

# Symmetries of Maxwell-Bloch Equations

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## Abstract

We study symmetries of the real Maxwell-Bloch equations. We give a Lax pair, bi-Hamiltonian formulations and we find a symplectic realization of the system. We have also constructed a hierarchy of master symmetries which is used to generate nonlinear Poisson brackets. In addition we have calculated the classical Lie point symmetries and variational symmetries.

## 1 Introduction

This paper is an attempt to understand the connection between two areas of Mathematics both dealing with differential equations. The first area is the application of Lie groups in the study of differential equations, most specifically the determination of symmetry groups. The second is restricted to a class of differential equations known as integrable Hamiltonian systems. The list of topics in this second area includes bi-Hamiltonian structure, recursion operators, symmetries, master symmetries, Lax formulations, Poisson and symplectic Geometry. This area of integrable systems has been studied extensively for infinite dimensional systems such as the KdV, Burgers, Kadomtsev-Petviashvili, Benjamin-Ono equations and for some finite-dimensional systems such as the Toda lattice for example. Symmetries and master symmetries for the Toda lattice were calculated in [3] and [4]. In this paper, we do a similar analysis for a three dimensional integrable system called the real Maxwell-Bloch equations.

A symmetry group of a system of differential equations is a Lie group acting on the space of independent and dependent variables in such a way that solutions are mapped into other solutions. Knowing the symmetry group allows one to determine some special types of solutions invariant under a subgroup of the full symmetry group, and in some cases one can solve the equations completely. The symmetry approach to solving differential equations can be found, for example, in the books of Olver [13], Bluman and Cole [1], Bluman and Kumei [2], Fushchych and Nikitin [8], and Ovsiannikov [14]. One method of finding symmetry groups is the use of recursion operators, an idea introduced by Olver [12]. The existence of a recursion operator provides a mechanism for generating infinite hierarchies of symmetries. Most of the well-known integrable equations, including the KdV, do have a recursion operator. In this paper, we attempt to construct a recursion operator for the Maxwell-Bloch system by finding a second Hamiltonian structure. Unfortunately, the second bracket is not compatible with the symplectic bracket and therefore the method fails. We overcome this difficulty by constructing master symmetries.

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Master symmetries were first introduced by Fokas and Fuchssteiner in [6] in connection with the Benjamin-Ono equation. Then in W. Oevel and B. Fuchssteiner [11], a master symmetry was found for the Kadomtsev-Petviashvili equation. General theory of master symmetries is discussed in Fuchssteiner [7].

Finally, in section 3, the method of the Lie theory of extended groups is applied to the Maxwell-Bloch equations. The calculations follow the spirit of [9].

Now a few words about the Maxwell-Bloch system. This system is obtained by applying  $S^1$  reduction to an invariant subsystem of a dynamical system on  $\mathbf{C}^3$ . In [5] it is shown that the system is bi-Hamiltonian and that it possesses several inequivalent Lie-Poisson structures parametrized by classes of orbits in the group  $SL(2, \mathbf{R})$ . In [16] the dynamics and Poisson structures of the real Maxwell-Bloch equations are studied with one control around the  $x_2$ -axis. We give a sketch of the results in [5] in order to make the presentation self-contained. We begin by considering the following Hamiltonian on  $\mathbf{C}^3$

$$H(u, v, w) = -\frac{1}{2}(\bar{u}vw + u\bar{v}\bar{w}) \quad (1)$$

The five-dimensional Maxwell-Bloch system [15] is obtained by reducing system (1) through the  $S^1$  group action. Using a change of variables, the system becomes

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= xz \\ \dot{z} &= -\frac{1}{2}(\bar{x}y + x\bar{y}) . \end{aligned} \quad (2)$$

The real Maxwell-Bloch equations are obtained by restricting to real-valued  $x$  and  $y$ . Thus, the dynamics of this invariant subsystem is confined to the zero level surface of the Hamiltonian function  $H$  in (1) with coordinates  $x_1 = Re(x)$ ,  $x_2 = Re(y)$ , and  $x_3 = z$ . The real Maxwell-Bloch equations are:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1x_3 \\ \dot{x}_3 &= -x_1x_2. \end{aligned} \quad (3)$$

So the system arises as an invariant subsystem of the Maxwell-Bloch equations for optical traveling-wave pulses in two-level media, a five (real) dimensional system on  $\mathbf{C}^2 \times \mathbf{R}$ . The later system itself originates from the 2:1:1 resonant nonlinear oscillator system (1) on  $\mathbf{C}^3$ . Equations (3) also appear as the large-Rayleigh-number limit of the famous Lorentz system; see Sparrow [17].

Equations (3) can be written as

$$\dot{x} = \nabla H_1 \times \nabla H_2, \quad (4)$$

where  $H_1 = \frac{1}{2}(x_2^2 + x_3^2)$ , and  $H_2 = x_3 + \frac{1}{2}x_1^2$  are the two conserved quantities. Equation (4) implies that the system is bi-Hamiltonian. In this paper, we obtain a new Lax pair and a new bi-Hamiltonian formulation. In fact, using the master symmetries, one can generate an infinite sequence of Hamiltonian structures. The theory of bi-Hamiltonian systems was developed by F. Magri [10].

Equation (4) may be re-expressed as

$$\dot{x} = \nabla H \times \nabla C, \quad (5)$$

where  $H$  and  $C$  are  $SL(2, \mathbf{R})$  combinations of  $H_1$  and  $H_2$ . In other words,  $H = \alpha H_1 + \beta H_2$ ,  $C = \mu H_1 + \nu H_2$ , with  $\alpha\nu - \beta\mu = 1$ . Then equations (5) are equivalent to (4). The Lie-Poisson structure of the system is easy to obtain. In local coordinates it is given by

$$\begin{aligned} \{x_1, x_2\} &= (\nu + \mu x_3) \\ \{x_1, x_3\} &= \mu x_2 \\ \{x_2, x_3\} &= \nu x_1. \end{aligned} \quad (6)$$

The corresponding Hamiltonian vector fields satisfy the following bracket relations

$$[X_1, X_2] = -\mu X_3 \quad (7)$$

$$[X_1, X_3] = \mu X_2 \quad (8)$$

$$[X_2, X_3] = -\nu X_1. \quad (9)$$

This Lie algebra depends on the two parameters  $\mu$  and  $\nu$ .

## 2 Bi-Hamiltonian structure and master-symmetries

The real Maxwell-Bloch equations can be written as a Lax pair:

$$\dot{L} = [B, L], \quad (10)$$

where

$$L = \begin{pmatrix} x_3 & \frac{1}{\sqrt{2}}(x_2 - \frac{1}{2}x_1^2) \\ \frac{1}{\sqrt{2}}(x_2 + \frac{1}{2}x_1^2) & \frac{1}{2}x_1^2 \end{pmatrix} \quad (11)$$

$$B = \begin{pmatrix} -x_1 & -\frac{1}{\sqrt{2}}x_1 \\ \frac{1}{\sqrt{2}}x_1 & 0 \end{pmatrix}. \quad (12)$$

Then  $H_1 = \text{Tr}L$  and  $H_2 = \frac{1}{2}\text{Tr}L^2$  are constants of motion and

$$H_1 = x_3 + \frac{1}{2}x_1^2 \quad (13)$$

$$H_2 = \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2. \quad (14)$$

We define two Poisson brackets  $\pi_1$  and  $\pi_2$  as follows:  $\pi_1$  is given by

$$\begin{aligned} \{x_1, x_2\}_1 &= 1 \\ \{x_1, x_3\}_1 &= 0 \\ \{x_2, x_3\}_1 &= x_1 \end{aligned} \quad (15)$$

and  $\pi_2$  is given by

$$\begin{aligned} \{x_1, x_2\}_2 &= -x_3 \\ \{x_1, x_3\}_2 &= x_2 \\ \{x_2, x_3\}_2 &= 0. \end{aligned} \quad (16)$$

Then we have

$$\pi_1 \nabla H_2 = \pi_2 \nabla H_1 \quad (17)$$

i.e, a bi-Hamiltonian system. For  $\pi_1$  bracket,  $H_2$  is the Hamiltonian and  $H_1$  is the Casimir. For  $\pi_2$  bracket  $H_1$  is the Hamiltonian and  $H_2$  is the Casimir.

If the vector

$$\vec{v} = \tau \frac{\partial}{\partial t} + A_1 \frac{\partial}{\partial x_1} + A_2 \frac{\partial}{\partial x_2} + A_3 \frac{\partial}{\partial x_3} \quad (18)$$

is a Lie point symmetry, then the following equations have to be satisfied:

$$\begin{aligned} \dot{A}_1 - \dot{\tau} x_1 - A_2 &= 0 \\ \dot{A}_2 - \dot{\tau} x_2 - x_1 A_3 - x_3 A_1 &= 0 \\ \dot{A}_3 - \dot{\tau} x_3 + x_1 A_2 + x_2 A_1 &= 0. \end{aligned} \quad (19)$$

One solution is the vector

$$\vec{X} = -t \frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial x_3}. \quad (20)$$

We can easily check that  $\vec{X}$  is actually a conformal symmetry satisfying

$$\begin{aligned} \vec{X}(H_1) &= 2H_1 \\ \vec{X}(H_2) &= 4H_2 \\ L_{\vec{X}}\pi_1 &= 3\pi_1 \\ L_{\vec{X}}\pi_2 &= \pi_2. \end{aligned} \quad (21)$$

Also the spatial part of  $\vec{X}$  is a master symmetry, as we should expect. There are also master symmetries sending  $\pi_1$  to  $\pi_2$ . These are given by

$$\vec{X}_p = p_1 \frac{\partial}{\partial x_1} + p_2 \frac{\partial}{\partial x_2} + p_3 \frac{\partial}{\partial x_3}, \quad (22)$$

where

$$\begin{aligned} p_1 &= k_1 x_1 + k_2 x_2 + k_3 x_3 \\ p_2 &= -k_1 x_2 + k_3 x_1 x_2 - x_2 x_3 \\ p_3 &= -k_1 x_1^2 - k_2 x_1 x_2 + \frac{1}{2} x_2^2 - k_3 x_1 x_3 + \frac{1}{2} x_3^2. \end{aligned} \quad (23)$$

For  $k_1 = 1$  and  $k_2 = k_3 = 0$ , we get

$$\vec{X}_1 = x_1 \frac{\partial}{\partial x_1} + (-x_2 - x_2 x_3) \frac{\partial}{\partial x_2} + \left( -x_1^2 + \frac{x_2^2}{2} + \frac{x_3^2}{2} \right) \frac{\partial}{\partial x_3}. \quad (24)$$

Similarly we have

$$\vec{X}_2 = x_2 \frac{\partial}{\partial x_1} - x_2 x_3 \frac{\partial}{\partial x_2} + \left( -x_1 x_2 + \frac{x_2^2}{2} + \frac{x_3^2}{2} \right) \frac{\partial}{\partial x_3} \quad (25)$$

and

$$\vec{X}_3 = x_3 \frac{\partial}{\partial x_1} + (x_1 x_2 - x_2 x_3) \frac{\partial}{\partial x_2} + \left( \frac{x_2^2}{2} - x_1 x_3 + \frac{x_3^2}{2} \right) \frac{\partial}{\partial x_3}. \tag{26}$$

The vector fields  $\vec{X}_1, \vec{X}_2$  and  $\vec{X}_3$  all send  $\pi_1$  to  $\pi_2$  and  $H_1$  to  $H_2$ . The master symmetries can be used to generate higher Poisson brackets. For example, one can define a quadratic bracket by taking the Lie derivative of  $\pi_2$  in the direction of  $X_1$ :

$$\begin{aligned} \{x_1, x_2\} &= 3x_3^2 - x_2^2 - 2x_1^2 \\ \{x_1, x_3\} &= 4x_2 + 2x_2 x_3 \\ \{x_2, x_3\} &= -4x_1 x_3 \end{aligned} \tag{27}$$

There is also a symplectic realization of the system. In  $R^4$  with coordinates  $(q_1, q_2, p_1, p_2)$ , we take as Hamiltonian

$$H = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + \frac{1}{8} q_1^4 - \frac{1}{2} q_1^2 p_2. \tag{28}$$

We have Hamilton's equations

$$\dot{q}_1 = p_1 \tag{29}$$

$$\dot{q}_2 = p_2 - \frac{1}{2} q_1^2 \tag{30}$$

$$\dot{p}_1 = q_1 p_2 - \frac{1}{2} q_1^3 \tag{31}$$

$$\dot{p}_2 = 0. \tag{32}$$

The mapping  $F : R^4 \rightarrow R^3$

$$F(q_1, q_2, p_1, p_2) = (q_1, p_1, -\frac{1}{2} q_1^2 + p_2) = (x_1, x_2, x_3) \tag{33}$$

gives the original Maxwell-Bloch equations

$$\dot{x}_1 = \dot{q}_1 = p_1 = x_2 \tag{34}$$

$$\dot{x}_2 = \dot{p}_1 = q_1 (p_2 - \frac{1}{2} q_1^2) = x_1 x_3 \tag{35}$$

$$\dot{x}_3 = -q_1 p_1 = -x_1 x_2. \tag{36}$$

The symplectic bracket is mapped onto the  $\pi_1$  bracket:

$$\{x_1, x_2\}_1 = \{q_1, p_1\} = 1 \tag{37}$$

$$\{x_1, x_3\}_1 = \{q_1, -\frac{1}{2} q_1^2 + p_2\} = 0 \tag{38}$$

$$\{x_2, x_3\}_1 = \{p_1, -\frac{1}{2} q_1^2 + p_2\} = q_1 = x_1. \tag{39}$$

We also have

$$\frac{1}{2} x_2^2 + \frac{1}{2} x_3^2 = H \tag{40}$$

and

$$x_3 + \frac{1}{2}x_1^2 = p_2 \quad (41)$$

which is conserved.

The bracket  $\pi_2$  is obtained as follows. Define a bracket  $s_1$

$$\begin{aligned} \{q_1, p_1\}_{s_1} &= \frac{1}{2}q_1^2 - p_2 \\ \{p_1, p_2\}_{s_1} &= p_2q_1 - \frac{1}{2}q_1^3 \\ \{q_1, p_2\}_{s_1} &= p_1 \\ \{q_2, p_2\}_{s_1} &= p_2 - \frac{1}{2}q_1^2 \\ \{q_2, p_1\}_{s_1} &= p_1. \end{aligned} \quad (42)$$

Then we have the relations

$$\{x_1, x_2\} = \{q_1, p_1\}_{s_1} = \frac{1}{2}q_1^2 - p_2 = -x_3 \quad (43)$$

$$\{x_1, x_3\} = \{q_1, -\frac{1}{2}q_1^2 + p_2\}_{s_1} = \{q_1, p_2\}_{s_1} = p_1 = x_2 \quad (44)$$

$$\{x_2, x_3\} = \{p_1, -\frac{1}{2}q_1^2 + p_2\}_{s_1} = 0. \quad (45)$$

Taking  $H = p_2$  as Hamiltonian, we obtain again Hamilton's equations (29)-(32).

If we represent by  $s_0$  the symplectic bracket, then we have the Lenard-type relations:

$$s_0 \nabla H = s_1 \nabla p_2, \quad (46)$$

but the two brackets are not compatible and they do not generate a recursion operator.

### 3 Group Symmetries

In this section, we study the symmetries of Newton's equations generated by the Hamiltonian

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{8}q_1^4 - \frac{1}{2}q_1^2 p_2. \quad (47)$$

From Hamilton's equations (29)-(32), we get Newton's equations by differentiation:

$$\ddot{q}_1 - q_1 \dot{q}_2 = 0 \quad (48)$$

$$\ddot{q}_2 + q_1 \dot{q}_1 = 0. \quad (49)$$

The above are also Lagrange's equations generated by the Lagrangian

$$L = \frac{\dot{q}_1^2}{2} + \frac{\dot{q}_2^2}{2} + \frac{1}{2}q_1^2 \dot{q}_2. \quad (50)$$

If

$$\vec{v} = \xi(q_1, q_2, t) \frac{\partial}{\partial t} + \eta_1(q_1, q_2, t) \frac{\partial}{\partial q_1} + \eta_2(q_1, q_2, t) \frac{\partial}{\partial q_2}$$

, the action of its second prolongation on Newton's equations gives:

$$\ddot{\eta}_1 - \dot{q}_1 \ddot{\xi} - q_1 \dot{q}_2 \ddot{\xi} - q_1 \dot{\eta}_2 - \dot{q}_2 \eta_1 = 0 \tag{51}$$

$$\ddot{\eta}_2 - \ddot{\xi} \dot{q}_2 + q_1 \dot{\xi} \dot{q}_1 + q_1 \dot{\eta}_1 + \dot{q}_1 \eta_1 = 0 \tag{52}$$

Equations (51),(52) are the conditions for  $\vec{v}$  to be a Lie point symmetry for Newton's equations. Expanding  $\dot{\xi}$ ,  $\ddot{\xi}$ ,  $\dot{\eta}_1$ ,  $\ddot{\eta}_1$ ,  $\dot{\eta}_2$ ,  $\ddot{\eta}_2$ , we will have:

$$\begin{aligned} &\eta_{1,tt} + 2\eta_{1,tq_1} \dot{q}_1 + 2\eta_{1,tq_2} \dot{q}_2 + \eta_{1,q_1 q_1} \dot{q}_1^2 + 2\eta_{1,q_1 q_2} \dot{q}_1 \dot{q}_2 + \eta_{1,q_2 q_2} \dot{q}_2^2 \\ &\quad + \eta_{1,q_1} q_1 \dot{q}_2 - \eta_{1,q_2} q_1 \dot{q}_1 - \xi_{tt} \dot{q}_1 - 2\xi_{tq_1} \dot{q}_1^2 - 2\xi_{tq_2} \dot{q}_2 \dot{q}_1 \\ &\quad - \xi_{q_1 q_1} \dot{q}_1^3 - 2\xi_{q_1 q_2} \dot{q}_1^2 \dot{q}_2 - \xi_{q_2 q_2} \dot{q}_2^2 \dot{q}_1 - 2\xi_{q_1} q_1 \dot{q}_1 \dot{q}_2 + \xi_{q_2} q_1 \dot{q}_1^2 \\ &\quad - \xi_t q_1 \dot{q}_2 - \xi_{q_2} q_1 \dot{q}_2^2 - \eta_{2,tq_1} - \eta_{2,q_1} q_1 \dot{q}_1 - \eta_{2,q_2} q_1 \dot{q}_2 - \eta_1 \dot{q}_2 = 0 \end{aligned} \tag{53}$$

and

$$\begin{aligned} &\eta_{2,tt} + 2\eta_{2,tq_1} \dot{q}_1 + 2\eta_{2,tq_2} \dot{q}_2 + \eta_{2,q_1 q_1} \dot{q}_1^2 + \eta_{2,q_2 q_2} \dot{q}_2^2 + 2\eta_{2,q_1 q_2} \dot{q}_1 \dot{q}_2 \\ &\quad + \eta_{2,q_1} q_1 \dot{q}_2 - \eta_{2,q_2} q_1 \dot{q}_1 - \dot{q}_2 \xi_{tt} - 2\xi_{tq_1} \dot{q}_1 \dot{q}_2 - 2\xi_{tq_2} \dot{q}_2^2 \\ &\quad - \xi_{q_1 q_1} \dot{q}_1^2 \dot{q}_2 - \xi_{q_2 q_2} \dot{q}_2^3 - 2\xi_{q_1 q_2} \dot{q}_1 \dot{q}_2^2 - \xi_{q_1} q_1 \dot{q}_2^2 + 2\xi_{q_2} q_1 \dot{q}_1 \dot{q}_2 \\ &\quad + q_1 \dot{q}_1 \xi_t + q_1 \dot{q}_1^2 \xi_{q_1} + q_1 \eta_{1t} + q_1 \dot{q}_1 \eta_{1,q_1} + q_1 \dot{q}_2 \eta_{1,q_2} + \dot{q}_1 \eta_1 = 0. \end{aligned} \tag{54}$$

Equations (53) and (54) must be satisfied identically in  $t, q_1, q_2, \dot{q}_1, \dot{q}_2$ , functions that are all independent. Equating the coefficients of  $\dot{q}_1^3, \dot{q}_1^2 \dot{q}_2, \dot{q}_1 \dot{q}_2^2$  to zero in both (53) and (54) and doing standard manipulations as in [9], for example, we obtain the overall result:

$$\xi = -c_1 t + c_2 \tag{55}$$

$$\eta_1 = c_1 q_1 \tag{56}$$

$$\eta_2 = c_1 q_2 + c_3 \tag{57}$$

where  $c_1, c_2, c_3$  are all constants.

For  $c_1 = c_3 = 0$ , but  $c_2$  different than zero, we get the time translation symmetry which generates the conservation of energy and for  $c_3$  different than zero we have translation in the cyclic  $q_2$  direction which is related to the conservation of  $p_2$ . Finally for  $c_2 = c_3 = 0$  and  $c_1$  different than zero, we have the symmetry

$$\vec{v} = -t \frac{\partial}{\partial t} + q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2} \tag{58}$$

The symmetries of Newton's equations form a 3-dimensional Lie algebra generated by

$$\vec{Y}_1 = -t \frac{\partial}{\partial t} + q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2} \tag{59}$$

$$\vec{Y}_2 = \frac{\partial}{\partial t} \tag{60}$$

$$\vec{Y}_3 = \frac{\partial}{\partial q_2} \tag{61}$$

with Lie algebra bracket multiplication given by:

$$\begin{aligned} [\vec{Y}_1, \vec{Y}_2] &= \vec{Y}_2 \\ [\vec{Y}_1, \vec{Y}_3] &= -\vec{Y}_3 \\ [\vec{Y}_2, \vec{Y}_3] &= 0. \end{aligned} \tag{62}$$

We can now find, which of the above Lie point symmetries are also variational symmetries using the Lagrangian  $L$  and Noether's theory: Both  $\vec{Y}_2$  and  $\vec{Y}_3$  are variational symmetries since they satisfy the condition

$$\text{pr}^{(1)}\vec{v}(L) + L \text{div}(\xi) = 0, \tag{63}$$

which is the necessary and sufficient condition for the vector  $\vec{v}$  to be a variational symmetry. On the other hand,  $\vec{Y}_1$  is not a variational symmetry. Using the general theorem of Noether, we can find the corresponding conserved quantities. For  $\vec{Y}_2$  we have

$$Q_1 E_1(L) + Q_2 E_2(L) = -\dot{q}_1 E_1(L) - \dot{q}_2 E_2(L) = -\frac{d}{dt} \left( \frac{1}{2} \dot{q}_1^2 + \frac{1}{2} \dot{q}_2^2 \right)$$

and so the conserved quantity is the Hamiltonian  $H$  as we should expect. Similarly for  $\vec{Y}_3$  using again the above theorem, we can prove that the conserved quantity is the momentum  $p_2$ .



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