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Symmetries of Euler Equations in Lagrangian Coordinates

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Abstract

The transition from Eulerian to Lagrangian coordinates is a nonlocal transformation. In general, isomorphism should not take place between basic Lie groups of studied equations. Besides, in the case of plane and rotational symmetric motion hydrodynamic equations in Lagrangian coordinates are partially integrated. This fact introduces arbitrary functions, initial data, to the resulting systems and makes cuurently central the problem of group classification. It is stated that under a transition to Lagrangian coordinates, the main group becomes infinite–dimensional as well in space coordinates. The exclusive values of arbitrary functions of Lagrange coordinates (vorticity, momentum), at which the further group widening takes place, are found in [1].

In this paper, the group properties of equations in Lagrange variables for nonuniform liquid are studied. The problem of group classification of these equations on functions of initial density is solved. Exact invariant solutions are obtained. As a rule, new obtained exact solutions describe nonstationary vortex motions. The solution representations in Lagrangian coordinates include arbitrary functions of time and space coordinates. This allows us to consider different invariant initial boundary problems. We emphasize that the majority of these solutions could hardly be found considering the equations of hydrodynamics in Euler variables.

1. Let us consider the equations of nonuniform heavy liquid

$$u_t + uu_x + vu_y + \frac{1}{\rho}p_x = 0, \quad v_t + uv_x + vv_y + \frac{1}{\rho}p_y = -g, \tag{1}$$

$$\rho_t + u\rho_x + v\rho_y = 0, \quad u_x + v_y = 0,$$

where $\rho(x, y, t)$ is the liquid density, $g = const > 0$ is the acceleration of gravity, (u, v) is the velocity vector, and $p(x, y, t)$ is the pressure of liquid. It is of common knowledge that there are nine infinitesimal generators of a finite symmetry group [1]

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_u + t\partial_x, \quad X_4 = \partial_v + t\partial_y, \quad X_5 = \partial_t,$$

$$X_6 = t\partial_t + 2x\partial_x + 2y\partial_y + u\partial_u + +v\partial_v + 2p\partial_p,$$

$$X_7 = \Big(y + \frac{1}{2}gt^2\Big)\partial_x - x\partial_y + (v + gt)\partial_u - u\partial_v, \tag{2}$$

$$X_8 = t\partial_t + x\partial_x + \Big(y - \frac{1}{2}gt^2\Big)\partial_y - gt\partial_v, \quad X_9 = \rho\partial_\rho + p\partial_p.$$

Additionaly, there is a generator $X_{10}(\varphi) = \varphi(t)\partial_p$ which depends on one arbitrary function of time.

2. Now we rewrite equations (1) in a more convenient form using the Lagrangian coordinates ξ and η which are defined by a solution of the Cauchy problem

$$\frac{dx}{dt} = u(x, y, t)\text{'} \quad \frac{dy}{dt} = v(x, y, t), x|_{t=0} = \xi, y|_{t=0} = \eta. \tag{3}$$

In such a case, the third equation of system (1) is integrable and we have

$$\rho(x(\xi, \eta, t), \quad y(\xi, \eta, t) = \rho_0(\xi, \eta) \equiv R^{-1}(\xi, \eta), \tag{4}$$

where $\rho_0(\xi, \eta)$ is an initial density.

By substituting the functions $x(\xi, \eta, t), \quad y(\xi, \eta, t), p(\xi, \eta, t)$ and (4) in Eq. (1), we get a new system

$$x_{tt} + R(\xi, \eta)(y_\eta p_\xi - y_\xi p_\eta) = 0, \qquad y_{tt} + R(\xi, \eta)(x_\xi p_\eta - x_\eta p_\xi) = 0, \tag{5}$$

$$x_\xi y_\eta - y_\xi x_\eta = 1.$$

Here we put $g = 0$.

In order to construct the group on the solution, one should first of all find the point transformation group that is admitted by the differential manifold given by system (5). An infinitesimal operator of this point group

$$Y = \mu^1\partial_t + \mu^2\partial_\xi + \mu^3\partial_\eta + \tau^1\partial_p + \tau^2\partial_x + \tau^3\partial_y$$

is calculated from the determining equation for coordinates μ^1 and τ^i following the scheme standard for the group analysis. However, here we have to solve the problem of group classification with respect to the function $R(\xi, \eta) > 0$. After some computations, from the determining equations, the following representations for μ^i and τ^i have been found

$$\mu^1 = c_1 + c_2t, \quad \mu^2 = \mu^2(\xi, \eta), \quad \mu^3 = \mu^3(\xi, \eta),$$

$$\tau^1 = \tau^1(t, x, y, p), \quad \tau^2 = c_3x + c_4y + g_1(t), \tag{6}$$

$$\tau^3 = c_3y - c_4x + g_2(t),$$

where $g_1(t), g_2(t) \in C^\infty$ are arbitrary functions, $c_1, ..., c_4$ are arbitrary constants. The functions μ^1, μ^2, τ^1 are the solution of the determining system

$$\mu^2_\xi + \mu^2_\eta = 2c_3, \quad \mu^2R_\xi + \mu^3R_\eta = (2c_3 - 2c_2 - \tau^1_p)R, \tag{7}$$

$$R\tau^1_x + g_1(t) = 0, \quad R\tau^1_y + g_2(t) = 0.$$

It can be easily shown that general equivalence transformations for system (5) are

$$\bar{t} = a_1 t + a_2, \quad \bar{\xi} = a_3\alpha(\xi,\eta), \quad \bar{\eta} = a_3\beta(\xi,\eta),$$

$$\bar{x} = a_3(x\cos a_4 + y\sin a_4) + a_5 t + a_6, \quad \bar{y} = a_3(y\cos a_4 - x\sin a_4) + a_7 t + a_8, \tag{8}$$

$$\bar{p} = a_1^{-2}a_3^2 a_9^{-1} p + \varphi(t), \quad \bar{R} = a_9 R(\bar{\xi},\bar{\eta}),$$

where $a_1, ..., a_9$ are constants and $a_2 > 0, a_3 > 0, a_9 > 0$.

Using the equivalence transformations (8), structures of solutions of the determining system (7) is investigated. The result of the group classification obtained here is shown in Table 1.

T a b l e 1

	$R(\xi,\eta)$	generators	remarks
1.	arbitrary	$L = Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_\varphi$	
2.	1	$L, Y_8, Y_{g_1}, Y_{g_2}, Y_*$	ideal liquid
3.	$\xi, (R_\xi \neq 0)$	L, Y_9, Y_{10}, Y_b	

$$Y_1 = \partial_t, Y_2 = t\partial_t - 2p\partial_p, Y_3 = y\partial_x - x\partial_y, Y_4 = t\partial_x,$$

$$Y_5 = \partial_x, Y_6 = t\partial_y, Y_7 = \partial_y, Y_\varphi = \varphi(t)\partial_p, Y_8 = \xi\partial_\xi +$$

$$+\eta\partial_\eta + x\partial_x + y\partial_y + 2p\partial_p, Y_{g_1}(t)\partial_x - \ddot{g}_1(t)x\partial_p,$$

$$Y_{g_2} = g_2(t)\partial_y - \ddot{g}_2(t)y\partial_p, Y_* = \psi_\eta\partial_\xi - \psi_\xi\partial_\eta,$$

$$Y_9 = \eta\partial_\eta - \xi\partial_\xi + p\partial_p, Y_{10} = 2\xi\partial_\xi - 2\eta\partial_\eta +$$

$$+x\partial_x + y\partial_y, Y_b = b(\xi)\partial_\eta,$$

where $\varphi(t), g_1(t), g_2(t), \psi(\xi,\eta), b(\xi)$ are arbitrary functions and $\dot{g}_1 \neq 0, \dot{g}_2 \neq 0$.

Now we shall claim the invariance of the initial data (3) $x = \xi, y = \eta, t = 0$. It gives us a more detailed group classification of system (5) with respect to the function $R(\xi,\eta)$.

Complete results are given in Table 2.

Of course, the cases 6 or 7 are more convenient to be studied in polar coordinates.

4. Let us consider several examples of exact solutions of equations (1) and (5).

Example 1. Let $R = R(\eta)$, then system (5) admits the two–dimensional subgroup $< \partial_\xi, \partial_x >$. The partially–invariant solution of rank 2 and defect 1 is sought in the form

$$x = x(\xi,\eta,t), \quad y = y(\eta,t), \quad p = p(\eta t). \tag{9}$$

By substituting the invariant forms of solution (10) into Eqs. (5), we obtain

$$x = [1 + a_0(\eta)t]\xi + b_0(\eta)t, \quad y = \int \frac{d\eta}{1 + a_0(\eta)t}, \tag{10}$$

$$p = -\int \frac{\rho_0(\eta)y_{tt}(\eta,t)d\eta}{1 + a_0(\eta)t} + \varphi(t)$$

with arbitrary functions $a_0(\eta), b_0(\eta), \rho_0(\eta) \equiv R^{-1}(\eta), \varphi(t)$. It describes unsteady rotational flows of liquid in the plane layer with one or two free surfaces.

Remark 1. The initial manifold is not invariant with respect to the subgroup $< \partial_\xi, \partial_x >$, however, as follows from (11), $x = \xi, y = \eta$ when $t = 0$.

Table 2

	$R(\xi,\eta)$	generators
1.	arbitrary	$L = \{Y_1, Y_4, Y_6, Y_\varphi\}$
2.	1 homogeneous liquid	L, Y_2, Y_3, Y_5, Y_7
3.	$F(\gamma_1\xi + \beta_1\eta)\exp(\gamma_2\xi + \beta_2\eta)$	$L, \beta_1 Y_5 - \gamma_1 Y_7 + (\gamma_1\beta_2 - \gamma_2\beta_1)Y_8$
4.	$\exp(\gamma\xi + \beta\eta)$	$L, Y_5 - \gamma Y_8, Y_7 - \beta Y_8$
5.	$(\gamma\xi + \beta\eta)^\alpha$	$L, Y_2 - \alpha Y_8, \gamma Y_7 - \beta Y_5$
6.	$F(\beta_1 \arctan(\xi/\eta) + \gamma_1 \ln(\xi^2 + \eta^2))(\xi^2 + \eta^2)^\gamma \exp(\beta_2 \arctan(\xi/\eta))$	$L, \beta_1 Y_5 - 2\gamma_1 Y_3 + 2(\gamma_1\beta_2 - \gamma_2\beta_1)Y_8$
7.	$(\xi^2 + \eta^2)^\gamma \exp(\beta \arctan(\xi/\eta))$	$L, Y_2 - 2\gamma Y_8, Y_3 - \beta Y_8$

$$Y_1 = t\partial_t - 2p\partial_p, Y_2 = \xi\partial_\xi + \eta\partial_\eta + x\partial_x + y\partial_y + 2p\partial_p,$$

$$Y_3 = \eta\partial_\xi - \xi\partial_\eta + y\partial_x - x\partial_y, Y_4 = t\partial_x, Y_5 = \partial_\xi + \partial_x, Y_6 = t\partial_x,$$

$$Y_6 = t\partial_y, Y_7 = \partial_\eta + \partial_y, Y_8 = p\partial_p, Y_\varphi = \varphi(t)\partial_p;$$

$F \in C^\infty$ is an arbitrary function; $\beta, \gamma, \beta_1, \gamma_1, \beta_2, \gamma_2, \alpha$ are constants, $\alpha \neq 0, \beta^2 + \gamma^2 \neq 0, \beta_1^2 + \gamma_1^2 \neq 0$; if $\beta_1 \cdot \gamma_1 \cdot \beta_2 \cdot \gamma_2 \neq 0$ then $\gamma_2\beta_1 + \gamma_1\beta_2 = 0$

Example 2. If $R = \exp(\gamma\xi + \beta\eta)$, then the subgroup $< \partial_\xi + \partial_x - \gamma p\partial p >$ is admitted (see the case 4 from Table 2, where $\gamma \neq 0$). The invariant solution has the form

$$x = \xi + X(\eta, t), \quad y = Y(\eta, t), \quad p = P(\eta, t)\exp(-\gamma\xi),$$

and, hence, we get from Eqs.(5)

$$Y = f(t) + \eta, \quad f(0) = 0, \quad X_{tt} - \gamma e^{\beta\eta} P = 0, \quad X_\eta X_{tt} + f_{tt} + e^{\beta\eta} P_\eta = 0.$$

Eliminating P from the third equation, we obtain

$$X_\eta X_{tt} - \frac{\beta}{\gamma} X_{tt} + \frac{1}{\gamma} X_{\eta tt} + f_{tt} = 0. \tag{11}$$

Let us assume that flow is uniform along the y axis, i.e. $f_t = v_0 =$ const. In such a, case $P = \mu(t)\exp(-\gamma X)$ with an arbitrary $\mu(t)$ and the function $Z = X - \beta\eta/\gamma$ is a solution of ODE

$$Z_{tt} - \gamma\mu(t)\exp(-\gamma Z) = 0.$$

For $\mu(t) = \mu_0 =$ const, the last equation has integrals depending on $X_t|_{t=0} = u_0(\eta)$:

$$X = \frac{\beta}{\gamma}\eta - \frac{1}{\gamma}\ln\Big\{\frac{d}{\mu_0[ch(\gamma\sqrt{d}(t + c)) - 1]}\Big\}, \tag{12}$$

if $d(\eta) \equiv u_0^2(\eta) - 2\mu_0 \exp(\beta\eta) > 0$;

$$X = \frac{\beta}{\gamma}\eta - \frac{1}{\gamma}\ln\Big\{-\frac{d}{\mu_0[\cos(\gamma\sqrt{-d}(t+c))+1]}\Big\}, \tag{13}$$

if $d(\eta) < 0$;

$$X = \frac{\beta}{\gamma}\eta - \frac{1}{\gamma}\ln\Big[\frac{4}{\mu_0\gamma^2(t+c)^2}\Big], \tag{14}$$

if $d(\eta) = 0$. The function $c(\eta)$ can be determined from the initial condition $X(\eta, 0) = 0$.

The density of liquid is constant on the lines $\gamma\xi + \beta\eta = \mathrm{const}$. During the time, these lines are deformed according the formula

$$\gamma[x - X(y - v_0 t, t)] + \beta(y - v_0 t) = \mathrm{const},$$

where X is defined by (13), (14), or (15).

The another solution can be obtained if we suppose that X is linear with respect to η. Integrating the system (5), we have

$$x = \xi + a_0 t\eta + b(t), \quad y = \eta + f(t), \quad b(0) = f(0) = 0, \tag{15}$$

$$p = \frac{1}{\gamma} b_{tt} \exp(-\gamma\xi - \beta\eta) + \varphi(t),$$

where a_0 is a constant and $f(t), b(t), \varphi(t)$ are arbitrary functions with $(a_0 t - \beta/\gamma)b_{tt} + f_{tt} = 0$. The line $\gamma\xi + \beta\eta = \mathrm{const}$ can be considered as a free surface one.

Example 3. For $R = R(\xi^2 + \eta^2)$, system (5) admits the subgroup of rotations $< \eta\partial_\xi - \xi\partial_\eta + y\partial_x - x\partial_y >$ (see the case 6, Table 2, where $\beta_1 = \beta_2 = 0$). Introducing polar coordinates $x = r\cos\Theta$, $y = r\sin\Theta$, $\xi = \beta\cos\alpha$, $\eta = \beta\sin\alpha$ in (5), we get a new system for the unknown functions $r(\beta,\alpha,t)$, $\Theta(\beta,\alpha,t)$, $p(\beta,\alpha,t)$

$$r_\beta(r_{tt} - r\Theta_t^2) + \Theta_\beta(r^2\Theta_t)_t + Rp_\beta = 0,$$

$$r_\alpha(r_{tt} - r\Theta_t^2) + \Theta_\alpha(r^2\Theta_t)_t + Rp_\alpha = 0, \quad r(r_\beta\Theta_\alpha - r_\alpha\Theta_\beta) = \beta. \tag{16}$$

The generator $\eta\partial_\xi - \xi\partial_\eta + y\partial_x - x\partial_y$ transforms to the generator $\partial_\Theta + \partial_\alpha$ and an invariant solution has the form

$$r = r(\beta, t), \quad \Theta = \alpha + \varphi(\beta, t), \quad p = p(\beta, t).$$

Equations (5) now imply

$$r = (\beta^2 + C(t))^{1/2}, \quad \varphi_t = B(\beta)(\beta^2 + C(t))^{-1}, \tag{17}$$

$$p = \int \frac{r_\beta}{R(\beta)}(r_{tt} - r\varphi_t^2)d\beta + \mu(t)$$

with arbitrary functions $C(t), C(0) = 0, B(\beta), \mu(t)$. This solution describes rotational ring motion with free surfaces

$$r_1(t) = (\beta_1^2 + C(t))^{1/2}, \quad r_2(t) = (\beta_2^2 + C(t))^{1/2}, \quad \beta_2 > \beta_1 \geq 0.$$

The function $B(\beta)$ is connected with vorticity $w_0(\beta)$ of liquid by the formula $B(\beta) = \int \rho w_0(\rho) d\rho$.

Example 4. Let us consider arbitrary Lagrangian coordinates (a, b) defined by $\xi = \alpha(a, b)$, $\eta = \beta(a, b)$. In terms of these variables, (5) become

$$x_a x_{tt} + y_a y_{tt} + y_a g + R p_a = 0, \quad x_b x_{tt} + y_b y_{tt} + y_b g + R p_b = 0,$$

$$x_a y_b - x_b y_a = S(a, b), \tag{18}$$

where $1/R = \rho_0(a, b)$ is an initial density and $S = \alpha_a \beta_b - \alpha_b \beta_a$ is the Jacobian of the map $(a, b) \to (\alpha(a, b), \beta(a, b))$.

The system (18) admits the subgroup $< \frac{1}{c}\partial_t - \partial_a - \partial_x >$ when $R = R(b)$ and $S = S(b)$. It is easy to verify that the invariant solution has the form

$$x = a + \frac{1}{k} \exp(kb) \sin k(a + ct), \quad y = b + \frac{1}{k} \exp(kb) \cos k(a + ct),$$

$$p = g \int_{b_0}^{b} \frac{1}{R(b)} (e^{2kb} - 1) db, \tag{19}$$

where k, c, b_0 are arbitrary constants, $c^2 = g/k$. The solution (19) describes famous Gerstner's waves on a free surface of nonhomogeneous fluid.

References

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