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Symplectic Symmetries of Hamiltonian Systems

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The goal of this paper is to describe some interesting phenomena which occur in Hamiltonian systems with symplectic (locally Hamiltonian) symmetries.

1. The phase space of an abstract Hamiltonian system is a symplectic manifold (M, ω^2), $\dim M = 2n$, with the symplectic structure ω^2 [1, 2].

Let $\Im : T^*M \mapsto TM$ be the fibre map induced by the symplectic structure:

$$\omega^2(\xi, \Im\alpha) = \alpha(\xi) := (\alpha, \xi), \quad \forall\alpha \in T^*_xM,\ \forall\xi \in T_xM.$$

Denote by $\Im dH$ the Hamiltonian vector field generated by the smooth function $H : M \mapsto \mathbf{R}$.

Definition 1 *The flow $(M, \{g^t\}_{t\in\mathbf{R}})$ is called a symplectic symmetry of the system $\Im dH$ if*

$$H \circ g^t = H, \qquad (g^t)^*\omega^2 = \omega^2.$$

The last equality is true iff the vector field $X(x) = \frac{d}{dt}|_{t=0} g^t x$ can be represented in the form $X = \Im\alpha$, where α is some closed 1-form on M.

Classical mechanics usually deals with the case, where α is exact, i.e., $\alpha = dF$ for some smooth function $F : M \mapsto \mathbf{R}$. In this case, F is the constant of motion for the Hamiltonian system $\Im dH$, so orbits of the latter lie down on the common level manifolds

$$H = \text{const}, \qquad F = \text{const}.$$

When α is not exact, the restrictions

$$H = \text{const}, \qquad \alpha = 0$$

are topologically more mild than in the classical case.

This circumstance may lead to some new qualitative effects in the behaviour of Hamiltonian systems with symplectic symmetries.

2. Our first example concerns the symplectic manifold admitting free smooth symplectic action of a k-dimensional torus T^k. The projection $\pi : M \mapsto N = M/T^k$ determines the structure of a principal T^k-fibre bundle (M, N, π). Consider the Hamiltonian system on M with the T^k-invariant Hamiltonian function $H = \tilde{H} \circ \pi$, $\tilde{H} : N \mapsto \mathbf{R}$. Our goal is to investigate motions of such a system in a neighbourhood of its relative equilibria [1, 2] in the case, where T^k-action does not admit a momentum map. To lower the order of the system $\Im dH$, we need to modify the well-known Marsden-Weinstein reduction procedure [1, 2].

Denote by $\mathbf{t}^k = \mathbf{R}^k$ the Lie algebra of T^k. The torus action on M defines the bilinear form $C : \mathbf{t}^k \times \mathbf{t}^k \mapsto \mathbf{R}$ in the following way

$$C(a,b) = \omega^2(X_a, X_b),$$

where the vector field X_a generates the action of a one-parameter group corresponding to $a \in \mathbf{t}^k$. Suppose C to be nontrivial but $\dim\ker C := k_0 > 0$.

The projection π gives rise to the reduced Poisson structure (r.P.s.) on N:

$$\{\cdot,\cdot\}_M \overset{\pi}{\mapsto} \{\cdot,\cdot\}_N.$$

Now one can reduce $\Im dH$ to the Hamiltonian (with respect to r.P.s.) vector field $\Im_N d\tilde{H}$ on N which is tangent to each symplectic leaf of r.P.s. There exists $(\ker C)^*$-valued closed 1-form $\tilde{\theta}$ on N such that

$$\omega^2(X_a,\cdot) = -(\pi^*\tilde{\theta}(\cdot), a) \quad \forall a \in \ker C.$$

It is not difficult to show that the above symplectic leaves are integral manifolds of the Pfaff equation $\tilde{\theta} = 0$.

Denote by $\ell(x_0)$ the symplectic leaf passing through $x_0 \in N$. Suppose x_0 be a critical point of $\tilde{H}|_{\ell(x_0)}$. Then x_0 is the equilibrium for the restriction of the system $\Im_N d\tilde{H}$ to $\ell(x_0)$, and at the same time the projection of a relative equilibrium of the initial system under the map π. We consider the so-called elliptic case, where the eigenvalues of the linear operator $D(\Im_N d\tilde{H}|_{\ell(x_0)})(x_0)$ are of the form

$$\pm i\lambda_1, \ldots, \pm i\lambda_m, \qquad m := \dim \ell(x_0)/2,$$

$\lambda_j \neq 0$ being real. Then in some neighbourhood of x_0, the set of equilibria

$$\mathcal{W} = \{x \in N : \ \Im_N d\tilde{H} = 0\}$$

forms the smooth manifold of dimension k_0 which transversally intersects symplectic leaves. In accordance with the KAM-theory, under the nonresonant conditions

$$k_1\lambda_1 + \cdots + k_m\lambda_m \neq 0, \quad k_j \in \mathbf{Z},\ 0 < \sum_{j=1}^{m} |k_j| \leq 2l$$

with sufficient $l \in \mathbf{N}$, there exist many quasi-periodic motions near $\mathcal{W}$. Let $z(t)$ be one of such motions. Generally it possesses m rationally independent frequencies: $\operatorname{Hull} z(t) = T^m$.

Consider now the reconstruction of $z(t)$ on M, that is the motion $x(t)$ on M generated by $\Im dH$ which covers the motion $z(t)$. It turns out that if the symplectic structure is "nonresonant" with respect to fibration by tori (this means that there exist no integer nonzero vectors orthogonal to $\ker C$) then the quasi-periodic motion $x(t)$ possesses the "unusual" property:

$$\operatorname{Hull} x(t) = T^r \text{ where } r > n.$$

Note that "classical" quasi-periodic motions of Hamiltonian systems cover tori of dimensions $\leq n$.

3. In our second example we show the appearance of so-called nilpotent flows in Hamiltonian systems with symplectic symmetries.

Let M be a 4-dimensional compact symplectic manifold admitting free symplectic circle action $S^1 \mapsto M \overset{\pi}{\mapsto} N$. Consider the Hamiltonian system with the S^1-invariant Hamiltonian $H = \tilde{H} \circ \pi$. There exists the closed but not exact 1-form θ on N such that the vector field $X = \Im \pi^* \theta$ generates S^1-action on M. Let c be a regular value of H, $M_c = H^{-1}(c)$, $N_c = M_c / S^1$. Denote by Ω the curvature 2-form of the principal fibre bundle (M, N, π).

Theorem 1 *Suppose that θ and $d\tilde{H}$ be linearly independent on N_c. Then the following statements are true:*

1) N_c is diffeomorphic to T^2.

2) M_c is diffeomorphic to T^3 if $n := \int_{N_c} \Omega = 0$.

3) M_c is diffeomorphic to the 3-dimensional nilmanifold $Nil_n^3 = N_n / D_n$ if $n \neq 0$. Here N_n is a Lie group of matrices

$$\begin{pmatrix} 1 & u & w/n \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}, \quad (u, v, w) \in \mathbf{R}^3,$$

D_n is its discrete subgroup of matrices

$$\begin{pmatrix} 1 & k & m/n \\ 0 & 1 & l \\ 0 & 0 & 1 \end{pmatrix}, (k, l, m) \in \mathbf{Z}^3.$$

Put $\nu_i = \int_{c_i} \theta$, where c_1, c_2 is the basis of cycles on N_c. The next theorem shows that the vector field $\Im dH$ can be straightened near M_c.

Theorem 2 *Suppose that for some $K > 0$, $L > 0$ the numbers ν_1, ν_2 satisfy Siegel's condition*

$$|k_1 \nu_1 + k_2 \nu_2| \geq K / (|k_1| + |k_2|)^L \quad \forall k_i \in \mathbf{Z}, \ |k_1| + |k_2| > 0.$$

Then the following statements are true:

1) In the case, where $n = 0$, there is a neighbourhood of M_c diffeomorphic to

$$\{y \in \mathbf{R}\} \times \{T^3 = \{(\varphi_1, \varphi_2, \varphi_3) | \bmod 1\}\}.$$

In (y, φ)-coordinates, the Hamiltonian function H depends only on y, and the corresponding equations of motion take the form

$$\dot{y} = 0, \quad \dot{\varphi}_i = \nu_i H'(y), \quad i = 1, 2, 3.$$

2) In the case, where $n \neq 0$, there is a neighbourhood of M_c diffeomorphic to

$$\{y \in \mathbf{R}\} \times \{N_n / D_n\}.$$

In (y, u, v, w)-coordinates, the Hamiltonian H also depends only on y, and the corresponding equations of motion take the form

$$\dot{y} = 0, \quad \dot{\mathrm{Z}} = H'(y) \mathrm{A} \mathrm{Z},$$

where

$$\mathrm{Z} = \begin{pmatrix} 1 & u & w/n \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathrm{A} = \begin{pmatrix} 0 & \nu_1 & 0 \\ 0 & 0 & \nu_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that the natural action of one-parameter group $\exp(\mathrm{A}t)$ on N_n/D_n defines the nilpotent flow in the sense of [3]. For the details of the above results, we refer to [4, 5].

4. In conclusion, we consider a Hamiltonian system which governs the motion of charged particle on the nilmanifold $Nil^3 = N_1/D_1$ under influence of magnetic field. We show that the phase space of such a system is stratified by 4-dimensional compact nilmanifolds, which are the ergodic components of system's flow.

Denote by $\omega_1, \omega_2, \omega_3$ the 1-forms on Nil^3 naturally generated by the right-invariant 1-forms du, dv, $dw - vdu$ on N_1. Let the kinetic energy of our system be determined by the metric

$$\frac{1}{2}\sum_{i,j=1}^{3} A_{ij}\omega_i\omega_j, \qquad A_{ij} \in \mathbf{R},$$

and the magnetic force form to be as follows:

$$\Gamma = \nu_1\omega_2 \wedge \omega_3 + \nu_2\omega_3 \wedge \omega_1 + \nu_3\omega_1 \wedge \omega_2.$$

The corresponding system on $TNil^3$ can be transformed into Hamiltonian one on the so-called twisted cotangent bundle $(T^*Nil^3, d\Lambda + \mathrm{pr}^*\Gamma)$ with the Hamiltonian function

$$T(\mathbf{m}) = \frac{1}{2}\sum_{i,j=1}^{3} A^{ij} m_i m_j, \qquad \left(\sum_{k=1}^{3} A^{ik}A_{kj} = \delta_{ij}\right).$$

Here Λ is a so-called Liouville 1-form which gives rise to the standard symplectic structure $d\Lambda$ on T^*Nil^3, m_1, m_2, m_3 are components of the momentum map $\mathbf{m}$ for the symplectic N_1-action on $(T^*Nil^3, d\Lambda)$, and $\mathrm{pr} : T^*Nil^3 \mapsto Nil^3$ is the natural projection.

The above system possesses local symplectic symmetries acting along the manifolds $\mathbf{m} = \mathbf{const}$. For this reason, the Poisson map

$$\mathbf{m} : T^*Nil^3 \mapsto \mathbf{R}^3$$

can be used to obtain a reduced integrable system on $\mathbf{R}^3$.

Theorem 3 *The system with the Hamiltonian $T(\mathbf{m})$ possesses the constant of motion $J(\mathbf{m}) = \sum_{i=1}^{3}\nu_i m_i + m_3^2/2$. The generic common level manifold of $T(\mathbf{m})$ and $J(\mathbf{m})$ is a 4-dimensional manifold M_c diffeomorphic to $S^1 \times Nil^3$. Some neighbourhood of M_c is diffeomorphic to*

$$\{(I, J) \in \mathcal{D}\} \times \{S^1 = \{\varphi | \bmod 1\}\} \times Nil^3,$$

where $\mathcal{D}$ is the open domain in $\mathbf{R}^2$. In (I, J, φ, u, v, w)-coordinates, the Hamiltonian $T(\mathbf{m})$ depends only on $I, J : T = \hat{T}(I, J)$. The corresponding equations of motion take the form

$$\dot{I} = 0, \quad \dot{J} = 0, \quad \dot{\varphi} = \frac{\partial \hat{T}(I, J)}{\partial I},$$

$$\dot{\mathrm{Z}} = \left(\frac{\partial \hat{T}(I, J)}{\partial J}\mathrm{A} + \frac{\partial \hat{T}(I, J)}{\partial I}\mathrm{B}\right)\mathrm{Z},$$

A *and* Z *being the same as in Theorem 2, and*

$$\mathrm{B} = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda \in \mathbf{R}.$$

Note that the flow on M_c is ergodic if ν_1, ν_2 are rationally independent, whereas in the case $\Gamma = 0$, the motions generally cover 2-dimensional tori.

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