# Perturbed Lie Symmetry and Systems of Non-Linear Diffusion Equations

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### Abstract

The method of one parameter, point symmetric, approximate Lie group invariants is applied to the problem of determining solutions of systems of pure one-dimensional, diffusion equations. The equations are taken to be non-linear in the dependent variables but otherwise homogeneous. Moreover, the matrix of diffusion coefficients is taken to differ from a constant matrix by a linear perturbation with respect to an infinitesimal parameter. The conditions for approximate Lie invariance are developed and are applied to the coupled system. The corresponding prolongation operator is derived and it is shown that this places a power law and logarithmic constraints on the nature of the perturbed diffusion matrix. The method is used to derive an approximate solution of the perturbed diffusion equation corresponding to impulsive boundary conditions.

## 1   Introduction

Our interest in diffusion lies in the extension of the Richards equation, which describes the movement of water in a homogeneous unsaturated soil, to cases describing the combined transport of water vapour, heat and solute under a combination of gradients of soil temperature, volumetric water content and solute concentration. The theory was formulated by Philip and De Vries in [1] and [2] and is presented here in the form of Jury *et al* [3] as applied to the case of pure coupled diffusion, so that gravitational advection terms are omitted. In particular, for a vertical column of soil:

$$Z\left(\mathbf{u}\right) = \frac{\partial \mathbf{Y}}{\partial t} - \frac{\partial}{\partial x}\left(K\left(\mathbf{Y}\right)\frac{\partial \mathbf{Y}}{\partial x}\right) = 0, \tag{1}$$

where:

$$\mathbf{u} \equiv \left(x, t, \mathbf{Y}, \mathbf{Y}', \mathbf{Y}'', \dot{\mathbf{Y}}\right) \tag{2}$$

and where $\mathbf{Y}\left(x, t\right)$ is a vector $\{y_i\left(x, t\right)\}$ of soil temperature, moisture content and solute concentration values as a function of soil depth $x$ and time $t$. In addition, the dash and dot indicate derivatives with respect to depth and time, respectively. Furthermore, $K\left(\mathbf{Y}\right)$ is a matrix, often diagonally dominant, expressing the homogeneous, but non-linear diffusive properties of the medium.

Our main aim here is to extend our mathematical understanding of equation (1) by exploring its approximate Lie symmetry group properties. In a recent paper, Wiltshire [4], developed classical Lie symmetries of (1) with $K(\mathbf{Y})$ constant and also for non-linear cases with:

$$K(\mathbf{Y}) = A \prod_i (y_i + \beta_i)^{R_i}, \tag{3}$$

where $A, \beta_i, R_i$ are constants. In fact, this is the only non-linear case admitting classical symmetry and further, there are no examples of non-classical symmetry as defined by Hill [5] .

In this paper, we consider the question of whether a choice of diffusion matrix $K(\mathbf{Y})$ which differs only slightly from a constant matrix $\Lambda$ will approximately admit symmetries which are of a different character from those given in [4]. In particular, we consider equation (1) with

$$K(\mathbf{Y}) = \Lambda + \varepsilon\lambda(\mathbf{Y}), \tag{4}$$

where $\varepsilon$ is a small parameter and we use the theory of Fushchych and Shtelen [6], and Baikov *et al* [7, 8] to determine the approximate Lie point symmetries. In this theory, the classical infinitesimal Lie generator for a given partial differential equation with $\varepsilon = 0$ is assumed to be known. The generator is then perturbed with respect to the small parameter $\varepsilon$ with a view to determining symmetries of the modified partial differential equation. We will consider first-order perturbations only in the analysis.

In the context of our problem, the partial differential equation with the known Lie point symmetry is

$$\mathbf{Z}_0(\mathbf{v}) = \frac{\partial \mathbf{Y}_0}{\partial t_0} - \Lambda\frac{\partial^2 \mathbf{Y}_0}{\partial x_0^2} = 0, \tag{5}$$

where

$$\mathbf{v} \equiv \left(x_0, t_0, \mathbf{Y}_0, \mathbf{Y}_0', \mathbf{Y}_0'', \dot{\mathbf{Y}}_0\right) \tag{6}$$

and where it will be supposed that $\mathbf{v}$ and $\mathbf{u}$ differ by a first-order term in $\varepsilon$ as folllows:

$$\mathbf{u} = \mathbf{v} + \varepsilon\mathbf{v}_1 + o(\varepsilon). \tag{7}$$

The infinitesimal generator for the Lie symmetries is given by Wiltshire [4] . Equation (1) with (5) may be written in the form:

$$\mathbf{Z}(\mathbf{u}) = \mathbf{Z}_0(\mathbf{u}) + \varepsilon\mathbf{Z}_1(\mathbf{u}) = 0, \tag{8}$$

where

$$\mathbf{Z}_1(\mathbf{u}) = -\frac{\partial}{\partial x}\left(\lambda(\mathbf{Y})\frac{\partial \mathbf{Y}}{\partial x}\right). \tag{9}$$

Before describing the symmetry properties of (1), it is first necessary to define the meaning of approximate point transformations.

## 2   First-Order Approximate Lie Point Group Transformations

The invariance properties of equation (1) will be analysed by defining the approximate one-parameter group of transformations through:

$$
\begin{aligned}
x_1 &\equiv f\left(x,t,\mathbf{Y},a,\varepsilon\right) = f_0\left(x,t,\mathbf{Y},a\right) + \varepsilon f_1\left(x,t,\mathbf{Y},a\right) + o\left(\varepsilon\right), \\
t_1 &\equiv g\left(x,t,\mathbf{Y},a,\varepsilon\right) = g_0\left(x,t,\mathbf{Y},a\right) + \varepsilon g_1\left(x,t,\mathbf{Y},a\right) + o\left(\varepsilon\right), \\
\mathbf{Y}_1 &\equiv \mathbf{H}\left(x,t,\mathbf{Y},a,\varepsilon\right) = \mathbf{H}_0\left(x,t,\mathbf{Y},a\right) + \varepsilon\mathbf{H}_1\left(x,t,\mathbf{Y},a\right) + o\left(\varepsilon\right),
\end{aligned}
\tag{10}
$$

where $a$ is a group parameter, so that for example, in the case of (10)

$$
f\left(x,t,\mathbf{Y},0\right) = x, \qquad f\left(f\left(x,t,\mathbf{Y},a\right),t,\mathbf{Y},b\right) = f\left(x,t,\mathbf{Y},a+b\right).
\tag{11}
$$

Similar rules also hold for second and third of (10). In addition, the familiar infinitesimal generator will be written in the form:

$$
\mathcal{L} = \xi\left(x,t,\mathbf{Y},\varepsilon\right)\frac{\partial}{\partial x} + \eta\left(x,t,\mathbf{Y},\varepsilon\right)\frac{\partial}{\partial t} + \pi\left(x,t,\mathbf{Y},\varepsilon\right)\cdot\nabla,\ \ \nabla \equiv \left\{\frac{\partial}{\partial y_i}\right\},
\tag{12}
$$

but for approximate transformations we write:

$$
\begin{aligned}
\xi\left(x,t,\mathbf{Y},\varepsilon\right) &= \xi_0\left(x,t,\mathbf{Y}\right) + \varepsilon\xi_1\left(x,t,\mathbf{Y}\right) + o\left(\varepsilon\right), \\
\eta\left(x,t,\mathbf{Y},\varepsilon\right) &= \eta_0\left(x,t,\mathbf{Y}\right) + \varepsilon\eta_1\left(x,t,\mathbf{Y}\right) + o\left(\varepsilon\right), \\
\pi\left(x,t,\mathbf{Y},\varepsilon\right) &= \pi_0\left(x,t,\mathbf{Y}\right) + \varepsilon\pi_1\left(x,t,\mathbf{Y}\right) + o\left(\varepsilon\right).
\end{aligned}
\tag{13}
$$

Moreover, by direct analogy with Lie's second theorem we can link the global and infinitesimal (expanded about $a = 0$) forms of the transformation (10) to (13) and obtain:

$$
\frac{dx_1}{da} = \xi\left(x_1,t_1,\mathbf{Y}_1,\varepsilon\right), \qquad \frac{dt_1}{da} = \eta\left(x_1,t_1,\mathbf{Y}_1,\varepsilon\right), \qquad \frac{d\mathbf{Y}_1}{da} = \pi\left(x_1,t_1,\mathbf{Y}_1,\varepsilon\right)
\tag{14}
$$

with $x = x_1,\ t = t_1, \pi = \pi_1$ at $a = 0$.

## 3   Conditions for First-Order Approximate Invariance

In this section, we describe the general relationships which must hold for approximate invariance of (1). This necessitates a description of the approximate prolongation operator.

First note that substitution of equations (7) into (8) gives rise to:

$$
\mathbf{Z}\left(\mathbf{u}\right) \equiv \mathbf{Z}_0\left(\mathbf{v}\right) + \varepsilon\left[\mathbf{Z}_1\left(\mathbf{v}\right) + \left(\mathbf{v}_1\cdot\mathcal{D}\right)\mathbf{Z}_0\left(\mathbf{v}\right)\right] + o\left(\varepsilon\right) = 0,
\tag{15}
$$

where

$$
\mathcal{D} = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial t},\nabla,\nabla_{Y'},\nabla_{\dot{Y}},\nabla_{Y''}\right),
\tag{16}
$$

$$
\nabla_{Y'} = \left\{\frac{\partial}{\partial y_i'}\right\},\ \nabla_{Y''} = \left\{\frac{\partial}{\partial y_i''}\right\},\ \nabla_{\dot{Y}} = \left\{\frac{\partial}{\partial \dot{y}_i}\right\}.
\tag{17}
$$

Clearly, from equation (5) it follows that:

$$
\mathbf{Z}_0\left(\mathbf{v}\right) = 0, \qquad\qquad \mathbf{Z}_1\left(\mathbf{v}\right) + \left(\mathbf{v}_1\cdot\mathcal{D}\right)\mathbf{Z}_0\left(\mathbf{v}\right) = 0.
\tag{18}
$$

Symmetry of equation (1) may be expressed with the aid of the prolongation operator:

$$
\begin{aligned}
\mathcal{P}\left(\mathbf{u}\right) \;\equiv\; & \xi\left(\mathbf{u},\varepsilon\right)\frac{\partial}{\partial x}+\eta\left(\mathbf{u},\varepsilon\right)\frac{\partial}{\partial t}+\pi\left(\mathbf{u},\varepsilon\right)\cdot\nabla+ \\
& \pi_x\left(\mathbf{u},\varepsilon\right)\cdot\nabla_{Y'}+\pi_t\left(\mathbf{u},\varepsilon\right)\cdot\nabla_{\dot{Y}}+\pi_{xx}\left(\mathbf{u},\varepsilon\right)\cdot\nabla_{Y''} \quad,
\end{aligned}
\tag{19}
$$

which may also be written in the form of an expansion about $\varepsilon$:

$$
\begin{aligned}
\mathcal{P}\left(\mathbf{u}\right) \;\equiv\; & \mathcal{P}_0\left(\mathbf{u}\right)+\varepsilon\mathcal{P}_1\left(\mathbf{u}\right)+o\left(\varepsilon\right) \\
= \; & \mathcal{P}_0\left(\mathbf{v}\right)+\varepsilon\left[\mathcal{P}_1\left(\mathbf{v}\right)+\left(\left(\mathbf{v}_1\cdot\mathcal{D}\right)\mathbf{p}_0\left(\mathbf{v}\right)\right)\cdot\mathcal{D}\right]+o\left(\varepsilon\right),
\end{aligned}
\tag{20}
$$

where

$$
\begin{aligned}
\mathcal{P}_i\left(\mathbf{v}\right) \;\equiv\; & \mathbf{p}_i\left(\mathbf{v}\right)\cdot\mathcal{D} \\
\mathbf{p}_i\left(\mathbf{v}\right) \;\equiv\; & \left[\xi\left(\mathbf{v}\right),\eta\left(\mathbf{v}\right),\pi\left(\mathbf{v}\right),\pi_x\left(\mathbf{v}\right),\pi_t\left(\mathbf{v}\right),\pi_{xx}\left(\mathbf{v}\right)\right]_i \quad,
\end{aligned}
\tag{21}
$$

and $i=0,1$. In this case, $\mathcal{P}_0\left(\mathbf{v}\right)$ is the prolongation operator generating the classical Lie symmetry of equation $(5)$.

The components of $\mathbf{p}_i\left(\mathbf{v}\right)$ may be calculated in the usual way with the aid of:

$$
\begin{aligned}
\pi_x \;=\; & \pi'-\dot{\mathbf{Y}}\eta'+\mathbf{Y}'\cdot\nabla\pi-\mathbf{Y}'\xi'-\mathbf{Y}'\left(\mathbf{Y}'\cdot\nabla\right)\xi-\dot{\mathbf{Y}}\left(\mathbf{Y}'\cdot\nabla\right)\eta, \\
\pi_t \;=\; & \dot{\pi}-\mathbf{Y}'\dot{\xi}+\dot{\mathbf{Y}}\cdot\nabla\pi-\dot{\mathbf{Y}}\dot{\eta}-\mathbf{Y}'\left(\dot{\mathbf{Y}}\cdot\nabla\right)\xi-\dot{\mathbf{Y}}\left(\dot{\mathbf{Y}}\cdot\nabla\right)\eta,
\end{aligned}
\tag{22}
$$

and

$$
\begin{aligned}
\pi_{xx} \;=\; & \pi''+2\mathbf{Y}'\cdot\nabla\pi'-\mathbf{Y}'\xi''-\dot{\mathbf{Y}}\eta''+\left(\mathbf{Y}'\cdot\nabla\right)^2\pi-2\mathbf{Y}'\left(\mathbf{Y}'\cdot\nabla\right)\xi'- \\
& 2\dot{\mathbf{Y}}\left(\mathbf{Y}'\cdot\nabla\right)\eta'-\mathbf{Y}'\left(\mathbf{Y}'\cdot\nabla\right)^2\xi-\dot{\mathbf{Y}}\left(\mathbf{Y}'\cdot\nabla\right)^2\eta- \\
& 2\dot{\mathbf{Y}}'\left(\eta'+\mathbf{Y}'\cdot\nabla\eta\right)+\mathbf{Y}''\cdot\left(\nabla\pi-\nabla\xi\mathbf{Y}'-\nabla\eta\dot{\mathbf{Y}}\right)- \\
& 2\xi'\mathbf{Y}''-2\left(\mathbf{Y}'\cdot\nabla\xi\right)\mathbf{Y}''.
\end{aligned}
\tag{23}
$$

The condition for first-order approximate invariance may be found by applying the prolongation operator to the differential equation (1) with the result that

$$
\begin{aligned}
\mathcal{P}\left(\mathbf{u}\right)\mathbf{Z}\left(\mathbf{u}\right) \;=\; & 0=\mathcal{P}_0\left(\mathbf{v}\right)\mathbf{Z}_0\left(\mathbf{v}\right)+\varepsilon\left[\mathcal{P}_1\left(\mathbf{v}\right)\mathbf{Z}_0\left(\mathbf{v}\right)+\right. \\
& \left(\left(\mathbf{v}_1\cdot\mathcal{D}\right)\mathbf{p}_0\left(\mathbf{v}\right)\right)\cdot\mathcal{D}\mathbf{Z}_0\left(\mathbf{v}\right)+\mathcal{P}_0\left(\mathbf{v}\right)\mathbf{Z}_1\left(\mathbf{v}\right)+ \\
& \left.\mathcal{P}_0\left(\mathbf{v}\right)\left(\left(\mathbf{v}_1\cdot\mathcal{D}\right)\mathbf{Z}_0\left(\mathbf{v}\right)\right)\right]+o\left(\varepsilon\right).
\end{aligned}
\tag{24}
$$

It follows, by equating coefficients of $\varepsilon$, that approximate invariance occurs whenever:

$$
\mathcal{P}_0\left(\mathbf{v}\right)\mathbf{Z}_0\left(\mathbf{v}\right)=0,
\tag{25}
$$

$$
\mathcal{P}_1\left(\mathbf{v}\right)\mathbf{Z}_0\left(\mathbf{v}\right)+\left(\mathbf{v}_1\cdot\mathcal{D}\right)\mathcal{P}_0\left(\mathbf{v}\right)\mathbf{Z}_0\left(\mathbf{v}\right)+\mathcal{P}_0\left(\mathbf{v}\right)\mathbf{Z}_1\left(\mathbf{v}\right)=0.
\tag{26}
$$

Equation (25) is, of course, the condition used to determine classical symmetry.

## 4    Equations for First-Order Lie Symmetry

Following the results given by Wiltshire [4], equation (25) has been solved for the linear coupled diffusion equation (5) with result that classical Lie symmetries may be summarized in two cases:

**Case (i):** $\Gamma\left(x,t\right)=\left(\kappa+\beta x\right)\Lambda^{-1}$,

$$\mathcal{P}_0\left(\mathbf{v}\right)=\left(\lambda+\frac{\mu}{2}x-2\beta t\right)\,_{\bar{\partial}}\partial x+\left(\alpha+\mu t\right)\,_{\bar{\partial}}\partial t+\Gamma\mathbf{Y}\cdot\nabla+$$
$$\mathbf{a}_x\cdot\nabla_{Y'}+\mathbf{a}_t\cdot\nabla_{\dot{Y}}+\mathbf{a}_{xx}\cdot\nabla_{Y''}, \tag{27}$$

where $\kappa,\beta,\alpha,\lambda$ and $\mu$ are constant.

**Case (ii):** $\Gamma\left(x,t\right)=qI$,where $q$ is constant. The corresponding generator may be obtained from (27) by setting $\beta=0$.

In both cases, it follows that

$$\mathbf{a}_x=\Gamma'\mathbf{Y}+\left(\Gamma-\frac{\mu}{2}\right)\mathbf{Y}',\ \ \mathbf{a}_t=2\beta\mathbf{Y}'+\left(\Gamma-\mu\right)\dot{\mathbf{Y}}, \tag{28}$$

$$\mathbf{a}_{xx}=2\Gamma'\mathbf{Y}'+\left(\Gamma-\mu\right)\mathbf{Y}''. \tag{29}$$

provided that $\Gamma'$ is set to zero in the second case. It follows that:

$$\mathcal{P}_0\left(\mathbf{v}\right)\mathbf{Z}_0\left(\mathbf{v}\right)=\mathcal{P}_0\left(\mathbf{v}\right)\left\{\dot{\mathbf{Y}}-\mathbf{\Lambda}\mathbf{Y}''\right\}=\left(\Gamma-\mu\right)\left\{\dot{\mathbf{Y}}-\mathbf{\Lambda}\mathbf{Y}''\right\}. \tag{30}$$

Clearly, this confirms that when the first of equations (18) is satisfied, then (25) also holds as required.

It follows from substitution of (30) and (18) into (25) that:

$$\mathcal{P}_1\left(\mathbf{v}\right)\mathbf{Z}_0\left(\mathbf{v}\right)+\mathcal{P}_0\left(\mathbf{v}\right)\mathbf{Z}_1\left(\mathbf{v}\right)=\left(\Gamma-\mu\right)\mathbf{Z}_1\left(\mathbf{v}\right). \tag{31}$$

The corresponding result for case (ii) may be obtained from (31) by putting $\mu=0$.

In both cases, the equation (31) may be expanded with the help of (28), (29), (5), (9) and also (22), (23) with the result that

$$\left[\pi_t-\Lambda\pi_{xx}\right]_1-\left(\Gamma\mathbf{Y}\cdot\nabla\right)\lambda\left(\mathbf{Y}\right)\mathbf{Y}''-\mathbf{Y}'\left(\Gamma\mathbf{Y}\cdot\nabla\right)\nabla\lambda\left(\mathbf{Y}\right)\mathbf{Y}'-\lambda\left(\mathbf{Y}\right)\mathbf{a}_{xx}-$$
$$\left(\mathbf{a}_x\cdot\nabla\right)\lambda\left(\mathbf{Y}\right)\mathbf{Y}'-\left(\mathbf{Y}'\cdot\nabla\right)\lambda\left(\mathbf{Y}\right)\mathbf{a}_x= \tag{32}$$
$$-\left(\Gamma-\mu\right)\left\{\lambda\left(\mathbf{Y}\right)\mathbf{Y}''+\left(\mathbf{Y}'\cdot\nabla\right)\lambda\left(\mathbf{Y}\right)\mathbf{Y}'\right\}.$$

Without loss of generality, the suffix 1 may now be dropped so that:

$$\left\{\dot{\pi}+\left(\dot{\mathbf{Y}}\cdot\nabla\right)\pi-\mathbf{Y}'\left[\dot{\xi}+\left(\dot{\mathbf{Y}}\cdot\nabla\right)\xi\right]-\dot{\mathbf{Y}}\left[\dot{\eta}+\left(\dot{\mathbf{Y}}\cdot\nabla\right)\eta\right]\right\}-$$

$$\Lambda\left\{\pi''+\left(\mathbf{Y}''\cdot\nabla\right)\pi+2\left(\mathbf{Y}'\cdot\nabla\right)\pi'+\left(\mathbf{Y}'\cdot\nabla\right)^2\pi-\right.$$

$$\mathbf{Y}'\left[\xi''+\left(\mathbf{Y}''\cdot\nabla\right)\xi+2\left(\mathbf{Y}'\cdot\nabla\right)\xi'+\left(\mathbf{Y}'\cdot\nabla\right)^2\xi\right]-2\mathbf{Y}''\left[\xi'+\left(\mathbf{Y}'\cdot\nabla\right)\xi\right]-$$

$$\dot{\mathbf{Y}}\left[\eta''+\left(\mathbf{Y}''\cdot\nabla\right)\eta+2\left(\mathbf{Y}'\cdot\nabla\right)\eta'+\left(\mathbf{Y}'\cdot\nabla\right)^2\eta\right]-2\dot{\mathbf{Y}}'\left[\eta'+\left(\mathbf{Y}'\cdot\nabla\right)\eta\right]\right\}-$$

$$\{(\Gamma \mathbf{Y} \cdot \nabla) \lambda (\mathbf{Y}) + [\lambda (\mathbf{Y}) ,\Gamma]\} \mathbf{Y}'' - (\Gamma \mathbf{Y} \cdot \nabla) (\mathbf{Y}' \cdot \nabla) \lambda \mathbf{Y}' - \qquad (33)$$

$$2\lambda \Gamma' \mathbf{Y}' - (\Gamma' \mathbf{Y} \cdot \nabla) \lambda (\mathbf{Y}) \mathbf{Y}' - (\mathbf{Y}' \cdot \nabla) \lambda (\mathbf{Y}) [\Gamma' \mathbf{Y} + \Gamma \mathbf{Y}'] +$$

$$\Gamma (\mathbf{Y}' \cdot \nabla) \lambda (\mathbf{Y}) \mathbf{Y}' - (\Gamma \mathbf{Y}' \cdot \nabla) \lambda (\mathbf{Y}) \mathbf{Y}' = \mathbf{0}.$$

## 5 Solution of Determining Equations

This equation may be solved by equating the derivatives of $\mathbf{Y}$ to give:

C1 No derivatives : $\dot{\pi} - \mathbf{\Lambda} \pi'' = 0$

C2 $\mathbf{Y}'$ :

$$-\dot{\xi} \mathbf{Y}' - \Lambda (2\mathbf{Y}' \cdot \nabla \pi' - \mathbf{Y}' \xi'') - 2\lambda \Gamma' \mathbf{Y}' - (\Gamma' \mathbf{Y} \cdot \nabla) \lambda \mathbf{Y}'$$
$$- (\mathbf{Y}' \cdot \nabla) \lambda \Gamma' \mathbf{Y} = 0$$

C3 $\dot{\mathbf{Y}}$ : $\left( \dot{\mathbf{Y}} \cdot \nabla \right) \pi - \dot{\eta} \dot{\mathbf{Y}} + \Lambda \dot{\mathbf{Y}} \eta'' = \mathbf{0}$

C4 $\mathbf{Y}', \mathbf{Y}'$ :

$$-\Lambda (\mathbf{Y}' \cdot \nabla)^2 \pi + 2\Lambda \mathbf{Y}' (\mathbf{Y}' \cdot \nabla) \xi' - (\Gamma \mathbf{Y} \cdot \nabla) (\mathbf{Y}' \cdot \nabla) \lambda \mathbf{Y}'$$
$$- (\mathbf{Y}' \cdot \nabla) \lambda \Gamma \mathbf{Y}' + \Gamma (\mathbf{Y}' \cdot \nabla) \lambda \mathbf{Y}' - (\Gamma \mathbf{Y}' \cdot \nabla) \lambda \mathbf{Y}' = \mathbf{0}$$

C5 $\mathbf{Y}', \mathbf{Y}', \mathbf{Y}'$ : $\Lambda \mathbf{Y}' (\mathbf{Y}' \cdot \nabla)^2 \xi = \mathbf{0}$

C6 $\mathbf{Y}', \dot{\mathbf{Y}}$ : $-\mathbf{Y}' \left( \dot{\mathbf{Y}} \cdot \nabla \right) \xi + 2\Lambda \dot{\mathbf{Y}} (\mathbf{Y}' \cdot \nabla) \eta' = \mathbf{0}$

C7 $\dot{\mathbf{Y}}, \dot{\mathbf{Y}}$ : $- \dot{\mathbf{Y}} \left( \dot{\mathbf{Y}} \cdot \nabla \right) \eta = \mathbf{0}$

C8 $\dot{\mathbf{Y}}, \mathbf{Y}', \mathbf{Y}'$ : $\Lambda \dot{\mathbf{Y}} (\mathbf{Y}' \cdot \nabla)^2 \eta = \mathbf{0}$

C9 $\mathbf{Y}''$ : $-\Lambda \{\mathbf{Y}'' \cdot [\nabla \pi - 2\xi']\} - [(\Gamma \mathbf{Y} \cdot \nabla) \lambda + [\lambda \quad \Gamma]] \mathbf{Y}'' = \mathbf{0}$

C10 $\mathbf{Y}'', \dot{\mathbf{Y}}$ : $\Lambda (\mathbf{Y}'' \cdot \nabla) \eta \dot{\mathbf{Y}} = \mathbf{0}$

C11 $\mathbf{Y}'', \mathbf{Y}'$ : $\Lambda \{\mathbf{Y}'' \cdot [\nabla \xi \mathbf{Y}' + 2 (\mathbf{Y}' \cdot \nabla) \xi]\} = \mathbf{0}$

C12 $\dot{\mathbf{Y}}'$ : $2\Lambda \dot{\mathbf{Y}}' \eta' = \mathbf{0}$

C13 $\dot{\mathbf{Y}}', \mathbf{Y}'$ : $2\Lambda \dot{\mathbf{Y}}' (\mathbf{Y}' \cdot \nabla) \eta = \mathbf{0}$.

These overdetermined equations, now including the suffix 1, may be solved by observing that conditions C5-C8 and C10-C13 are uniquely satisfied by $\eta_1 = \eta_1 (t)$ and $\xi_1 = \xi_1 (x, t)$. In addition, from C3:

$$\pi_1 = \dot{\eta}_1 \mathbf{Y} + \mathbf{b} (x, t), \qquad (34)$$

where $\mathbf{b} (x, t)$ is an arbitrary function of $x$ and $t$. Condition C9 may be solved with the help of the following transformations:

$$\mathbf{Y} = P\mathbf{X}, \qquad \Gamma = PUP^{-1}, \qquad \nabla_X = P^T \nabla, \qquad \nabla_X = \left\{ \frac{\partial}{\partial x_i} \right\}, \qquad (35)$$

where the defined transformation produces the diagonalized matrix $U$ with nonzero elements $u_i$. It also follows that

$$(U\mathbf{X}) \cdot \nabla_X = (\Gamma \mathbf{Y}) \cdot \nabla. \qquad (36)$$

Two distinct cases arise and these impose restrictions on the form of the matrix $\lambda\left(\mathbf{Y}\right)$:

    **(i):** $\dot{\eta}_1 - 2\xi_1' = 0$, any $\Gamma$ :

$$\lambda = A \prod_i x^{r_i} + B\left(x,t\right), \qquad \sum_i u_i r_i = 0,$$
$$\left[\Lambda, A\right] = 0, \qquad \left[\Lambda, B\right] = 0, \tag{37}$$

where $\{r_i\}$ are constants and $A, B\left(x,t\right)$ are matrices where the former is constant and the latter varies with $x, t$.

    **(ii):** $\dot{\eta}_1 - 2\xi_1' \neq 0$, $\Gamma = \Lambda^{-1}\left(\kappa + \beta x\right)$:

$$\lambda\left(\mathbf{Y}\right) = \Lambda \sum_i r_i \ln y_i + B\left(x,t\right), \qquad \dot{\eta}_1 - 2\xi_1' + \sum_i u_i r_i = 0, \qquad \left[\Lambda, B\right] = 0. \tag{38}$$

No further restrictions are imposed as a result of C4. However, consideration of C1 followed by C2 shows that $\beta = 0$, and in addition:

$$\Gamma = \kappa \Lambda^{-1}, \qquad \eta_1\left(t\right) = \bar{\alpha} + \bar{\mu} t, \qquad \dot{\mathbf{b}} - \Lambda \mathbf{b}'' = 0,$$
$$\xi_1\left(x,t\right) = \frac{\left(\bar{\mu} + \sum_i u_i r_i\right)}{2} x + \bar{q}. \tag{39}$$

Finally, the approximate Lie symmetry vector field or infinitesimal generator may be found using:

$$\mathcal{L}\left(x, t, \mathbf{Y}, \varepsilon\right) = \left[\xi_0 + \varepsilon \xi_1\right] \frac{\partial}{\partial x} + \left[\eta_0 + \varepsilon \eta_1\right] \frac{\partial}{\partial t} + \left[\pi_0 + \varepsilon \pi_1\right] \cdot \nabla + o\left(\varepsilon\right), \tag{40}$$

$$\Longrightarrow \mathcal{L}\left(x, t, \mathbf{Y}, \varepsilon\right) = \left[\left(\lambda + \varepsilon \bar{q}\right) + \left(\frac{\mu + \varepsilon\left(\bar{\mu} + \sum_i u_i r_i\right)}{2}\right) x\right] \frac{\partial}{\partial x} + \tag{41}$$

$$\left[\left(\alpha + \varepsilon \bar{\alpha}\right) + \left(\mu + \varepsilon \bar{\mu}\right) t\right] \frac{\partial}{\partial t} + \left[\Gamma \mathbf{Y} + \varepsilon \mathbf{b}\left(x,t\right)\right] \cdot \nabla + o\left(\varepsilon\right).$$

## 6   Solution of a Coupled Diffusion Equation

By way of a particular example, we consider (41) with:

$$\lambda + \varepsilon \bar{q} = 0, \alpha + \varepsilon \bar{\alpha} = 0, \bar{\mu} = 0, \mathbf{b}\left(x,t\right) = \mathbf{0}, \tag{42}$$
$$\Gamma = qI, \quad P = I, \quad u_i = q, \quad \sum_i r_i = 0.$$

The corresponding differential equation is:

$$\frac{\partial \mathbf{Y}}{\partial t} = \frac{\partial}{\partial x} \left\{ \left(\Lambda + \varepsilon A \prod_i y_i^{r_i}\right) \frac{\partial \mathbf{Y}}{\partial x} \right\}. \tag{43}$$

The symmetry vector field for (43) is:

$$\mathcal{L} = \frac{\mu}{2}x\frac{\partial}{\partial x} + \mu t\frac{\partial}{\partial t} + q\mathbf{Y}\cdot\nabla, \tag{44}$$

so that the global transformation (14) is defined through:

$$\frac{dx_1}{da} = \frac{\mu}{2}x_1, \qquad \frac{dt_1}{da} = \mu t_1, \qquad \frac{d\mathbf{Y}_1}{da} = q\mathbf{Y}_1, \tag{45}$$

which means that:

$$x_1 = x\exp\frac{\mu}{2}a, \qquad t_1 = t\exp\mu a, \qquad \mathbf{Y}_1 = \mathbf{Y}\exp qa. \tag{46}$$

It follows that the similarity variable $\omega$ and similarity ansatz are given by:

$$\omega = \frac{x}{t^{\frac{1}{2}}}, \qquad \mathbf{Y}(x,t) = t^{\frac{q}{\mu}}\mathbf{\Phi}(\omega), \tag{47}$$

and so (43) becomes:

$$\frac{q}{\mu}\mathbf{\Phi} - \frac{\omega}{2}\frac{d\mathbf{\Phi}}{d\omega} = \frac{d}{d\omega}\left\{\left(\Lambda + \varepsilon A\prod_i \phi_i^{r_i}\right)\frac{\partial\mathbf{\Phi}}{\partial\omega}\right\}, \tag{48}$$

where $\phi_i \in \mathbf{\Phi}$. This may be solved by writing:

$$\mathbf{\Phi} = \mathbf{\Phi}_0 + \varepsilon\mathbf{\Phi}_1 + o(\varepsilon), \tag{49}$$

and solving:

$$\frac{q}{\mu}\mathbf{\Phi}_0 - \frac{\omega}{2}\frac{d\mathbf{\Phi}_0}{d\omega} = \Lambda\frac{d^2\mathbf{\Phi}_0}{d\omega^2}, \tag{50}$$

$$\frac{q}{\mu}\mathbf{\Phi}_1 - \frac{\omega}{2}\frac{d\mathbf{\Phi}_1}{d\omega} = \Lambda\frac{d^2\mathbf{\Phi}_1}{d\omega^2} + \frac{d}{d\omega}\left\{A\prod_i(\phi_0)_i^{r_i}\frac{\partial\mathbf{\Phi}_0}{\partial\omega}\right\}. \tag{51}$$

If we take the particular case $\mu = -2q$, then it is easy to show that

$$\mathbf{\Phi}_0(\omega) = \exp\left(-\frac{\Lambda^{-1}\omega^2}{4}\right)\mathbf{b}, \tag{52}$$

$$\mathbf{\Phi}_0(\omega) = \exp\left(-\frac{\Lambda^{-1}\omega^2}{4}\right)\left\{\left(\frac{\Lambda^{-1}\omega}{2}\right)^2 A\prod_i b_i^{r_i}\mathbf{b}\right\} + \mathbf{c} \tag{53}$$

where $\mathbf{b}, \mathbf{c}$ are constant and $b_i \in \mathbf{b}$.

Finally, it may be noted that the initial condition

$$\mathbf{Y}(x,0) = \mathbf{Y}_0\delta(x), \tag{54}$$

where the Dirac delta-function satifies:

$$\delta(\lambda x) = \lambda^{-1}\delta(x), \tag{55}$$

has a transformation rule identical to $(46)$. It follows that our solution satisfies an impulsive initial condition.

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