, $\downarrow g_2 \in \downarrow U$, $\downarrow g = \downarrow g_1 \vee \downarrow g_2 \in \downarrow U$ by Claim 1. Define $\downarrow H: I \to \downarrow U$ by $H(t)=(1-t)g_1+tg$, then $\downarrow H$ is well defined and a path from $\downarrow g_1$ to $\downarrow g$ (cf. proof of [5, Lemma 6]). Similarly, there exists a path joining $\downarrow g$ and $\downarrow g_2$ in $\downarrow U$. We conclude that $\downarrow C_1(X)$ is relatively LC^0. Similarly, it can be proved that $\downarrow C_1(X)$ is path connected.

It follows from the above claims, Lemmas 1 and 8 that Lemma 9 holds. We are done.

Lemma 10. $\downarrow USC_1(X)$ has disjoint-cells property.

Proof. For every $n \in N$, let $\phi: I^n \to \downarrow USC_1(X)$ and $\varphi: I^n \to \downarrow USC_1(X)$ are two continuous maps. By Lemma 9, there exists a homeotopy $H: \downarrow USC_1(X) \times I \to \downarrow USC_1(X)$ such that $H_0=\mathrm{id}$, $H_t(\downarrow USC_1(X)) \subseteq \downarrow C_1(X)$ for every $t \in (0,1]$. For every $1 > \varepsilon > 0$, let $\Phi, \Psi: I^n \to \downarrow USC_1(X)$ be two maps defined by $\Phi(m)=H(\phi(m), \varepsilon/2)$ and $\Psi(m)=H(\varphi(m), \varepsilon/2)$ for every $m \in I^n$, respectively. Choose a non-isolated point $x_0 \in X$ arbitrarily and a point $x_1 \in U^0(x_0, \varepsilon/3)$. Let the map $M: \downarrow C_1(X) \to I$ defined by $M(\downarrow f)= \max\{f(x): d(x_0, x) \leq \varepsilon\}$. Then M is continuous by [10, Lemma 14]. Let $S: \downarrow C_1(X) \to I$ defined by $S(\downarrow f)= \max\{\varepsilon, M(\downarrow f)\}$. It is trivial that S is continuous. Define two maps $\downarrow \alpha$, $\downarrow \beta: \downarrow C_1(X) \to \downarrow USC_1(X)$

by $\alpha(f)(x)=\begin{cases} S(\downarrow f), & x = x_0 \\ 0, & x \in U^0(x_0, \varepsilon/2) \\ f(x), & \text{others} \end{cases}$ and

$\beta(f)(x)=\begin{cases} S(\downarrow f), & x = x_1 \\ 0, & x \in U(x_0, \varepsilon/2) \setminus \{x_1\} \\ f(x), & \text{others} \end{cases}$ for every $\downarrow f \in \downarrow C_1(X)$ and $x \in X$.

Claim 1. $\downarrow \alpha$, $\downarrow \beta$ are well-defined and continuous.

It is easy to check that $\downarrow \alpha$ (f) and $\downarrow \beta$ (f) are upper semi-continuous for every $f \in C_1(X)$. For every $f \in C_1(X)$, there exists $y \in X$ such that f(y)=1. If $y \in U(x_0, \varepsilon/2)$, then α (f)(x_0)= α (f)(x_1)=1; if $y \in X \setminus U(x_0, \varepsilon/2)$, then α (f)(y)= β (f)(y)=f(y)=1. Thus, $\downarrow \alpha$, $\downarrow \beta$ are well-defined. For every $\downarrow f \in \downarrow C_1(X)$ and every $\gamma > 0$, we shall prove that there exists $\tau > 0$, such that if $d_H(\downarrow f, \downarrow g) < \tau$, then $|\downarrow \alpha$ (f) - $\downarrow \alpha$ (g)$| < \gamma$. By the continuity of S, there exists $\tau_1 > 0$, such that if $d_H(\downarrow f, \downarrow g) < \tau_1$, then $|S(\downarrow f) - S(\downarrow g)| < \gamma$. Since X is compact and f $\in C_1(X)$, f is uniformly continuous on X, that is, there exits $\tau_2 > 0$ such that $|f(x)-f(y)| < \gamma/2$ for every x, y $\in X$ with $\rho(x, y) < \tau_2$. Choose $\tau = \min\{\gamma/2, \tau_1, \tau_2\}$, it suffices to prove that $d_H(\downarrow \alpha$ (f), $\downarrow \alpha$ (g))$< \gamma$ for every $\downarrow g \in \downarrow C_1(X)$ with $d_H(\downarrow f, \downarrow g) < \tau$.

On one hand, for every $(x, \lambda) \in \downarrow \alpha$ (g), we shall find a point $(y, z) \in \downarrow \alpha$ (f) such that d $((x, \lambda), (y, z)) < \gamma$. Consider three cases.

Case 1. If x=x_0, then it follows from the continuity of the map S.

Case 2. If x $\in U^0(x_0, \varepsilon/2)$, then (x, λ)=(x,0) $\in \downarrow \alpha$ (f). Choose (y, z)= (x, 0) as required.

Case 3. If x$\in X \setminus U(x_0, \varepsilon/2)$, then $(x, \lambda) \in \downarrow g$, there exists $(x', \lambda') \in \downarrow f$ such that d $((x, \lambda), (x', \lambda')) < \tau$. If $x' \in X \setminus U(x_0, \varepsilon/2)$, then $(x', \lambda') \in \downarrow \alpha$ (f). We can just choose (y, z)= (x', λ') as required. If $x' \in U(x_0, \varepsilon/2)$, then $|f(x)-f(x')| < \gamma/2$ since $d(x, x') < \tau < \tau_2$. Hence there exists λ_1 such that $|\lambda_1 - \lambda'| < \gamma/2$ and $(x, \lambda_1) \in \downarrow f$. Thus $|\lambda - \lambda'| \leq |\lambda - \lambda'| + |\lambda_1 - \lambda'| < \tau + \gamma/2 \leq \gamma$, hence $d((x, \lambda), (x, \lambda_1)) < \gamma$. Choose (y, z)= (x, λ_1), then (y, z) is as required.

Therefore, $\downarrow \alpha$ (g)$\subset B_d(\downarrow \alpha$ (f), γ).

On the other hand, we can proved that $\downarrow \alpha$ (f)$\subset B_d(\downarrow \alpha$ (g), γ) similarly. Thus $d_H(\downarrow \alpha$ (f), $\downarrow \alpha$ (g))$< \gamma$. That is, $\downarrow \alpha$ is continuous. It is a similar proof for the continuity of $\downarrow \beta$.

Put $\phi': I^n \rightarrow \downarrow USC_1(X)$ and $\varphi': I^n \rightarrow \downarrow USC_1(X)$ by $\phi' = \downarrow \alpha \circ \Phi$ and $\varphi' = \downarrow \beta \circ \Psi$. It is easy to see that $\phi'(\downarrow USC_1(X)) \cap \varphi'(\downarrow USC_1(X))= \emptyset$. Thus, it suffices to prove the following claims.

Claim 2. ϕ' and φ' are continuous.

It follows from the continuity of the maps $\downarrow \alpha$, $\downarrow \beta$, Φ and Ψ.

Claim 3. $d_H(\varphi (m), \varphi'(m)) < \varepsilon$ and $d_H(\phi (m), \phi'(m)) < \varepsilon$ for every m$\in I^n$.

By [6, Proposition 4.1.7], without loss of generliy, we can assume that $d_H(\phi(m), \Phi(m))$=$d_H(\phi(m), H(\phi(m), \varepsilon/2)) < \varepsilon/2$ and $d_H(\varphi(m), \Psi(m))$= d_H

$(\varphi(m), H(\varphi(m), \varepsilon/2)) < \varepsilon/2$. It is trival that $d_H(\phi(m), \Phi(m)) < \varepsilon/2$ and $d_H(\Psi(m), \varphi'(m)) < \varepsilon/2$. Thus $d_H(\varphi(m), \varphi'(m)) < \varepsilon$ and $d_H(\phi(m), \phi'(m)) < \varepsilon$.

Proof of Theorem 2. It follows from Lemmas 2, 6, 9 and 10 that $\downarrow USC_1(X) \approx Q$. By Lemmas 5 and 7, we conclude that Theorem 2 holds.

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