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Lie Algebras of Approximate Symmetries

Rafail K. GAZIZOV

Ufa State Aviation Technical University
12 K.Marx Str., 450025 Ufa, Russia
e-mail: gazizov@mathem.uaicnit.bashkiria.su

Abstract

Properties of approximate symmetries of equations with a small parameter are discussed. It turns out that approximate symmetries form an approximate Lie algebra. A concept of approximate invariants is introduced and the algorithm of their calculating is proposed.

A concept of approximate symmetry of an equation with a small parameter and algorith of calculating such symmetries were proposed in [1] (see also [3*–7*]) Examples of the approximate symmetries show that such symmetries usually do not form a Lie algebra, but form a so–called *approximate Lie algebra* in sense of definition given in [2].

In this paper, we continue investigation of properties of approximate transformation groups and correspoding Lie algebras. In §1, the concept of the approximate Lie algebra introduced in [2] is discussed. Some properties of approximate symmetries are investigated in §2. The §3 is devoted to approximate invariants and algorithms of their calculating for one- and multiparameter groups.

The following notation is used: $z = (z^1, ..., z^N)$ is an independent variable; ε is a small parameter; all functions under consideration are assumed to be locally analytic in their arguments. We write $F(z, \varepsilon) = o(\varepsilon^p)$ if $\lim_{\varepsilon\to 0} \dfrac{F(z,\varepsilon)}{\varepsilon^p} = 0$ or, equivalently, if $F(z, \varepsilon) = \varepsilon^{p+1}\varphi(z, \varepsilon)$, where $\varphi(z, \varepsilon)$ is an analytic function defined in a neighborhood of $\varepsilon = 0$ and p is an arbitrary positive integer. If $f(z, \varepsilon) - g(z, \varepsilon) = o(\varepsilon^p)$, we write briefly $f \approx g$.

1 Approximate Lie algebras

Definition 1. A class of first-order differential operators

$$X = \xi^i(z, \varepsilon)\frac{\partial}{\partial z^i}$$

such that

$$\xi^i(z, \varepsilon) \approx \xi^i_0(z) + \varepsilon\xi^i_1(z) + \cdots + \varepsilon^p\xi^i_p(z), \quad i = 1, \ldots, N,$$

with some fixed functions $\xi^i_0(z),\ \xi^i_1(z), \ldots,\ \xi^i_p(z), \quad i = 1, \ldots, N$, is called *an approximate operator*.

Definition 2. An approximate commutator of the approximate operators X_1 and X_2 is an approximate operator denoted by $[X_1, X_2]$ and is given by

$$[X_1, X_2] \approx X_1 X_2 - X_2 X_1.$$

The approximate commutator satisfies the usual properties, namely:
a) linearity: $[aX_1 + bX_2, X_3] \approx a[X_1, X_3] + b[X_2, X_3], \quad a, b = \text{const},$
b) skew-symmetry: $[X_1, X_2] \approx -[X_2, X_1],$
c) Jacobi identity: $[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] \approx 0.$

Definition 3. A vector space L of approximate operators is called *an approximate Lie algebra of operators* if it is closed (in approximation of the given order p) under the approximate commutator, i.e., if

$$[X_1, X_2] \in L$$

for any $X_1, X_2 \in L$. Here the approximate commutator $[X_1, X_2]$ is calculated to the precision indicated.

Example. Consider the approximate (up to $o(\varepsilon)$) operators

$$X_1 = \frac{\partial}{\partial x} + \varepsilon x \frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial y} + \varepsilon y \frac{\partial}{\partial x}.$$

Their linear span is not a Lie algebra in the usual (exact) sense. For instance, the (exact) commutator

$$[X_1, X_2] = \varepsilon^2 \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \right)$$

is not a linear combination of the above operators.

However, these operators span an approximate Lie algebra in the first-order of precision.

2 Algebraic properties of approximate symmetries

Consider a one-parameter approximate group G_1 of transformations

$$z'^i \approx f^i(z, a, \varepsilon) = f^i_0(z, a) + \varepsilon f^i_1(z, a) + \cdots + \varepsilon^p f^i_p(z, a) + o(\varepsilon^p), \quad i = 1, \ldots, N, \quad (2.1)$$

in R^N ($a \in R$ is a group parameter) with the generator

$$X = \xi^i(z, \varepsilon)\frac{\partial}{\partial z^i}. \quad (2.2)$$

Definition 4. The approximate equation

$$F(z, \varepsilon) \approx 0 \quad (2.3)$$

is said to be *invariant with respect to the approximate group of transformation* (2.1) if

$$F(f(z, a, \varepsilon), \varepsilon) \approx 0 \quad (2.4)$$

for all z satisfying (2.3).

Theorem 1. *Let the function $F(z,\varepsilon) = (F^1(z,\varepsilon),\ldots,F^n(z,\varepsilon)), n < N$, satisfy the condition*

$$\operatorname{rank} F'(z,0)\Big|_{F(z,0)=0} = n,$$

where $F'(z,\varepsilon) = \|\partial F^\nu(z,\varepsilon)/\partial z^i\|$ for $\nu = 1,\ldots,n$ and $i = 1,\ldots,N$.

Then the equation (2.3) is approximately invariant under the approximate group G_1 with the generator (2.2) if and only if

$$XF(z,\varepsilon)\Big|_{(2.3)} = o(\varepsilon^p). \tag{2.5}$$

Equation (2.5) is called *the determining equation* for approximate symmetries. If the determining equation (2.5) is satisfied, we also say that X *is an approximate symmetry of equation* (2.3).

Approximate symmetries satisfy the following properties:

Theorem 2. *A set of approximate symmetries of an equation forms an approximate Lie algebra.*

Theorem 3. *If X is an approximate symmetry of some equation, then εX is also an approximate symmetry of the same equation.*

Let Lie algebra L_r of approximate symmetries be spanned by the following r approximate operators

$$\begin{aligned} X_{\alpha_0} &= X_{\alpha_0,0} + \varepsilon X_{\alpha_0,1} + \ldots + \varepsilon^p X_{\alpha_0,p},\\ X_{\alpha_1} &= \varepsilon X_{\alpha_1,0} + \ldots + \varepsilon^p X_{\alpha_1,p-1},\\ &\quad\cdot \qquad \cdot \qquad \cdot \\ X_{\alpha_p} &= \varepsilon^p X_{\alpha_p,0}. \end{aligned} \tag{2.6}$$

Here $\alpha_i = 1,\ldots,r_i,\quad r_0 + \ldots + r_p = r,\quad X_{\alpha_l,k} = \xi^i_{\alpha_l,k}(z)\dfrac{\partial}{\partial z^i}.$

Theorem 4. *The exact operators $X_{\alpha_0,0}, X_{\alpha_1,0},\ldots,X_{\alpha_l,0}$ generate an exact Lie algebra for any $l = 0,\ldots,p$. For $l = p$, it is a Lie algebra of exact symmetries of the exact equation $F(z,0) = 0$.*

Theorem 5. *The approximate operators*

$$\begin{aligned} Y_{\alpha_0} &= X_{\alpha_0,0} + \varepsilon X_{\alpha_0,1} + \ldots + \varepsilon^l X_{\alpha_0,l},\\ Y_{\alpha_1} &= X_{\alpha_1,0} + \varepsilon X_{\alpha_1,1} + \ldots + \varepsilon^l X_{\alpha_1,l},\\ &\quad\cdot \qquad \cdot \qquad \cdot \\ Y_{\alpha_{p-l}} &= X_{\alpha_{p-l},0} + \varepsilon X_{\alpha_{p-l},1} + \ldots + \varepsilon^l X_{\alpha_{p-l},l},\\ Y_{\alpha_{p-l+1}} &= \varepsilon X_{\alpha_{p-l+1},0} + \varepsilon^2 X_{\alpha_{p-l+1},1} + \ldots + \varepsilon^l X_{\alpha_{p-l+1},l-1},\\ &\quad\cdot \qquad \cdot \qquad \cdot \\ Y_{\alpha_{p-1}} &= \varepsilon^{l-1} X_{\alpha_{p-1},0} + \varepsilon^l X_{\alpha_{p-1},1},\\ Y_{\alpha_p} &= \varepsilon^l X_{\alpha_p,0} \end{aligned}$$

form an approximate (up to $o(\varepsilon^l)$) Lie algebra of approximate symmetries.

3 Approximate invariants

Consider a set of the approximate transformations $\{T_a\}$:

$$T_a : \quad z'^i \approx f^i(z,a,\varepsilon) = f_0^i(z,a) + \varepsilon f_1^i(z,a) + \cdots + \varepsilon^p f_p^i(z,a) + o(\varepsilon^p), \; i = 1,\ldots,N, \tag{3.1}$$

in R^N generating an approximate r-parameter group G_r of transformations with respect to the group parameter $a \in R^r$. Let

$$X_\alpha = \xi_\alpha^i(z,\varepsilon)\frac{\partial}{\partial z^i} \tag{3.2}$$

be basic generators of the correspoding approximate Lie algebra.

Definition 5. An approximate function $I(z,\varepsilon)$ is called *an approximate invariant* of the approximate group G_r of transformations (3.1), if for each $z \in R^N$ and an admissible $a \in R^r$

$$I(z',\varepsilon) \approx I(z,\varepsilon). \tag{3.3}$$

Theorem 6. *The approximate function $I(z,\varepsilon)$ is an approximate invariant of the group G_r with the basic generators (3.2) if and only if the approximate equations*

$$XF(z,\varepsilon) \approx 0 \tag{3.4}$$

hold.

Remark. The equations (3.4) are approximate linear first-order partial differential equations with the coefficients depending on a small parameter.

Consider the case of a one-parameter approximate transformation group with the generator

$$X = \xi^i(z,\varepsilon)\frac{\partial}{\partial z^i}, \tag{3.5}$$

where

$$\xi^i(z,\varepsilon) \approx \varepsilon^l \left(\xi_0^i(z) + \varepsilon\xi_1^i(z) + \ldots + \varepsilon^{p-l}\xi_{p-l}^i(z)\right) + o(\varepsilon^p), \qquad l = 0,\ldots,p, \tag{3.6}$$

and vector $\xi_0(z) = (\xi_0^1(z),\ldots,\xi_0^N(z)) \neq 0$.

Theorem 7. *Any one-parameter approximate group G_1 with the generator (3.5), (3.6) has exactly $N-1$ functionally independent (when $\varepsilon = 0$) approximate invariants of the form*

$$I^k(z,\varepsilon) \approx I_0^k(z) + \varepsilon I_1^k(z) + \ldots + \varepsilon^{p-l} I_{p-l}^k(z), \qquad k = 1,\ldots,N-1,$$

and any approximate invariant of G_1 can be represented in the form

$$I(z,\varepsilon) = \varphi_0(I^1,\ldots,I^{N-1}) + \varepsilon\varphi_1(I^1,\ldots,I^{N-1}) + \cdots + \varepsilon^{p-l}\varphi_{p-l}(I^1,\ldots,I^{N-1}) + o(\varepsilon^{p-l}),$$

where $\varphi_0, \varphi_1, \ldots, \varphi_p$ are arbitrary functions.

For multiparameter approximate groups, we consider a case when the corresponding approximate Lie algebra is a Lie algebra of approximate symmetries, i.e., it is obtained as a solution of some determining equation and has the form (2.6). Let

$$\operatorname{rank} \begin{Vmatrix} \xi^i_{\alpha_0,0}(z) \\ \xi^i_{\alpha_1,0}(z) \\ \vdots \\ \xi^i_{\alpha_l,0}(z) \end{Vmatrix} = r^*_l.$$

Here $r^*_0 \le r^*_1 \le \ldots \le r^*_p$. Let

$$s_0 = N - r^*_p, \quad s_1 = N - r^*_{p-1}, \ldots, \quad s_p = N - r^*_0.$$

Theorem 8. *In this case, the multiparameter group has s_p approximate invariants*

$$I^1(z,\varepsilon) \approx I^1_0(z) + \varepsilon I^1_1(z) + \ldots + \varepsilon^p I^1_p(z) \equiv J^1,$$

$$\cdot \qquad \cdot \qquad \cdot$$

$$I^{s_0}(z,\varepsilon) \approx I^{s_0}_0(z) + \varepsilon I^{s_0}_1(z) + \ldots + \varepsilon^p I^{s_0}_p(z) \equiv J^{s_0},$$

$$I^{s_0+1}(z,\varepsilon) \approx \varepsilon \left(I^{s_0+1}_0(z) + \varepsilon I^{s_0+1}_1(z) + \ldots + \varepsilon^{p-1} I^{s_0+1}_{p-1}(z) \right) \equiv \varepsilon J^{s_0+1},$$

$$\cdot \qquad \cdot \qquad \cdot$$

$$I^{s_p}(z,\varepsilon) \approx \varepsilon^p I^{s_p}_0(z) \equiv \varepsilon^p J^{s_p},$$

with functionally independent functions $I^k_0(z), k = 1, \ldots, p$ and any approximate invariant of G_r can be represented in the form

$$I(z,\varepsilon) \approx \varphi_0(J^1, \ldots, J^{s_0}) + \varepsilon \varphi_1(J^1, \ldots, J^{s_1}) + \ldots + \varepsilon^p \varphi_p(J^1, \ldots, J^{s_p}),$$

where $\varphi_0, \varphi_1, \ldots, \varphi_p$ are arbitrary functions.

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