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Parasupersymmetric Quantum Mechanics with Arbitrary p and N

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Abstract

The generalization of parasupersymmetric quantum mechanics generated by an arbitrary number of parasupercharges and characterized by an arbitrary order of paraquantization is given. The relations for parasuperpotentials are obtained. It is shown that parasuperpotentials can be explicitly expressed via one arbitrary function.

The notion of parasupersymmetry as symmetry between particles which obey parastatistics of different orders was introduced in [1]. Such a symmetry is peculiar to quantum-mechanical particles with high spins in magnetic field. Parasupersymmetry is described not by ordinary Lie algebras or their supersymmetric extentions but by the polynomial algebras, which were called parasuperalgebras [2].

The physical theory with supersymmetry which was called parasupersymmetric quantum mechanics (PSSQM) was proposed in [1], independent version was worked out in [3]. The results of [1] were generalized to arbitrary order p [4–7].

In the present paper, we generalize PSSQM proposed in [3] to the case of an arbitrary number of parasupercharges N and arbitrary order of paraquantization p.

Let us note that the model of PSSQM of order p which was proposed by Beckers-Debergh for $N = 2$ is characterized by the following parasuperalgebra:

$$[\,H, Q_a] = 0, \tag{1a}$$

$$[\,Q_a, [\,Q_b, Q_c]\,] = \delta_{ab} Q_c H - \delta_{ac} Q_b H, \tag{1b}$$

$$(Q_a \pm i Q_b)^{p+1} = 0, \tag{1c}$$

where H is a Hamiltonian, Q_1, Q_2 are Hermitian parasupercharges, $a, b, c = 1, 2$. The generalization of PSSQM to the case of N parasupercharges was proposed in [8], where the generalized parasuperalgebra had the form (1) with $a, b, c = 1, \ldots, N$.

We shall show how many superpotentials we can introduce for any N and p and we shall obtain relations which are satisfied by these superpotentials. We shall also prove the statement that superpotentials can be explicitly expressed via one arbitrary function, and this statement is true either for the Beckers–Debergh approach [3] or for the Rubakov–Spiridonov one [1].

Let us first consider the case $N = 2$. For $p = 2$ the results of this problem are known [3, 9]. Generalization to the case of arbitrary p is proposed to be done in the following way. As so-called $N = 2$ — PSSQM is closely related to the Lie algebra $so(3)$ [8], then we choose parasupercharges Q_1 and Q_2 in the form:

$$Q_1 = \frac{1}{\sqrt{2}}((S_1 + S_2)P + i(S_1\eta W(x) + S_2\eta\widetilde{W}(x))), \tag{2a}$$

$$Q_2 = \frac{1}{\sqrt{2}}((S_1 - S_2)P + i(S_1\eta W(x) - S_2\eta\widetilde{W}(x))), \tag{2b}$$

where $S_1,\ S_2$ are so-called "ladder" matrices of a $so(3)$ representation, in other words,

$$S_1 = \frac{1}{2}\sum_{j=1}^{p}\sqrt{jp - j(j-1)}e_{j,j-1}, \tag{3a}$$

$$S_2 = \frac{1}{2}\sum_{j=1}^{p}\sqrt{jp - j(j-1)}e_{j-1,j}, \tag{3b}$$

$W(x) = \mathrm{diag}\,(0, W_1(x), \ldots, W_p(x))$, $\widetilde{W}(x) = \mathrm{diag}\,(W_1(x), \ldots, W_p(x), 0)$ are matrices of dimension $(p+1)\times(p+1)$, $W_1(x)$, $\ldots$, $W_p(x)$ are parasuperpotentials, $e_{j,k}$ is a matrix of dimension $(p+1)\times(p+1)$, which has zeros everywhere except the j-row and k-column intersection, $P = -i\dfrac{\partial}{\partial x}$, p is the order of paraquantization. We also demand for the matrix η satisfy the conditions

$$\{\eta, S_a\} = 0, \quad a = 1, 2 \tag{4a}$$

and

$$\eta^2 = 1. \tag{4b}$$

Then the Hamiltonian

$$H = \mathrm{diag}\,(H_1, H_2, \ldots, H_{p+1}), \tag{5}$$

where

$$H_r = \frac{1}{2}(P^2 + (-1)^{r+1}W_r' + W_r^2), \quad r = 1, 2, \ldots, p, \tag{6a}$$

$$H_{p+1} = \frac{1}{2}(P^2 + (-1)^{p}W_p' + W_p^2), \tag{6b}$$

commutes with the parasupercharges Q_1 and Q_2, and the following relations must be true

$$(-1)^r W_r^2 + W_r' = (-1)^r W_{r+1}^2 + W_{r+1}', \quad r = 1, \ldots, p-1. \tag{7}$$

Let us note that the potentials $W_1(x) = W_2(x) = \ldots = W_p(x) = W(x)$ satisfy the relations (7). If $W(x) = \omega x$ (oscillator-like interaction) and the matrices S_1, S_2 are chosen in realization (3), then the Hamiltonian has the following form:

$$H = \frac{1}{2}(P^2 + \omega^2 x^2 + \eta\omega), \tag{8}$$

where

$$\eta = \mathrm{diag}\,(1, -1, 1, \ldots, (-1)^p). \tag{9}$$

It is not difficult to obtain the spectrum of H:

$$\begin{array}{lll} E_n = \omega(n+1) & \text{for} \quad \eta = 1, & \\ E_n = \omega n & \text{for} \quad \eta = -1, & n = 0, 1, 2, \ldots \end{array} \tag{10}$$

If $p = 2$, then the Hamiltonian (8) can be interpreted as one of a three-level system and one boson mode [10]. In this case $\eta = \text{diag}\,(1, -1, 1)$, and such a configuration corresponds to the $V(\Lambda)$-type of three-level systems [10, 11].

Let us remind that for the generalized PSSQM of Rubakov–Spiridonov for arbitrary p (with the corresponding choice of parasupercharges) the relations on the potentials have the form [6, 7]:

$$(-1)^r W_r^2 + W_r' + c_r = (-1)^r W_{r+1}^2 + W_{r+1}' + c_{r+1}, \quad r = 1, \ldots, p-1. \tag{11}$$

Statement. *Superpotentials $W_1, \ldots, W_p$, which satisfy conditions (7) or (11), can be explicitly expressed via one arbitrary function.*

The statement is proved by induction. Let us only note that for the case $p = 2$ two superpotentials can be expressed in the form:

$$W_1 = \frac{u' + u^2 + c_1 - c_2}{2u}, \qquad W_2 = \frac{u' - u^2 + c_1 - c_2}{2u}, \tag{12}$$

where $u = W_1 - W_2 \neq 0$ (see also [12]). Three potentials can be expressed in the form (12) and

$$W_3 = \frac{-v' - v^2 + c_3 - c_2}{2v}, \tag{12a}$$

where

$$u = \frac{-a' + \varepsilon\sqrt{(a')^2 - 4((c_1 - c_2)a - a^2)(c_2 - c_3 - a)}}{2(c_2 - c_3 - a)}, \quad \varepsilon = \pm 1, \tag{13a}$$

$$v = \frac{a}{u}, \tag{13b}$$

if $W_1 - W_2 \neq 0,\ W_2 - W_3 \neq 0,\ a$ is an arbitrary function.

Let us now consider the case $N = 3$. The parasuperalgebra has the form (1), where $a, b, c = 1, 2, 3$. We search for parasupercharges Q_1, Q_2, Q_3 in the form:

$$Q_a = \frac{1}{\sqrt{2}} A_a (P + i\eta W_a(x)), \quad a = 1, 2, 3 \tag{14}$$

where A_a [8] are matrices of a representation of the $so(4)$ algebra, the matrix η satisfies (4).

For the case $p = 2$, if we choose the parasupercharges in the form (14), where A_a are 4×4 or 6×6 matrices, which realize a representation of $so(4)$, then we obtain the following conditions on the potentials W_1, W_2, W_3:

$$W_1' - W_1^2 = W_2' - W_2^2 = W_3' - W_3^2. \tag{15}$$

If A_a realize the representation $D(\frac{1}{2}, \frac{1}{2})$ of $so(4)$

$$A_1 = S_{41} = i(e_{4,1} + e_{1,4}),$$

$$A_2 = S_{42} = i(e_{4,2} + e_{2,4}), \qquad (16)$$
$$A_3 = S_{43} = i(e_{4,3} + e_{3,4}),$$

than the matrix η has the form

$$\eta = \mathrm{diag}\,(1, 1, 1, -1). \qquad (17)$$

Let us consider the oscillatorlike interaction: $W_1(x) = W_2(x) = W_3(x) = \omega x$. The Hamiltonian has the form

$$H = \frac{1}{2}(P^2 + \omega^2 x^2 + \eta\omega),$$

where η is the matrix (17). The spectrum of H can be easily found and it has the form (10). The Hamiltonian can be interpreted in this case as the Hamiltonian of 4-level system, besides the configuration of levels corresponds to the f-type [10].

Let us note that superpotentials W_1, W_2, W_3 which satisfy (15) are explicitly expressed via one arbitrary function u:

$$\begin{aligned} W_1 &= \frac{u'' + (u')^2}{2u'}, \qquad W_2 = \frac{u'' - (u')^2}{2u'}, \\ W_3 &= \frac{u''}{2u'} - u'\frac{1 + \exp(u)}{-1 + \exp(u)}. \end{aligned} \qquad (18)$$

Let us now consider the general case with an arbitrary number of parasupercharges N and arbitrary order of paraquantization p. We search for parasupercharges in the following form:

$$Q_a = \frac{1}{\sqrt{2}} S_{N+1,a}(P + i\eta W_a(x)), \quad a = 1, 2, \ldots, N \qquad (19)$$

where W_a are parasuperpotentials, $S_{N+1,a}$ are the generators of the algebra $so(N+1)$:

$$[S_{\mu\nu}, S_{\lambda\sigma}] = i(\delta_{\mu\lambda}S_{\nu\sigma} + \delta_{\nu\sigma}S_{\mu\lambda} - \delta_{\mu\sigma}S_{\nu\lambda} - \delta_{\nu\lambda}S_{\mu\sigma}), \qquad (20)$$

where $\delta_{\mu\nu}$ is the Kronecker symbol, $\mu, \nu, \lambda, \sigma = 1, \ldots, N+1$. Using the relations (1) $(a, b, c = 1, \ldots, N)$, we obtain the following result:

a) $W_1 = W_2 = \ldots = W_N = W$

or

b) superpotentials $W_1, \ldots, W_N$ satisfy the relations

$$S_{N+1,a}(1 - S_{N+1,b}^2)(\eta W_a' - \eta^2 W_a^2 - \eta W_b' + \eta^2 W_b^2) = 0, \qquad (21)$$
$$a, b = 1, 2, \ldots, N,$$

and the matrices $S_{N+1,a}$ satisfy the Kemmer algebra

$$S_{N+1,a}S_{N+1,b}S_{N+1,c} + S_{N+1,c}S_{N+1,b}S_{N+1,a} = \delta_{ab}S_{N+1,c} + \delta_{bc}S_{N+1,a}. \qquad (22)$$

Note, that in the case b) the superpotentials can be also expressed via one arbitrary function (the statement is analogous to the previous one).

So we see that in the general case we have one superpotential. If the matrices $S_{N+1,a}$ satisfy the Kemmer algebra (22), then we have the potentials $W_1, \ldots, W_N$, which can be explicitly expressed via one arbitrary function (in accordance with the statement).

Thus, we generalize PSSQM of Beckers–Debergh to the case of an arbitrary number of parasupercharges N and arbitrary order p, and show that in the general case as well as in the case of $N = 2, p = 2$ parasuperpotentials can be explicitly expressed via one arbitrary function.

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References

[1] Rubakov V.A. and Spiridonov V.P., On Pararelativistic Quantum Mechanics, *Mod. Phys. Lett. A,* 1988, V.3, 1337.

[2] Durand S. and Vinet L., Dynamic Parasuperalgebras of Parasupersymmetric Harmonic Oscillator, Cyclotron Motion and Morse Hamiltonian, *J. Phys. A,* 1990, V.23, 3661.

[3] Beckers J. and Debergh N., Parastatistics and Supersymmetry in Quantum Mechanics, *Nucl. Phys. B,* 1990, V.340, 767.

[4] Spiridonov V.P., Hierarchy of the Parasupersymmetric Hamiltonians in Quantum Mechanics, *J. Phys. A,* 1991, V.24, L529.

[5] Tomiya M., Comment on Generalized Parasupersymmetric Quantum Mechanics, *J. Phys. A,* 1992, V.25, 4699.

[6] Khare A., Parasupersymmetric Quantum Mechanics of Arbitrary Order, *J.Phys. A,* 1992, V.25, L749.

[7] Khare A., Parasupersymmetry in Quantum Mechanics, *J. Math. Phys.,* 1993, V.34, 1277.

[8] Debergh N. and Nikitin A.G., Parasupersymmetric Quantum Mechanics with an Arbitrary Number of Parasupercharges and Orthogonal Lie Algebras, *Helv. Phys. Acta.,* 1995, V.68, 19.

[9] Beckers J. and Debergh N., A Note on Recent Lie Parasuperstructures, *J. Phys. A,* 1990, V.23, L1073.

[10] Semenov V.V. and Chumakov S.M., Generalizations of the Superalgebra Other than the Parasuperalgebra, *Phys. Lett. B,* 1991, V.262, 451.

[11] Yoo H.I and Eberly J.H., Dynamical Theory of an Atom with Two or Three Levels Interacting with Quantized Cavity Fields, *Phys. Rep.,* 1985, V.118, 241.

[12] Beckers J., Debergh N. and Nikitin A.G., More on Parasupersymmetries of the Schrödinger Equation, *Mod. Phys. Lett. A,* 1993, V.8, 435.