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Determining Equations and Differential Constraints

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1 Introduction

The classical Lie method for finding of exact solutions is based on the defining equations. By now there is a large body of papers on symmetry properties and group invariant solutions. An extensive bibliography can be found in Ovsiannikov [1], Ibragimov [2], Olver [3].

The key idea of the method of differential constraints was proposed by Yanenko in [6]. The general formulation of the method of differential constraints requires that the original system of partial differential equations

$$F^1 = 0, ..., \quad F^m = 0 \tag{1.1}$$

be enlarged by appending additional differential equations (diffential constraints or DC)

$$h_1 = 0, ..., \quad h_p = 0, \tag{1.2}$$

such that the overdetermining system (1.1), (1.2) satisfies some conditions of compatibility. Applications of this method to gas dynamics can be found in Yanenko, Sidorov, Shapeev [7].

Meleshko [8], Olver and Rosenau [5], Olver [9], Kaptsov [10], Levi and Winternitz [11] show that many reduction methods such as partial invariance, separation of variables, the Clarkson–Kruskal direct method, etc., can be included into the method of differential constraints.

In practice, methods based on the Riquier–Ritt theory of overdetermined systems of partial differential equations may be difficult. To simplify a search for DC, it was proposed in [12] to use B–determining equations. The B–determining equations are slightly more general than the classical defining equations. It is easy to introduce more general determining equations. Briefly speaking, the general determining equations for finding DC (1.2) of the system (1.1) are given by

$$L_i(h_1, ..., h_p) = 0, \quad i = 1, ..., q \tag{1.3}$$

where L_i are nonlinear operators and (1.3) must be satisfied for every smooth solution of (1.1).

For example, it is easy to check that the intermediate integrals

$$I_1 = u_y - \varphi(u_x),$$

$$I_2 = u - xu_x - yu_y + \varphi(u_x),$$

for the Monge–Ampere equation

$$u_{xy}^2 - u_{xx}u_{yy} = 0,$$

satisfy the determining equations

$$u_{xy}D_x(I_i) - u_{xx}D_y(I_i) = 0,$$

where D_x, D_y are total derivatives.

In this paper we introduce new algebraic structures and special classes of determining equations. We apply the determining equations to find DC of the first order to the Prandtl equation and the Zabolotskaya–Khokhlov one. In Section 3, we propose the determining equations for higher order DC and consider an evolution equation of the third order.

2 General Notions and Some Examples

Before considering particular differential equations, it is useful to introduce some notations and prove auxiliary statements. Let A be the algebra of smooth functions depending on commutative variables x_i, u_α, where $1 \le i \le n$, $1 \le k \le m$, $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi–index, and $\alpha_j \ge 0$.

The total derivatives $D_1, ..., D_n$ act on x_i, u_α^k

$$D_i(x_i) = 1, \quad D_i(x_j) = 0, \qquad i \ne j,$$

$$D_i(u^k_{\alpha_1,...,\alpha_n}) = u^k_{\alpha_1,\alpha_i+1,...\alpha_n}.$$

The action of D_i can be extended on A (see Olver [3]). The partial derivative of $f \in A$ with respect to u_α^k is denoted by $f_{u_\alpha^k}$

Consider the space A^m and introduce the Lie bracket for any $F, G \in A^m$

$$[F, G] = \sum_{\substack{1 \le i \le m \\ |\alpha| \ge 0}} F_{u_\alpha^i} D^\alpha(G^i) - G_{u_\alpha^i} D^\alpha(F^i), \tag{2.1}$$

where F^i and G^i are components of F and G, $D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n}$.

Given a system of partial differential equations

$$F^1 = 0, ..., F^m = 0, \tag{2.2}$$

$$F^i \in A,$$

a ∞ – prolongation of the system consists of equations

$$D^\alpha(F^i) = 0, \quad i = 1, ..., m, \quad |\alpha| \ge 0. \tag{2.3}$$

We will say that a equation $f = 0$ belongs to differential consequences of (2.2) and write $f \in [F]$ if every smooth solution of (2.2) satisfies to $f = 0$.

It follows from the definition that if $f \in [F]$, then $D_i(f), fg \in [F]$, for every $g \in A$.

A system of partial differention equations

$$f_1 = 0, ..., f_N = 0, \quad f_i \in A \tag{2.4}$$

is called solvable if this system has at least one smooth solution.

From the above definitions, we have the following assertions.

Proposition 1. If the system (2.4) is solvable, then the system

$$g_1 = 0, ..., g_M = 0,$$

$$g_i \in [f]$$

is also solvable.

Corollary. If the system

$$f_1 = 0, \quad f_2 = 0$$

is solvable, then the system

$$f_1 + gD_i(f_2) = 0, \quad f_2 = 0, \quad g \in A$$

is solvable too.

This corollary allows us to construct new differential equations with fixed DC $f_2 = 0$. Some examples can be found in Fokas, Liu [13].

Proposition 2. Suppose the system (1.1) with DC (1.2) be solvable. Then this system is compatible with DC

$$\alpha_1 h_1 + ... + \alpha_p h_p = 0, \qquad \alpha_i \in R.$$

Lemma 1. Assume that (2.2) be a system for a single unknown function u. Then any function u which satisfies this system must also satisfy all of the equations

$$[F^i, F^j] = 0 \qquad i, j = 1, ..., k.$$

Proof. According to definition, every term of the Lie brackets belongs to $[F]$. It is obvious that the proof follows from Proposition 1.

Given system (1.1), we can define symmetries of (1.1) as follows:

$$Sym = \{h : h \in A^m, \quad [F, h] \subset [F]\}.$$

Equating to zero $h \in Sym$, we get DC.

It should be noted that the condition $[F, h] \subset [F]$ is equivalent to the classical defining equations

$$\sum_{\substack{1 \le i \le m \\ |\alpha| \ge 0}} D^\alpha(h^i) F_{u^i_\alpha}|_{[F]} = 0$$

To generalize the previous construction, we introduce the set

$$B_I = \{h : h \in A^m, \quad [F,h] + B_0 h + \sum_{i=1}^{n} B_i D_i h \subset [F]\},$$

where B_j is an $m \times m$ matrix. Elements of B_j are functions of $x_i, u^k (1 \le i \le n, \quad 1 \le k \le m)$.

Example 1. Consider the Prandtl equation

$$\psi_{yyy} = \psi_{ty} + \psi_y \psi_{xy} - \psi_x \psi_{yy},$$

which arises in the theory of viscous fluid. For convenience, we rewrite the above equation in the following way:

$$u_{xxx} = u_{tx} + u_x u_{xy} - u_y u_{xx}. \tag{2.5}$$

In this case, the condition

$$[F,h] + B_0 h + \sum_{i=1}^{n} B_i D_i h \subset [F] \tag{2.6}$$

implies the equation

$$D_x^3 h - D_t D_x h + u_y D_x^2 h + u_{xx} D_y h - u_x D_x D_y h - u_{xy} D_x h +$$
$$+ b_0 h + b_1 D_x h + b_2 D_y h + b_3 D_t h = 0, \tag{2.7}$$

where h, b_0, b_1, b_2, b_3 are functions to be determined. The B_I – determining equation (2.7) must be satisfied for every smooth solution of (2.5). Suppose the functions b_i depend on t, x, y and u; the function h depends on t, x, y, u, u_t, u_x, and u_y. The problem is to find solutions h and corresponding functions b_0, b_1, b_2, b_3. The method for finding solutions is very similar to the standard procedure applied in the group analysis of differential equations [1], [3] and omitted here for the sake of brevity.

The Lie algebra of infinitesimal symmetries of the Prandtl equations was found by Pukhnachov [12]. Moreover, the function

$$h_1 = u_x + kx + r(t,y) \tag{2.8}$$

is a solution of (2.7). Here r is an arbitrary smooth function. The corresponding b_0, b_1, b_2, and b_3 are of the form

$$b_0 = 0, \quad b_1 = -r_y, \quad b_2 = b_3 = 0.$$

Another solution of (2.7) is

$$h_2 = v u_x + u_y + u v_x + w,$$

where the functions $v(t,y)$ and $w(t,x,y)$ must satisfy the following equations

$$v w_x + w_y - v_t = 0, \quad -w w_{xx} - w_{tx} + w_{xxx} + w_x^2 = 0.$$

The corresponding functions b_i are

$$b_0 = -w_{xx}, \quad b_1 = w_x, \quad b_2 = 0, \quad b_3 = 0.$$

If we set $h_1 = 0$, then we have the equation

$$u_x + kx + r(t, y) = 0.$$

Solving the last equation, we find

$$u = -kx^2/2 - rx + q, \tag{2.9}$$

where q depends on t, y. Substitution of (2.9) into (2.5) yields

$$-r_t + rr_y + kq_y = 0.$$

If the function r is given, then we may find q by integrating the previous equation.

Example 2.2. The Zabolotskaya–Khokhlov equation (ZKE).
Consider the stationary, two–dimensional ZKE

$$u_{xxx} + uu_{xx} + u_x^2 + u_{yy} = 0. \tag{2.10}$$

From condition (2.6), we obtain the equation

$$D_x^3 h + uD_x^2 h + u_{xx}h + 2u_x D_x h + D_y^2 h + b_0 h + b_1 D_x h + b_2 D_y h = 0, \tag{2.11}$$

where b_0, b_1, and b_2 are functions of x, y, u. This equation must be satisfied whenever u satisfies (2.10).

Assume h to depend on x, y, u, u_x, u_y. One can show by a straightforward analysis that h is of the form

$$h = a_1 u_x + a_2 u_y + a_3, \tag{2.12}$$

where a_1, a_2 only depend on x, y and a_3 depends on x, y, u.

The classical and nonclassical symmetry groups of ZKE were found by Clarkson and Hood [14].

It is possible to check that in addition to the classical symmetries, there exist three solutions of (2.11)

$$h_1 = xu_x + (y + c_0)u_y + u + \frac{x^2}{(y + c_0)^2}, \tag{2.13}$$

$$h_2 = -(c_1 y + c_0)u_x + u_y + 2c_1(c_1 y + c_0), \tag{2.14}$$

$$h_3 = u_x + 2xv + w, \tag{2.15}$$

where $c_0, c_1 \in R$ and w, v must satisfy ordinary differential equations

$$w'' = 6w^2,$$

$$v'' = 6wv.$$

The corresponding functions b_0, b_1, b_2 for (2.13), (2.14), (2.15) are

$$b_0 = -2/(y+c_0)^2, \quad b_1 = -x/(y+c_0)^2, \quad b_2 = 1/(y+c_0), \tag{2.16}$$

$$b_0 = 0, \qquad b_1 = 2c_1, \qquad b_2 = 0, \tag{2.17}$$

$$b_0 = -4v, \quad b_1 = -2xv - w, \quad b_2 = 0. \tag{2.18}$$

Using the direct method, Clarkson and Hood [14] found the following nonclassical symmetries

$$h_4 = u_x - 2u/x, \quad h_5 = u_x - 2u/x + 6/x^2,$$

besides the classical symmetries and (2.13)-(2.15). The functions h_4 and h_5 do not satisfy equation (2.11) if b_0, b_1, b_2 only depend on x, y, u. But h_4 and h_5 are solutions of (2.11) for some b_0, b_1 and b_2 depending on x, y, u and u_x.

3 Higher–Order Differential Constraints

The examples of DC considered so far in this paper have all been differential constraints of the first order. To find DC, we used the conditions (2.6). On the other hand, it can be shown [12] that system formed by an equation

$$u_t = F(t, x, u, u_1, u_2) \tag{3.1}$$

and a higher order DC

$$h = u_n + g(t, x, u, u_1, ..., u_{n-1}) = 0, \tag{3.2}$$

where $n \geq 5$ and $u_i = \frac{\partial_i u}{\partial x_i}$, is compatible if and only if h satisfies the determining equation

$$D_t h = F_{u_2} D_x^2 h + [F_{u_1} + nD_x F_{u_2}] Dh +$$

$$[F_u + nD_x F_{u_1} + \frac{n(n-1)}{2} D_x^2 F_{u_2} + F_{u_2} h h_{u_{n-1}u_{n-1}} - h_{u_{n-1}} D_x F_{u_2} - 2F_{u_2} D_x h_{u_{n-1}}] h,$$

for every smooth solution of (3.1).

Using the method described in [12], it is possible to obtain determining equations which correspond to higher order DC and partial differential equations more general than (3.1). It is clear that these determining equations may differ from ones corresponding to the first order DC and are more complicated. To simplify these equations, we will reject nonlinear terms with respect to h. That is why for finding higher order differential constraints (3.2) of evolution equations

$$u_t = F(t, x, u, u_1, ..., u_p),$$

we can propose the following determining equations

$$D_t h = \sum_{r=0}^{p} \sum_{k=0}^{r} C_n^{r-k} D_x^{r-k} (F_{u_{p-k}}) D_x^{p-r}(h), \tag{3.3}$$

where $n \geq k, C_n^k = \frac{n!}{k!(n-k)!}$.

Example. Consider the equation

$$u_t = u_3 + F(u, u_1, u_2). \tag{3.4}$$

If $n = 3$, then equation (3.3) is

$$D_t h = D_x^3 h + F_{u_2} D_x^2 h + (F_{u_1} + 3D_x F_{u_2}) D_x h + \\ h(F_u + 3D_x F_{u_1} + 3D^2 F_{u_2}). \tag{3.5}$$

We will seek solutions of the form

$$h = u_3 + f(u, u_1),$$

where f is a function to be determined. It is easy to see that the left and right sides of (3.5) are polynomials in u_4 and u_3.

Equating coefficients of like powers (of u_4 and u_3) yields four equations

$$F_{u_2 u_2} = F_{u_1 u_2} = 0, \quad f_{u_1 u_1} = f_{u u_1} = 0.$$

It follows that

$$F = \alpha u_2 + \mu, \quad f = cu_1 + s, \tag{3.6}$$

where $c \in R$, α and s may only depend on u, μ may depend on u and u_1. Substituting into (3.5) and equating the coefficient of u_2^3 , we have

$$\alpha_{u_1 u_1 u_1} = 0$$

Hence

$$\alpha = \eta^2 u_1^2 + \eta^1 u_1 + \eta^0,$$

where η^2, η^1 and η^0 are functions of u. The coefficients of $u_2^2 u_1, u_2^2, u_2 u_1^3,$
$u_2 u_1, u_2,$ and u_1^3 show that

$$4\eta_u^2 + \alpha_{uu} = 0, \quad \eta_u^1 = 0, \\ 9\eta_{uu}^2 + \alpha_{uuu} = 0, \quad s(3\eta^2 + 2\alpha) = 0, \\ -2\eta^2 c + \eta_{uu}^0 - 2\alpha_u c - s_{uu} = 0, \\ -6\eta^2 c + \eta_{uuu}^0 - 3\alpha_{uu} c - s_{uuu} = 0. \tag{3.7}$$

Assume that $s = 0$. Further analysis of the remaining terms in (3.5) proves that

$$F = (-2b_1 u^2 + a_1 u + a_0) u_2 + (b_1 u + b_0) u_1^2 + k u_1 - b_1 c u^3 + c(a_1 + b_0) u^2 + c_1 u + c_0,$$

in which a_i, b_i and c_i are arbitrary constants.

Since h takes the form

$$h = u_3 + cu_1,$$

it follows that the corresponding DC is

$$u_3 + cu_1 = 0. \tag{3.8}$$

From (3.7) we have

$$u = \alpha(t)\sin\sqrt{c}x + \beta(t)\cos\sqrt{c}t + \gamma(t) \qquad for \quad c > 0; \tag{3.9}$$

$$u = \alpha(t)\exp(\sqrt{-c}x) + \beta(t)\exp(\sqrt{-c}t) + \gamma(t) \qquad for \quad c < 0; \tag{3.10}$$

$$u = \alpha(t)x^2 + \beta(t)x + \gamma(t) \qquad for \quad c = 0. \tag{3.11}$$

Using the representation (3.9)-(3.11), it is easy to derive corresponding reduced equations for the functions α, β and γ.

For example, the reduced system

$$\alpha_t = \alpha(c_1 + 1 - 3\gamma),$$

$$\beta_t = \beta(c_1 - 1 - 3\gamma),$$

$$\gamma_t = 4\alpha\beta + c_1\gamma - 2\gamma^2,$$

is obtained from the equation

$$u_t = u_3 + uu_2 + u_1^2 - 2u^2 + c_1 u.$$

If $s \neq 0$ for the system (3.7), then it can be shown that

$$F = (k_1 u + k_0)u_2 - \frac{2}{3}k_1 u_1^2 + n_1 u_1 + n_2(k_1 u + k_0)^{-4/3} + n_3,$$

$$h = u_3 + n_2(k_1 u + k_0)^{-1/3} + n_3,$$

where k_1, k_0, n_1, n_2, n_3 are arbitrary constants.

To generalize the above construction, we introduce, for any $F, G \in A^m$, the following bracket

$$\{F, G\} = \sum_{\alpha,\beta,j} c^{\beta}_{\alpha,j}(D^{\beta}(F^j)D^{\alpha-\beta}(G_{u^j_\alpha}) - D^{\beta}(G^j)D^{\alpha-\beta}(F_{u^j_\alpha})) \tag{3.12}$$

where $c^{\beta}_{\alpha,j} \in R$, the sum runs over all n–multi–indices $|\alpha| \geq |\beta|$, and $1 \leq j \leq m$. It is evident that the Lie bracket is a special case of (3.12).

Given system (1.1) then by B_g we denote the set

$$\{h : h \in A^m, \{F, h\} \subset [F]\}.$$

Every element $h \in B_g$ generates DC

$$h = 0. \tag{3.13}$$

But there is no guarantee in general that the corresponding system (1.1), (3.13) will be consistent. However, as noted above, there exist conditions which guarantee compatibility of (1.1), (3.13) (see [10],[12]).

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