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On Methods of Finding Bäcklund Transformations in Systems with More than Two Independent Variables

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Abstract

Bäcklund transformations, which are relations among solutions of partial differential equations–usually nonlinear–have been found and applied mainly for systems with two independent variables. A few are known for equations like the Kadomtsev-Petviashvili equation [1], which has three independent variables, but they are rare. Wahlquist and Estabrook [2] discovered a systematic method for searching for Bäcklund transformations, using an auxiliary linear system called a prolongation structure. The integrability conditions for the prolongation structure are to be the original differential equation system, most of which systems have just two independent variables. This paper discusses how the Wahlquist-Estabrook method might be applied to systems with larger numbers of variables, with the Kadomtsev-Petviashvili equation as an example. The Zakharov-Shabat method is also discussed. Applications to other equations, such as the Davey-Stewartson and Einstein equation systems, are presented.

I Introduction

The following discussion reports some completed work and some work in progress. In the study of various nonlinear partial differential equations (pde) that has taken place in the last 30 years, it has become recognized that certain pde's admit what are called "Bäcklund transformations" (BT's). These can be given a formal definition [3], but I shall speak of them here only as equations that allow one to find a new solution of the given partial differential equation from an old one, often by simple quadratures (integrations.) These can also give rise to BT "superposition" relations that enable one to find new solutions algebraically after one has performed the first integration steps. It should be noted in the following that, because of the variety of mathematical expressions, some latin and greek letters will be used more than once; generally the notation within any one section is unique.

II Sine-Gordon Equation

A simple, standard example of a BT is that for the sine-Gordon equation, written as [4]

$$\phi_{uv} = \sin\phi, \tag{1}$$

where subscripts indicate differentiation. The BT is

$$\begin{aligned} \phi'_u &= \phi_u + 2k \sin\Big(\frac{\phi' + \phi}{2}\Big), \\ \phi'_v &= -\phi_v + 2k^{-1} \sin\Big(\frac{\phi' - \phi}{2}\Big), \end{aligned} \tag{2}$$

where ϕ is an old (seed) solution of Eq. (1), ϕ' is a new solution, and k is a parameter. Eqs. (2) are to be integrated for ϕ'. This is particularly easy if the seed solution is simply the zero solution; then integration gives

$$\phi' = 4\arctan(\exp(ku + k^{-1}v)), \tag{3}$$

the single soliton solution. If ϕ_0 is a beginning solution, and ϕ_1 and ϕ_2 are solutions obtained from applying the BT to ϕ_0, with parameters k_1 and k_2, respectively, then ϕ_3, defined by

$$\tan\Big(\frac{\phi_3 - \phi_0}{4}\Big) = \frac{k_1 + k_2}{k_1 - k_2} \tan\Big(\frac{\phi_1 - \phi_2}{4}\Big), \tag{4}$$

is also a solution of Eq. (1) (BT superposition.) [5]

How does one obtain the BT? One general systematic method for doing this is due to Wahlquist and Estabrook (WE) [2]. We illustrate it here for the sine-Gordon equation.

First we define new variables in order to write the given pde(s) as first order equations. For our case, we define

$$r = \phi_u \tag{5a}$$

then

$$r_v = \sin\phi. \tag{5b}$$

One can search for a potential z of these equations by writing

$$dz = f(r,\phi)du + g(r,\phi)dv \tag{6}$$

and requiring integrability, substituting from Eq. (5) where possible. That gives

$$f_r \sin\phi + f_\phi \phi_v = g_r r_u + r g_\phi. \tag{7}$$

Since r_u and ϕ_v are independent, we must put $f_\phi = g_r = 0$. All equations can now be solved to find

$$f = r^2, \quad g = -2\cos\phi. \tag{8}$$

Thus z, defined as an integral, is

$$z = \int r^2 du - 2\cos\phi\, dv, \tag{9}$$

which is always defined because of the conditions imposed, provided that ϕ is a solution of Eq. (1). Equivalently, we can write Eq. (5) as 2-forms

$$\begin{aligned} \alpha &= d\phi dv - r du dv, \\ \beta &= dr du + \sin\phi \, du dv, \end{aligned} \tag{10}$$

which are the basis of an ideal I of 2-forms. (We often suppress the symbol $\wedge$ in the hook product of forms.) Then we write a 1-form

$$\sigma = -dz + f(r,\phi)du + g(r,\phi)dv \tag{11}$$

and require

$$d\sigma \subset I \tag{12}$$

giving the same equations as before.

The WE method generalizes these equations by letting the potential become a "pseudopotential" z, the equations for which are generalized by letting z be a dependent variable:

$$dz = F(r,\phi,z)du + G(r,\phi,z)dv. \tag{13}$$

Integrability and substitution from Eq. (13) now give

$$-F_r \sin\phi + rG_\phi + FG_z - GF_z = 0. \tag{14}$$

Equivalently, we can write

$$\omega = -dz + F(r,\phi,z)du + G(r,\phi,z)dv \tag{15}$$

and require

$$d\omega \subset \{I, \omega\}. \tag{16}$$

A sufficiently general solution of these equations is

$$F = ra + b, \quad G = c\sin\phi + e\cos\phi, \tag{17}$$

where a, b, c, and e are functions of z satisfying a certain set of ordinary differential equations. These may be solved to give (where k is a constant)

$$\omega = -dz + (k\sin z - r)du + k^{-1}\sin(z+\phi)dv. \tag{18}$$

This method can in turn be generalized to many pseudopotentials ζ^μ. We write

$$d\zeta^\mu = F^\mu(r,\phi,\zeta^\nu)du + G^\mu(r,\phi,\zeta^\nu)dv. \tag{19}$$

While these equations may be written in general, an extremely important special case is that in which the equations are linear in the pseudopotentials:

$$d\zeta^\mu = F^\mu{}_\nu(r,\phi)\zeta^\nu du + G^\mu{}_\nu(r,\phi)\zeta^\nu dv. \tag{20}$$

We may write Eq. (20) in terms of column vectors and matrices. If ζ is a column vector of the ζ^μ and ω is a column vector of 1-forms, we write

$$\omega = -d\zeta + (Fdu + Gdv)\zeta, \tag{21}$$

where now F and G are matrix functions of r and ϕ. Solution gives

$$F = rA + B, \quad G = C\sin\phi + E\cos\phi, \tag{22}$$

where A, B, C, and E are constant matrices. These matrices now satisfy an incomplete Lie algebra:

$$\begin{aligned} [A, C] &= -E, \qquad [B, E] = 0, \\ [A, E] &= C, \qquad [B, C] = -A. \end{aligned} \tag{23}$$

Such a set of equations, or the associated linear equations, is often called a "prolongation structure" (PS). A particular representation of the matrices, satisfying Eq. (23), is

$$B = k^2 E = \frac{k}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \frac{1}{2k}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{24}$$

Then the ζ^μ satisfy the equations (to be annulled)

$$\begin{aligned} \omega^1 &= -d\zeta^1 + \frac{1}{2}(-k\zeta^1 + r\zeta^2)du + \frac{1}{2k}(\zeta^2\sin\phi - \zeta^1\cos\phi)dv, \\ \omega^2 &= -d\zeta^2 + \frac{1}{2}(k\zeta^2 - r\zeta^1)du + \frac{1}{2k}(\zeta^1\sin\phi + \zeta^2\cos\phi)dv. \end{aligned} \tag{25}$$

The original pseudopotential z is now given by

$$z = 2\arctan(\zeta^2/\zeta^1). \tag{26}$$

To find the BT, we propose the *Ansatz* that there are new field variables r', ϕ' which are functions of the old variables r, ϕ and the pseudopotential(s). It is sufficient just to use z. The new variables are substituted in the original pde and equations for the functions are found, given that the old variables also satisfy the original pde and that z satisfies $\omega = 0$ in Eq. (18). Equivalently, we write the differential form relations

$$\{\alpha', \beta'\} \subset \{\alpha, \beta, \omega\} \tag{27a}$$

or

$$I' \subset \{I, \omega\}. \tag{27b}$$

We find

$$r' = r - 2k\sin z, \quad \phi' = -\phi - 2z. \tag{28}$$

The first of these is the first of Eqs. (2); the v-derivative of the second is the second of Eqs. (2).

The essential steps can be formulated as: (1) put the pde(s) into a first–order form; (2) assume a PS, find the incomplete Lie algebra, and find a representation for it; (3) assume new solution variables which are functions of the old ones and the pseudopotentials (or prolongation variables), and solve the resulting equations. In the results, it may be possible to eliminate the pseudopotentials as above.

We may write the equations for the PS in the general approach as, where x and t are now taken to be the independent variables,

$$\zeta_x = -F\zeta, \quad \zeta_t = -G\zeta. \tag{29}$$

Equivalently,

$$\omega = d\zeta + (Fdx + Gdt)\zeta \tag{30}$$

is to be annulled. Then the requirement of $(\zeta_x)_t = (\zeta_t)_x$, substituting where appropriate from Eq. (29), and setting the coefficient of ζ to zero, or alternatively requiring

$$d\omega \subset \{I, \omega\}, \tag{31}$$

gives

$$-F_t + G_x + [F, G] = 0 \quad (\text{mod } S), \tag{32}$$

where S means the set of original field equations, or

$$dFdx + dGdt + [F, G]dxdt \subset I \tag{33}$$

in terms of the ideal I of forms.

The number of pseudopotentials may often be taken as two, so that the dimension of the matrices is 2×2. In other cases, a higher number (say, n) may be needed. Usually, ratios of the pseudopotentials may be used in step (3), thus yielding $n - 1$ nontrivial variables. In the PS for the Ernst equation of general relativity [6], it was also necessary to allow the matrices to be functions of a particular combination of independent variables (invariant under the equation symmetry group).

This method was successfully applied to the Korteweg-deVries equation [2,7], the nonlinear Schrödinger equation [8], the Ernst equation as already noted, and other equations. These equations all have just two independent variables; in that case, the maximum rank of the basis forms in the ideal I (from the pde's) is two.

III Generalization to Three Variables

A natural question is: can the method be applied to systems with more than two independent variables?

To attempt to generalize the basic method to three variables [4], we model it after the two–dimensional case. We first put the equations in a first–order form. Then we assume a vector of pseudopotentials ζ, defined as follows, where x, y, and t are taken as the independent variables. These coordinates and equations are adapted for the KP equation which follows below.

$$\begin{aligned} \zeta_x &= -F\zeta - A\zeta_y \\ \zeta_t &= -G\zeta - B\zeta_y. \end{aligned} \tag{34}$$

Integrability, $(\zeta_x)_t = (\zeta_t)_x$, and setting the coefficients of ζ, ζ_y, and ζ_{yy} to zero now give:

$$[A, B] = 0 \tag{ 0.35}$$

$$[A, G] - [B, F] = 0 \tag{ 0.36}$$

and

$$-F_t + G_x + AG_y - BF_y + [F, G] = 0 \quad (\text{mod } S). \tag{37}$$

Equivalently, we define a column of 2-forms ω, a matrix of 2-forms α, and a matrix of 1-forms β, with

$$\omega = \alpha\zeta + \beta \wedge d\zeta, \tag{38}$$

where ζ is a column of 0-forms (pseudopotentials) as before. We require

$$d\omega \subset \{\omega, I\}, \tag{39}$$

which suggests

$$d\omega - \rho \wedge \omega \subset I, \tag{40}$$

where ρ is a matrix 1- form. This gives

$$d\omega - \rho \wedge \omega = \alpha \wedge d\zeta + (d\alpha)\zeta + d\beta \wedge d\zeta - \rho \wedge (\alpha\zeta + \beta \wedge d\zeta) \subset I$$

or

$$\alpha - \rho \wedge \beta + d\beta \subset I \tag{41}$$

and

$$d\alpha - \rho \wedge \alpha \subset I \tag{42}$$

which together give

$$(d\rho - \rho \wedge \rho) \wedge \beta \subset I \tag{43}$$

($\rho \wedge \rho \neq 0$ in general because ρ is a matrix.)

This requires some supplementing in order to get Eqs. (35) to (37). We write

$$\beta = 1\, dy - Adx - Bdt \tag{44}$$

(1 is the unit matrix) and require A and B to be constant, so that $d\beta = 0$. We also require $\beta \wedge \beta = 0$, which gives Eq. (35). We also take

$$\rho = -Fdx - Gdt. \tag{45}$$

A term in dy would be redundant, since we could replace it with β. Then

$$\alpha = \rho \wedge \beta = -Fdxdy - Gdtdy + (FB - GA)dxdt. \tag{46}$$

We require

$$\rho \wedge \beta + \beta \wedge \rho = 0, \tag{47}$$

giving Eq. (36). Finally, Eq. (42) or Eq. (43) gives Eq. (37). We note that $d\zeta$ may be written as

$$d\zeta = \rho\zeta + \beta\zeta_y\, ; \tag{48}$$

then (mod I)

$$\beta \wedge d\zeta = \beta \wedge \rho\zeta = -\rho \wedge \beta\zeta = -\alpha\zeta$$

or $\omega = 0$. It will be seen that, as in the terms AG_y and BF_y in Eq. (37), individual matrix products now appear, not just in commutators as in the two-variable case.

For step (3), we assume functional dependence of the new field variables on the pseudopotentials and old field variables as before, but now it appears necessary to assume dependence also on some pseudopotential first derivatives, as will be seen in the treatment of the KP equation discussed below.

IV Kadomtsev-Petviashvili (KP) equation

This is the equation [9, 1]

$$3u_{tt} + 6(uu_x)_x + u_{xxxx} + 3u_{xy} = 0. \tag{49}$$

We write the ideal of forms as

$$\begin{gathered}
(dudt - pdxdt)dy \\
(dpdt - rdxdt)dy \\
(dwdt - zdxdt)dy \\
(dpdx - \frac{4}{3}dzdt)dy \\
\Big(dwdx + \frac{3}{2}updxdt + \frac{1}{4}drdt\Big)dy + \frac{3}{4}dudxdt,
\end{gathered} \tag{50}$$

where we have defined $p = u_x$, $r = p_x = u_{xx}$, and $z = -\frac{3}{4}u_t = w_x$. If we annul these forms and treat all dependent variables as functions of x, t, and y, we get

$$\begin{aligned}
w_t &= \frac{3}{2}uu_x + \frac{1}{4}u_{xxx} + \frac{3}{4}u_y \\
w_x &= -\frac{3}{4}u_t,
\end{aligned}$$

the integrability conditions of which yield the KP equation, Eq. (49).

A PS for the KP equation was given by Morris several years ago [10]:

$$\begin{aligned}
{\zeta^1}_x &= \zeta^2 \\
{\zeta^2}_x &= \zeta^3 - \frac{3}{4}u\zeta^1 \\
{\zeta^3}_x &= -\frac{3}{4}u\zeta^2 - (w-\mu)\zeta^1 - \frac{3}{4}{\zeta^1}_y \\
{\zeta^1}_t &= -\zeta^3 - \frac{1}{4}u\zeta^1 \\
{\zeta^2}_t &= \frac{1}{2}u\zeta^2 + \Big(w - \mu - \frac{1}{4}p\Big)\zeta^1 + \frac{3}{4}{\zeta^1}_y \\
{\zeta^3}_t &= -\Big(\frac{r}{4} + \frac{9u^2}{16}\Big)\zeta^1 + \Big(w - \mu + \frac{1}{4}p\Big)\zeta^2 - \frac{1}{4}u\zeta^3 + \frac{3}{4}{\zeta^2}_y,
\end{aligned} \tag{51}$$

where μ is a constant. These can be written in terms of forms, but that will not be done here.

How does one find such a PS? It seems sufficient to assume a lower triangular structure for A and B in Eqs. (35-37), then to explore that effect on the other matrices. The values of the matrices for the PS given here are (from [10], corrected)

$$A = \frac{3}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad B = -\frac{3}{4}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{52a}$$

and

$$F = \begin{pmatrix} 0 & -1 & 0 \\ (3/4)u & 0 & -1 \\ w-\mu & (3/4)u & 0 \end{pmatrix}, \quad G = \begin{pmatrix} u/4 & 0 & 0 \\ -w+\mu+p/4 & -u/2 & 0 \\ r/4+9u^2/16 & -w+\mu-p/4 & u/4 \end{pmatrix}. \tag{52b}$$

Morris does not give a BT for the KP equation. This is possibly because he may have assumed functional dependence on just the ζ^μ, whereas one must also include the ${\zeta^\mu}_y$, as shown below.

To try to find a BT, we can simply assume that there exist new variables u', p', r', w', and z', which are functions of u, p, r, w, and z, the ζ^μ, and ${\zeta^\mu}_y$, and which satisfy the original KP equation (Eq. (49)). This assumption leads to a straightforward, but very messy, set of equations, which can indeed be solved to give a BT. But one can also simplify the process by some reasonable assumptions based on scale.

We first define new variables which are ratios of the ζ^μ and the ${\zeta^\mu}_y$ to ζ^1:

$$\alpha = \frac{\zeta^2}{\zeta^1},\ \beta = \frac{\zeta^3}{\zeta^1},\ \gamma = \frac{{\zeta^1}_y}{\zeta^1},\ \delta = \frac{{\zeta^2}_y}{\zeta^1},\ \epsilon = \frac{{\zeta^3}_y}{\zeta^1}. \tag{53}$$

The reason for doing this is that previously found expressions for BT's seem to be functions of only such ratios, not the separate pseudopotentials. Since the PS is linear in the pseudopotentials, we may consider one of them to set the scale, while the ratios appear in the BT.

These ratios satisfy certain relations, derived from Eq. (51), and here expressed most economically in terms of 1- and 2-forms:

$$d\alpha = \Big(\beta - \alpha^2 - \frac{3}{4}u\Big)dx + \Big(w - \mu - \frac{p}{4} + \frac{3}{4}u\alpha + \alpha\beta + \frac{3}{4}\gamma\Big)dt + (\delta - \alpha\gamma)dy, \tag{54a}$$

$$\begin{aligned} d\beta = &-\Big(w - \mu + \frac{3}{4}u\alpha + \alpha\beta + \frac{3}{4}\gamma\Big)dx + \Big(\epsilon - \beta\gamma\Big)dy \\ &+\Big[-\frac{r}{4} - \frac{9u^2}{16} + \Big(w - \mu + \frac{p}{4}\Big)\alpha + \frac{3}{4}\delta + \beta^2\Big]dt, \end{aligned} \tag{54b}$$

$$dy d\gamma = (\alpha\gamma - \delta)dxdy + (\epsilon - \beta\gamma)dtdy - \frac{1}{4}dudt + \frac{1}{4}pdxdt, \tag{54c}$$

and

$$\begin{aligned} dyd\delta = &\Big(\frac{3}{4}u\gamma - \epsilon + \alpha\delta\Big)dxdy + \Big(-\frac{3}{4}\delta + \frac{r}{4} - \frac{1}{2}p\alpha + \frac{3}{4}\alpha\gamma\Big)dxdt \\ &+\Big[\Big(w - \mu - \frac{p}{4}\Big)\gamma + \frac{3}{4}u\delta + \delta\beta + \frac{3}{4}\gamma^2\Big]dydt \\ &-\frac{3}{4}dtd\gamma - \frac{3}{4}dudx + \Big(dw - \frac{1}{4}dp + \frac{1}{2}\alpha du\Big)dt. \end{aligned} \tag{54d}$$

Now we note that we may consider the field variables and the new variables from Eq. (53) to have particular scales, as may be derived by inspection from Eqs. (49), (50), and (54). Finally, by dimensional considerations we may note that combinations of variables having the same dimension as u or u' are simply $u, \beta, \alpha^2, m\alpha$, and m^2, where m is a dimensional constant. Since we do not expect u' to be a function of any derivatives higher than u itself, we simply assume that u' has the form

$$u' = au + b\beta + c\alpha^2 + em\alpha + fm^2, \tag{55}$$

where the coefficients a, b, c, e, f are simply numbers. Had we assumed that u' was a function of higher derivatives of u, then ultimately derivatives of order too high to accommodate would appear. Thus with this sort of argument we may work out the expected form for u' in terms of the given variables.

We also make the assumption, using the same type of analysis, that (k is a dimensional constant)

$$w' = gw + h\gamma + lk + np + qm\beta + vmu + sm\alpha^2 + t\alpha\beta + \lambda\alpha^3 + \theta\alpha u + \psi m^2\alpha + \zeta m^3, \quad (56)$$

where $g, h, l, n, q, v, s, t, \lambda, \theta, \psi$, and ζ are numbers. Now we use these quantities to define the derivative quantities p', r', and z', and require them satisfy the KP equation. We find only one nontrivial set of values, that the constants in Eq. (55) are given by $a = -(1/2), b = -c = 2, e = f = 0$. Then the BT takes the form

$$u' = -\frac{1}{2}u + 2\beta - 2\alpha^2 \quad (57)$$

which may be written as (if we put ϕ for ζ^1)

$$u' = u + \frac{2\phi_{xx}}{\phi} - \frac{2{\phi_x}^2}{\phi^2} = u + 2(\ln\phi)_{xx}. \quad (58)$$

The equations for the pseudotentials may be written in the original form Eq. (51). These equations may be boiled down to the form:

$$\begin{aligned} \phi_t &= -u\phi - \phi_{xx}, \\ \phi_y &= -p\phi - \frac{4}{3}(w-\mu)\phi - 2u\phi_x - \frac{4}{3}\phi_{xxx}. \end{aligned} \quad (59)$$

Thus, if a (seed) solution u (with corresponding p and w) to the KP equation is known, and if ϕ is a solution of the linear Eqs. (59), then a BT is given by Eq. (58). Essentially, these equations are the same as found by Chen [1], who found the BT somewhat by inspection. The function ϕ is the τ-function used by some authors.

If we take the trivial seed solution $u = 0$, we get an instant set of solutions given by, for some function $f(a)$,

$$\phi = \int da f(a) \exp[ax + \frac{4}{3}(\mu - a^3)y - a^2 t]. \quad (60)$$

Solutions found in this manner may be compared with the solutions given by Ablowitz and Segur [11] and by Chen [1].

It may be remarked that inspection is much easier than this method. That is true enough, but one may not always be able to find a BT by inspection.

V. Davey Stewartson (DS) equation(s)

We write the DS equation(s) [12] in this form:

$$\begin{aligned} qr &= 4\gamma_{uv} \\ q_t &= 2(q_{uu} + q_{vv}) + 4\epsilon q(\gamma_{uu} + \gamma_{vv}) \\ -r_t &= 2(r_{uu} + r_{vv}) + 4\epsilon r(\gamma_{uu} + \gamma_{vv}), \end{aligned} \quad (61)$$

where $\epsilon = \pm 1$. Generally, in this notation, γ is taken as real, $r = q*$, and t is pure imaginary. u and v may be real or may be mutual complex conjugates. Eq. (61) may be converted to forms treated by other authors, as shown in the Appendix.

Again we know a linear PS, written as follows (generalized from [13])

$$\begin{aligned} A_v &= \frac{1}{2}(\zeta A + qB) \\ B_u &= \frac{1}{2}(\mu B - \epsilon r A) \\ A_t &= 2A_{uu} - 2\mu A_u + Bq_v - qB_v \\ &\quad + 4\epsilon A\gamma_{uu} + \Big(\lambda + \frac{\zeta^2}{2}\Big)A + \frac{\zeta}{2}qB \\ -B_t &= 2B_{vv} - 2\zeta B_v + \epsilon r A_u - \epsilon A r_u \\ &\quad + 4\epsilon B\gamma_{vv} + \Big(-\lambda + \frac{\mu^2}{2}\Big)B - \frac{\epsilon\mu}{2}rA \end{aligned} \tag{62}$$

or alternatively (because the original equations are invariant with respect to interchange of u and v):

$$\begin{aligned} C_u &= \frac{1}{2}(\tau C + qD) \\ D_v &= \frac{1}{2}(\psi D - \epsilon r C) \\ C_t &= 2C_{vv} - 2\psi C_v + Dq_u - qD_u \\ &\quad + 4\epsilon C\gamma_{vv} + \Big(\chi + \frac{\tau^2}{2}\Big)C + \frac{\tau}{2}qD \\ -D_t &= 2D_{uu} - 2\tau D_u + \epsilon r C_v - \epsilon C r_v \\ &\quad + 4\epsilon D\gamma_{uu} + \Big(-\chi + \frac{\psi^2}{2}\Big)D - \frac{\epsilon\psi}{2}rC, \end{aligned} \tag{63}$$

where $\zeta, \mu, \lambda, \tau, \psi$, and χ are constant.

Can we find a BT by using either of these PS, using the above method? If we assume q', r' and γ' to be functions of $q, r, \gamma, A, B, A_u, B_v, C, D, C_v, D_u, \gamma_u, \gamma_v$, we find that does not work.

However, there is a treatment of the DS equation in the literature [14], [15], using the Zakharov-Shabat (ZS) "dressing method" [16], which gives partial results. We outline a version of the ZS method briefly as follows. Assume we have given two equations for a variable ϕ (which may be regarded as a column pseudopotential in the WE method), in terms of two independent linear operators T_1 and T_2 (α and β are constant):

$$T_1\phi = \alpha\phi, \quad T_2\phi = \beta\phi. \tag{64}$$

Thus, $T_1T_2\phi = \alpha\beta\phi = T_2T_1\phi$, so

$$[T_1, T_2]\phi = 0. \tag{65}$$

We require

$$[T_1, T_2] = 0, \tag{66}$$

independent of ϕ. Eq. (66) is the (set of) field equation(s) which we wish to study.

We now assume a new $\bar{\phi}$ related to the old ϕ by

$$\bar{\phi} = G\phi \tag{67}$$

with new operators $\bar{T}_1, \bar{T}_2$, with $\bar{T}_1\bar{\phi} = \bar{\alpha}\bar{\phi}$, $\bar{T}_2\bar{\phi} = \bar{\beta}\bar{\phi}$. Then $[\bar{T}_1, \bar{T}_2] = 0$ is the set of field equations in terms of new field variables, and is to be satisfied when Eq. (66) is satisfied. We can arrange this by writing (where ρ and ξ are constant)

$$\bar{T}_1 G = \xi G T_1, \quad \bar{T}_2 G = \rho G T_2 \tag{68}$$

and then we have

$$\begin{aligned}[\bar{T}_1, \bar{T}_2]G &= \bar{T}_1(\bar{T}_2 G) - \bar{T}_2(\bar{T}_1 G) \\ &= \bar{T}_1(\rho G T_2) - \bar{T}_2(\xi G T_1) \\ &= \xi\rho G[T_1, T_2] = 0\end{aligned}$$

as desired. Eqs. (68) constitute a BT, relating $\bar{T}_1$ and $\bar{T}_2$ to T_1 and T_2 and the old and new field variables. If one is given the old operators T_i in terms of the old field variables, and the new operators $\bar{T}_i$ are to have the same form in terms of the new field variables, then these equations in principle can be solved for G and for the BT field variable relations.

The operators T_1 and T_2 may have matrix, derivative, or integral terms. Especially with the latter, global properties are usually considered in the literature. However, we treat these equations purely locally here. Where integrals of variables occur, we treat them as indefinite integrals and usually define them as new variables.

The ZS approach easily allows the use of more than two independent variables. However, if one is confronted with the field equations themselves, there is no automatic prescription which describes how to find the linear operators T_1 and T_2. The absence of this prescription makes the method somewhat different from the WE method discussed above. However, if one is able to find T_1 and T_2, then this method has some definite advantages, as will be seen below for the DS equation.

The following prescription gives the DS equation(s):

$$\begin{aligned}T_1 &= \frac{\partial}{\partial x} + A\frac{\partial}{\partial y} - P, \\ T_2 &= \frac{\partial}{\partial t} - B + 2P\frac{\partial}{\partial y} - 2A\frac{\partial^2}{\partial y^2},\end{aligned} \tag{69a}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & q \\ -\epsilon r & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 4\epsilon\gamma_{vv} & 2q_v \\ 2\epsilon r_u & -4\epsilon\gamma_{uu} \end{pmatrix}. \tag{69b}$$

In working this out, we have defined the new variable γ defined by $qr = 4\gamma_{uv}$ so that we can express the integrals $\int qrdu$ and $\int qrdv$ in terms of derivatives of γ. We also have defined new independent variables $u = x + y$ and $v = x - y$. If one expands Eq. (65), one finds the earlier PS Eq. (62), but with the specialization $\alpha = \beta, \xi = \rho = 1$.

We now find by writing out Eq. (68) and going through some calculation that we can write G as

$$G = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\frac{\partial}{\partial y} + \begin{pmatrix} \epsilon\eta_v & (1/2)(\lambda\bar{q} - q) \\ (\epsilon/2)(\bar{r} - \lambda r) & -\lambda\epsilon\eta_u \end{pmatrix}, \tag{70}$$

where λ is a constant and where $\eta = \bar{\gamma} - \gamma$. Furthermore, we have the following equations relating old and new field variables; these constitute the formal BT. The first two of these, constituting the "spatial part" of the BT, may be found in ref. [14], [15]; the last one, the "time part", does not appear to be written out and thus is presented for the first time here.

$$\begin{aligned} \bar{q}_u &= -\epsilon\bar{q}\eta_u + \lambda^{-1}(q_v - \epsilon q\eta_v) \\ \bar{r}_v &= -\epsilon\bar{r}\eta_v + \lambda(r_u - \epsilon r\eta_u) \\ \eta_t &= -2\eta_{uu} + 2\eta_{vv} - 2\epsilon{\eta_u}^2 + 2\epsilon{\eta_v}^2 \\ &\quad - 4\gamma_{uu} + 4\gamma_{vv} + \lambda^{-1}q\bar{r} - \lambda r\bar{q}. \end{aligned} \tag{71}$$

From the first two of these, one can express $\bar{q}$ and $\bar{r}$ as follows:

$$\begin{aligned} \bar{q} &= (1/\lambda\psi)\int du\,(\psi q_v - q\psi_v) \\ \bar{r} &= (\lambda/\psi)\int dv\,(\psi r_u - r\psi_u), \end{aligned} \tag{72}$$

where $\eta = \epsilon\ln\psi$. One now sees from the third equation that ψ satisfies the integro-differential equation

$$\begin{aligned} \psi_t &= 2(\psi_{vv} - \psi_{uu}) + 4\epsilon\psi(\gamma_{vv} - \gamma_{uu}) \\ &\quad + \epsilon q\int dv\,(\psi r_u - r\psi_u) - \epsilon r\int du\,(\psi q_v - q\psi_v). \end{aligned} \tag{73}$$

This equation has not yet been explored; however, one can see that it is the analog to Eq. (59) for the KP equation.

One sees that the assumption made in the WE method, that the new field variables are functions of the old variables and the pseudopotentials, apparently does not apply here. The more general form Eq. (72) is suggestive of possible approaches to other equations, but does not seem to provide an unambiguous method of searching for BTs of arbitrary equations.

VI. General relativity

We already know two BTs for the Ernst equation, the equation found if one assumes two symmetries (stationary axisymmetry) for the general relativity equations [6], [17].

In general relativity, if one writes out the full equations, there are many variables–the 10 metric coefficients, the 40 Christoffel symbols, and any variables associated with the presence of matter or nongravitational fields. So in some ways it is more economical to work with differential forms instead of the individual variables. It is possible to write the vacuum equations, for the cases of one symmetry and no symmetry (general) in terms of sets of forms. The forms for one symmetry are given in [3], [18], those for the general vacuum equations in [19]. Interestingly, there is not an obvious relation between the two.

One then hopes to be able to use some such scheme as WE or ZS in order to work with differential forms.

A PS is known for the general case [19]. It was used to find a very restricted BT, which produced only a limited class of already known solutions (Kerr-Schild). It can be generalized slightly [20]. A PS is not known for the single symmetry case.

In summary: the Wahlquist-Estabrook method, including the *Ansatz* of functional dependence to get the BT, works well for two–variable systems. It may also work for three–variable systems, but apparently not consistently. The ZS approach may need to be used; it may be possible to combine the two approaches. The WE approach can be formulated in terms of differential forms; while is probably not necessary for most systems, it may be important for use in general relativity; but results in that area have not yet been obtained.

VII. Appendix

For the Davey-Stewartson equation(s), we present various versions here.

In Eq. (61), we first change the variables as follows. Put $\gamma = -\eta\beta$, $t = -i\eta\tau$, $\epsilon = \eta\lambda$, $x = a^2(u-v)/2$, $y = a(u+v)/2$, $a^2 = \xi$, $\kappa = -\xi\eta$, where $\xi, \eta, \kappa, \lambda = \pm 1$. Then Eq. (61) takes the form

$$\begin{aligned} qr &= \eta\beta_{xx} + \kappa\beta_{yy} \\ iq_\tau &= \eta q_{xx} - \kappa q_{yy} - 2\lambda q(\eta\beta_{xx} - \kappa\beta_{yy}) \\ -ir_\tau &= \eta r_{xx} - \kappa r_{yy} - 2\lambda r(\eta\beta_{xx} - \kappa\beta_{yy}). \end{aligned} \tag{74}$$

In the following, we always have $r = q*$ and $\eta = 1$.

Ablowitz and Segur [11] is obtained by putting $\phi = -4\beta_x$, $\sigma = \lambda$, $\tau = -t$, $q = A/\sqrt{2}$ and interchanging x and y.

Cheng, Li, and Tang [21] is recovered by writing $\alpha^2 = \lambda = -\kappa, \tau = t$, and

$$\begin{aligned} \psi &= \alpha^2\beta_{xx} + \beta_{yy} \\ a &= i(\beta_{xx} + 2\alpha\beta_{xy} + \alpha^2\beta_{yy}) \\ b &= i(-\beta_{xx} + 2\alpha\beta_{xy} - \alpha^2\beta_{yy}). \end{aligned} \tag{75}$$

Here, DS I is the case $\alpha = \lambda = -\kappa = 1$, while DS II is the case $\alpha = i, \lambda = -\kappa = -1$.

Levi, Pilloni, and Santini [22] is found by putting $\kappa = 1, z = y, T = \tau$, and

$$\Gamma = -\lambda(\beta_{xx} - \beta_{zz} + 2i\beta_{xz}). \tag{76}$$

Morris [13] results from $q = A, t = -\tau, \lambda = -\kappa = 1$, and

$$\begin{aligned} \phi &= (1/2)(\beta_{xx} + 2\beta_{xy} + \beta_{yy}) \\ \psi &= (1/2)(-\beta_{xx} + 2\beta_{xy} - \beta_{yy}). \end{aligned} \tag{77}$$

Finally, Dodd, Eilbeck, Gibbon, and Morris [23] is recovered by putting $\lambda = 1$, $\phi = -(4/3)\beta_x$, $\zeta = \pm(2/3)q$, $\tau = -(1/2)t$ and by interchanging x and y.

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