

Similarity Solutions for a Nonlinear Model of the Heat Equation

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Abstract

We apply the similarity method based on a Lie group to a nonlinear model of the heat equation and find its Lie algebra. The optimal system of the model is constructed from the Lie algebra. New classes of similarity solutions are obtained.

Introduction

The paper consists of two sections. In the first section we use the Lie similarity method to find similarity solutions of the inhomogeneous nonlinear diffusion equation in the form

$$f(x) \frac{\partial u(x, t)}{\partial t} = \frac{\partial}{\partial x} (g(x) u^n u_x^q), \quad (1)$$

when $f(x) = x^p$, $g(x) = x^m$, and p, m and q are arbitrary constants. Eq.(1) is of considerable interest both in physics and mathematics as well as its special cases have been used to successfully model physical situations. For example, the case $p = m = 0$, $n = -1/2$ and $q = 1$ arises in models of plasma diffusion [9] and the thermal expulsion of liquid helium [5,10]. The homogeneous form of Eq.(1) has been used by [6] to discuss the spread of lava from volcanic eruptions, the case $p = m = 0$, $q = n = 1$ arises in the other physical phenomena besides heat or chemical diffusion, for example, in the isothermal percolation of a perfect gas through a microporous medium [10].

The reduction of the case $q = 1$, $p = m = 0$ and $n = -1/2, -1, -3/2$ is discussed in [4] and for $q = 1$, p, m and n are arbitrary, is discussed in [8]. When $f = \text{const}$, $g = 1$ and n, q are arbitrary, the transient temperature distribution is determined without the thermal relaxation effect [7].

In the second section we use the method of group-invariant solution [3-11] to determine new classes of similarity reduction of the case $f = a^2 = \text{const}$, $g = 1$ and n, q are arbitrary, in addition to the previously known ones [7].

Section 1

When $f = x^p$ and $g = x^m$, then Eq.(1) becomes

$$x^p \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} (x^m u^n u_x^q). \quad (2)$$

Classical similarity [2,3] determines transformations which leave the differential equation invariant. In the infinitesimal representation, the corresponding generator of the transformation is written as

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$$v = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \eta(x, t, u) \frac{\partial}{\partial u}, \quad (3)$$

where ξ, τ and η are unknown functions of x, t and u . The condition of invariance of Eq.(2) is

$$\text{Pr}^{(2)}v(\Delta) \Big|_{\Delta=0} = 0, \quad (4)$$

where

$$\Delta = mx^{m-1}u^n u_x^q + nx^m u^{n-1} u_x^{q-1} + qx^m u^n u_x^{q-1} u_{xx} - x^p u_t, \quad (5)$$

and

$$\text{Pr}^{(2)}v = v + \eta^x \frac{\partial}{\partial u_x} + \eta^t \frac{\partial}{\partial u_t} + \eta^{xx} \frac{\partial}{\partial u_{xx}} \quad (6)$$

is the second prolongation of the vector field v and $\eta^x, \eta^t, \eta^{xx}$ are expressed in terms of ξ, τ, η and their derivatives. From Eq.(4) by equating the coefficients of the various monomials of u and inserting $u_{xx} = \frac{x^{p-m}}{qu^n u_x^{q-1}} u_t - \frac{m}{qx} u_x - \frac{n}{qu} u_x^2$, we get the following set of determining equations

$$\begin{aligned} \xi &= \xi(x), \quad \tau = \tau(t), \quad \eta = \eta(u), \\ m\xi - mx\xi_x + qx^2\xi_{xx} &= 0, \\ n\eta - nu\eta_u - qu^2\eta_{uu} &= 0, \\ \frac{(m-p)}{x}\xi + \frac{n}{u}\eta + \tau' - (q+1)\xi_x + (q-1)\eta_u &= 0. \end{aligned} \quad (7)$$

Solving Eq.(7), we get

$$\xi(x) = [2c_2 + c_1(1-n-q)]x/r, \quad \tau(t) = 2c_2t + c_3, \quad \eta(u) = -c_1u \quad (8)$$

where c_1, c_2, c_3 are arbitrary constants and $r = p - m + q + 1$. Then we have the three symmetry vector fields, namely

$$\begin{aligned} v_1 &= \frac{\partial}{\partial t}, \\ v_2 &= \frac{1-n-q}{r} x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}, \\ v_3 &= 2t \frac{\partial}{\partial t} + \frac{2}{r} x \frac{\partial}{\partial x}. \end{aligned} \quad (9)$$

These fields form a Lie algebra. We find $[v_1, v_3] = 2v_1$ and all other commutations vanish. By using the adjoint algebra [2], we can find four different kinds of solutions corresponding to the basic fields of an optimal system given by $v_2, v_3, v_1 + v_2$ and $v_2 + v_3$. Also we can obtain further solutions of Eq.(2) through the characteristic equation

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}. \quad (10)$$

Group invariant solutions of Eq.(2)

(a) For the vector field v_2 .

The general reduction of this subgroup can be obtained by the similarity representation

$$s = t, \quad u = x^{-r/(1-n-q)} F(s). \quad (11)$$

From Eq.(11) and Eq.(2), we get

$$F' = \left[\frac{r^{q+1} + r^q(1+p)(n+q-1)}{(n+q-1)^{(q+1)}} \right] F^{(n+q)}. \quad (12)$$

Direct integration gives

$$F = \left[c - \frac{r^{q+1} + r^q(1+p)(n+q-1)}{(n+q-1)^{(q+1)}} s \right]^{1/(1-n-q)}, \quad (13)$$

where c is arbitrary constant, $(1-n-q) \neq 0$ and $r = p - m + q + 1$. Then the solution of Eq.(2) is

$$u(x, t) = x^{-r/(1-n-q)} \left[c - \frac{r^{q+1} + r^q(1+p)(n+q-1)}{(n+q-1)^{(q+1)}} t \right]^{1/(1-n-q)}. \quad (14)$$

(b) For the linear combination $v_1 + kv_2$, k is constant.

The finite transformation for this combination can be written as

$$s = x \exp\left(-\frac{(1-n-q)t}{r}\right), \quad u = \exp(-kt) F(s). \quad (15)$$

Then Eq.(2) becomes

$$-ks^p \left(F - \frac{n+q-1}{r} s F' \right) = \frac{d}{ds} (s^m F^n (F')^q), \quad (16)$$

if we put $(n+q-1) = -r/(1+p)$, then Eq.(16) becomes

$$-\frac{k}{(1+p)} \frac{d}{ds} (s^{(1+p)} F) = \frac{d}{ds} (s^m F^n (F')^q). \quad (17)$$

Integrating Eq.(17) once, we get

$$F^{(n+q-1)/q} = (-k)^{1/q} \left(\frac{1}{1+p} \right)^{1/q} \left(\frac{n+q-1}{r} \right) s^{r/q} + c, \quad (18)$$

where c is the constant of integration, $q \neq 0, r \neq 0$ and $p \neq -1$.

(c) For the linear combination $v_3 + kv_2$, k is constant.

The similarity representation is given by

$$s = xt^{-1/2M}, \quad u = t^{-k/2} F(s), \quad (19)$$

where $M = r/[2 + k(1 - q - n)]$. Then Eq.(2) becomes

$$-s^p \left(\frac{k}{2} F + \frac{s}{2M} F' \right) = \frac{d}{ds} (s^m F^n (F')^q). \quad (20)$$

For $k = \frac{p+1}{M}$ Eq.(20) becomes

$$-\frac{1}{2M} \frac{d}{ds} (s^{p+1} F) = \frac{d}{ds} (s^m F^n (F')^q). \quad (21)$$

Integrating Eq.(21), we obtain

$$-\frac{1}{2M} s^{p+1} F = s^m F^n (F')^q + c, \quad (22)$$

where c is the constant of integration. For $c = 0$, integrating Eq.(22) gives

$$F^{(n+q-1)/q} = c_1 + \frac{(n+q-1)}{r} \left(-\frac{1}{2M}\right)^{1/q} s^{r/q}, \quad r \neq 0, \quad (23)$$

$$F^{(n+q-1)/q} = c_1 + \frac{(n+q-1)}{q} \left(-\frac{1}{2M}\right)^{1/q} \ln s, \quad r = 0, \quad (24)$$

where c_1 is the constant of integration and $q \neq 0$. For $c \neq 0$ consider the case $g = 2$, $m = 1$, $n = 0$ and $p = -1$. Then Eq.(22) becomes $s(F')^2 = -F + c$, where c is a constant, and this equation has the solution $F = -s + c$, i.e., $u = -xt^{-1} + c$. which is a solution of Eq.(2), when $q = 2$, $m = 1$, $n = 0$ and $p = -1$.

(d) For the vector field v_3 .

The similarity variable s and the similarity solution are

$$s = x^r/t, \quad u = F(s). \quad (25)$$

From Eq.(25) and Eq.(2), we get

$$r^{(q+1)} s^q \frac{d}{ds} (F^n (F')^q) + r^q (qr - q - m) s^{q-1} F^n (F')^q + s F' = 0. \quad (26)$$

Consider the case $q = m = 2$, $n = -2$ and p is arbitrary. Then Eq.(26) becomes

$$F'' = \frac{1}{F} (F')^2 - \frac{1}{s} F' + \frac{a}{s} F^2, \quad (27)$$

where $a = -\frac{1}{2r^3}$. Eq.(27) is equivalent to a special case of one of the Painlevé equations [1].

Section 2

When $f = a^2 = \text{const}$ and $g = 1$, then Eq.(1) becomes

$$a^2 u_t = \frac{\partial}{\partial x} (u^n u_x^q). \quad (28)$$

This equation was discussed in [7] and one similarity solution was obtained for it, but in this section we shall find new classes of similarity solutions for Eq.(28). By using Eq.(4), in this case we have

$$\Delta = qu^n u_x^{q-1} u_{xx} + nu^{n-1} u_x^{q+1} - a^2 u_t. \quad (29)$$

This will lead to a system of determining equations involving x, t, u and the derivatives of u with respect to x and t as well as ξ, τ, η and their derivatives with respect to x, t, u . The solution of these determining equations is Eq.(8) itself at $p = m = 0$, i.e.,

$$\begin{aligned}\xi(x) &= [2c_2 + c_1(1 - n - q)]x/(q + 1), \\ \tau(t) &= 2c_2t + c_3, \quad \eta(u) = -c_1u.\end{aligned}\tag{30}$$

The vector field (3) is spanned by the three vector fields

$$\begin{aligned}v_1 &= \frac{\partial}{\partial t}, \\ v_2 &= \frac{1 - q - n}{q + 1}x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}, \\ v_3 &= 2t \frac{\partial}{\partial t} + \frac{2}{q + 1}x \frac{\partial}{\partial x}.\end{aligned}\tag{31}$$

These vector fields satisfy the commutator table and by using the adjoint algebra [2], we find four different kinds of solutions corresponding to the basic fields of an optimal system given by $v_2, v_3, v_1 + v_2, v_2 + v_3$. Further solutions can be obtained by using the characteristic equation $\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}$ and the solution which was obtained in [7] can be found by setting $c_1 = \alpha = \text{const}$, $c_2 = 1/2$, $c_3 = 0$ and $p = m = 0$ in the characteristic equation.

Group of similarity solutions of Eq.(28)

(a) For the vector field v_2 .

The similarity variable s and the similarity solution are

$$s = t, \quad u = x^{-(q+1)/((1-n-q))} F(s).\tag{32}$$

From Eq.(32) and Eq.(28), we get

$$F' = \frac{1}{a^2} \left[\frac{(q+1)^{(q+1)} + (q+1)^q(n+q-1)}{(n+q-1)^{(q+1)}} \right] F^{n+q}.\tag{33}$$

After integration and by using Eq.(32), we get

$$u = x^{-h/(1-n-q)} \left[c - \frac{1}{a^2} \left(\frac{h^h + h^q(n+q-1)}{(n+q-1)^h} \right) t \right]^{1/(1-n-q)}\tag{34}$$

which is a solution of Eq.(28), where $h = q + 1$ and c is the constant of integration and $n + q \neq 1$.

(b) For the linear combination $v_1 + kv_2$, k is a constant.

The corresponding new similarity representation is given by

$$s = x \exp \left(\frac{k(n+q-1)t}{q+1} \right),$$

$$u = \exp(-kt)F(s). \quad (35)$$

Then from Eq.(35) and Eq.(28), we get

$$-a^2k \left(F - \frac{n+q-1}{q+1} sF' \right) = \frac{d}{ds} (F^n(F')^q). \quad (36)$$

If we put $n = -2q$, then Eq.(36) becomes

$$-a^2k \frac{d}{ds} (sF) = \frac{d}{ds} (F^{-2q}(F')^q). \quad (37)$$

Integrating Eq.(37), we get

$$F^{-(q+1)/q} = ba^{2/q} s^{(q+1)/q} + C, \quad (38)$$

where $b = -(-k)^{1/q}$ and C is the constant of integration.

(c) For the linear combination $v_3 + kv_2$, k is a constant.

The similarity representation is

$$s = xt^{-1/2M}, \quad u = t^{-k/2}F(s), \quad (39)$$

where $M = (1+q)/[2+k(1-q-n)]$. Then Eq.(28) becomes

$$-a^2 \left(\frac{k}{2}F + \frac{s}{2M}F' \right) = \frac{d}{ds} (F^n(F')^q). \quad (40)$$

If we put $k = \frac{1}{M}$, after integration we get

$$-\frac{a^2}{2M} sF = F^n(F')^q + C. \quad (41)$$

Let $C = 0$, then the solution of Eq.(41) is

$$F^{(n+q-1)} = c_1 + \frac{(n+q-1)}{(q+1)} \left(-\frac{a^2}{2} \right)^{1/q} M^{-1/q} s^{(q+1)/q}, \quad (42)$$

where c_1 is a constant and $q \neq 0, -1$. For $q = -1$ then we have

$$F^{(2-n)} = c_2 - \frac{2M}{a^2} (2-n) \ln s, \quad (43)$$

where c_2 is a constant.

(d) For the vector field v_3 .

The similarity variable s and the similarity solution are

$$s = \frac{x^{q+1}}{t}, \quad u = F(s), \quad (44)$$

then Eq.(28) becomes

$$r^r s^q \frac{d}{ds} (F^n(F')^q) + r^{(r-1)} q^2 s^{q-1} F^n(F')^q + a^2 s F' = 0, \quad (45)$$

where $r = q + 1$. Eq.(45) can be written in the form

$$F'' + \frac{n(F')^2}{qF} + \frac{q}{rs} F' + \frac{a^2}{qr^r} s^{1-q} F^{-n} (F')^{(2-q)} = 0. \quad (46)$$

Consider the case $q = 2$, then Eq.(46) becomes a special case of one of the Painlevé equations [1].

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