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Symmetry Reduction and Exact Solutions of the Euler–Lagrange–Born–Infeld, Multidimensional Monge–Ampere and Eikonal Equations

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Abstract

Using the subgroup structure of the generalized Poincaré group $P(1,4)$, ansatzes which reduce the Euler–Lagrange–Born–Infeld, multidimensional Monge–Ampere and eikonal equations to differential equations with fewer independent variables have been constructed. Among these ansatzes there are ones which reduce the considered equations to linear ordinary differential equations. The corresponding symmetry reduction has been done. Using the solutions of the reduced equations, some classes of exact solutions of the investigated equation have been presented.

Let us consider the following equations:

$$\Box u \,(1 - u_\nu u^\nu) + u_{\mu\nu} u^\mu u^\nu = 0, \tag{1}$$

$$\det \|u_{\mu\nu}\| = 0, \tag{2}$$

$$u^\mu u_\mu \equiv (u_0)^2 - (u_1)^2 - (u_2)^2 - (u_3)^2 = 1, \tag{3}$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in \mathbf{R}_4$, $u_\mu \equiv \dfrac{\partial u}{\partial x_\mu}$, $u^\mu = g^{\mu\nu} u_\nu$, $u_{\mu\nu} \equiv \dfrac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $\Box$ is the d´Alembertian, $\mu, \nu = 0, 1, 2, 3$.

Equations of the type (1) – (3) have wide applications [1 – 6].

From the results of [7 – 10] it follows that the symmetry groups of the equations (1) – (3) contain the generalized Poincaré group $P(1,4)$ as a subgroup. The subgroup structure of the group $P(1,4)$ has been studied in [11 – 15]. On the basis of the subgroup structure of the group $P(1,4)$ we have constructed the ansatzes which reduce the equations (1) – (3) to differential equations with fewer independent variables. The corresponding symmetry reductions have been done. Having solved some of the reduced equations, we have found classes of exact solutions of the investigated equations. Some of these results have been presented in [16 – 19].

Below we give a brief review of the results obtained. Here, we only consider the ansatzes which reduce the equations (1) – (3) to linear ordinary differential equations (ODE). Let us give examples of ansatzes of such a type.

$$2x_0 \cdot \omega - \left(x_1^2 + x_2^2 + x_3^2\right) = -\varphi(\omega), \qquad \omega = x_0 + u. \tag{4}$$

This ansatz reduces the equations (1) – (3) to the following linear ODEs, respectively:

$$\varphi''\omega^2 - 8\omega\varphi' + 8\varphi - 6\omega^2 = 0;$$

$$\tfrac{1}{2}\omega^2\varphi'' - \omega\varphi' + \varphi = 0;$$

$$\omega\varphi' - \varphi + \omega^2 = 0.$$

Solving these reduced equations, we obtain the exact solutions of the equations (1) – (3), respectively:

$$C_2(x_0+u)^8 - (x_0+u)^2 + (2x_0+C_1)(x_0+u) - (x_1^2+x_2^2+x_3^2) = 0;$$

$$C_2(x_0+u)^2 + (2x_0+C_1)(x_0+u) - (x_1^2+x_2^2+x_3^2) = 0;$$

$$(x_0+u)^2 - (2x_0+C_1)(x_0+u) + (x_1^2+x_2^2+x_3^2) = 0.$$

In these formulae C_1 and C_2 are arbitrary constants. As we can see, the left–hand part of the ansatz (4) is a polynomial in the variable ω. Other ansatzes which reduce the equations (1) – (3) to the linear ODEs as well as the corresponding linear ODEs have been presented in [17]. These ansatzes also are polynomials in the variable $\omega = x_0 + u$, i.e., they can be written in the form

$$\sum_{k=0}^{n} \omega^k \cdot f_k(x,u) = f(x) \cdot \varphi(\omega), \quad n = 1, 2, 3, \tag{5}$$

where $f_k(x,u)$ $(k = \overline{0,n})$, $f(x)$ are given functions, $\varphi(\omega)$ is an unknown function, $\omega = x_0 + u$ is an invariant of subgroups of the group $P(1,4)$. Recently we have found also two ansatzes of this type for $n = 4$. Let us note that all of the ansatzes mentioned above are invariant under subgroups of the extended Galilei group $G(1,3) \subset P(1,4)$. The corresponding symmetry reduction of the equations (1) – (3) to the linear ODEs has been performed. Having solved the corresponding linear ODEs, we have constructed some exact solutions of the equations (1) – (3). These solutions are polynomials in the variable $x_0 + u$, i.e., they can be written in the form:

$$\sum_{k=0}^{n} (x_0+u)^k \cdot \tilde{f}_k(x,u) = 0; \tag{6}$$

In (6) $\tilde{f}_k(x,u)$ $(k = 0, \ldots, n)$ are given functions. The set of values of n depends on the equation. For example, in the case of equation (1), $n = 2, 4, 6, 7, 8, 9$.

Now we give ansatzes which are valid (in the sense of a desired kind of symmetry reduction) for each of the investigated equations separately. Let us give examples of ansatzes which are valid for equation (1)

$$u = \varphi(\omega), \quad \omega = x_3; \tag{7}$$

$$u = \exp\left(\varphi(\omega) + \frac{x_1}{a}\right) - x_0, \quad \omega = x_2, \quad a \neq 0. \tag{8}$$

These ansatzes reduce the equation (1) to the linear ODE

$$\varphi'' = 0.$$

Solving the reduced equation, we find the following exact solutions for the equation (1)

$$u = C_1 x_3 + C_2; \qquad u = \exp\left(\frac{x_1}{a} + C_1 x_2 + C_2\right) - x_0.$$

Let us note that the ansatz (7) is invariant under subgroups of the group $G(1,3)$, but the ansatz (8) is not invariant under subgroups of the group $G(1,3)$.

Below we present an example of the ansatz of the same type for the equation (2)

$$u = \varphi(\omega), \quad \omega = \left(x_1^2 + x_2^2 + x_3^2 - x_0^2\right)^{1/2}. \tag{9}$$

This ansatz reduces the equation (2) to the linear ODE:

$$\varphi'' = 0.$$

This reduced equation gives us the following exact solution of the equation (2)

$$u = C_1 \left(x_1^2 + x_2^2 + x_3^2 - x_0^2\right)^{1/2} + C_2.$$

Let us give an example of the ansatz of the same type for the equation (3)

$$u = \varphi(\omega) - x_0, \quad \omega = x_1^2 + x_2^2 + x_3^2. \tag{10}$$

This ansatz reduces the equation (3) to the linear ODE:

$$\varphi' = 0.$$

The ansatzes (7) – (10) can be written in the form:

$$h(u) = f(x) \cdot \varphi(\omega) + g(x), \tag{11}$$

where $h(u)$, $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is an unknown function. $\omega = \omega(x)$ are invariants of subgroups of the group $P(1,4)$.

Let us note that there exist other types of ansatzes which reduce the equation (1) to linear ODEs. For example,

$$\begin{gathered}\frac{\left(2\omega + \frac{2\alpha}{\varepsilon} x_3\right)^{3/2}}{3\alpha^2} - \frac{\varepsilon x_3}{\alpha} \cdot \left(2\omega + \frac{2\alpha}{\varepsilon} x_3\right)^{1/2} - \frac{\left(2\omega + \frac{2\alpha}{\varepsilon} x_3\right)^{1/2}}{2} = \\ = -\varphi(\omega) - x_0,\end{gathered} \tag{12}$$

$$\omega = \frac{1}{2}\left(x_0 + u\right)^2 - \frac{\alpha}{\varepsilon} x_3, \quad \varepsilon = \pm 1, \quad \alpha > 0.$$

This ansatz reduces the equation (1) to the linear ODE:

$$2\omega\varphi'' - \varphi' = 0.$$

Solving this reduced equation, we find the following exact solutions for equation (1) in the form:

$$\frac{(x_0+u)^3}{3\alpha^2} - \frac{\varepsilon x_3}{\alpha}(x_0+u) + \frac{x_0-u}{2} = -\frac{C_1\left((x_o+u)^2 - \frac{2\alpha}{\varepsilon}x_3\right)^{3/2}}{3} - C_2.$$

As we see, the left hand part of the ansatz (12) is not a polynomial in the variable ω. This ansatz is also invariant under subgroups of the group $G(1,3)$. The ansatzes of the same type as (12) can be written in the form:

$$h(\omega,x) = f(x)\cdot\varphi(\omega) + g(x), \tag{13}$$

where $h(\omega,x)$, $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is an unknown function. $\omega=\omega(x,u)$ are invariants of subgroups of the group $(1,4)$.

Let us finally say some words about generalizations of the results considered:
1. If we consider in the ansatz (5) $\omega=\omega(x,u)$, $f_k(x,u)$ $(k=0,\dots,n)$ and $f(x)$ as arbitrary (sufficiently) smooth functions, we obtain the natural generalization of this type of ansatz. For these generalized ansatzes (n is also arbitrary) we have found necessary and sufficient conditions in order that they reduce the equations (1) and (3) to linear ODEs.
2. If we put in the formula (6) instead of x_0+u an arbitrary (sufficiently) smooth function $\omega(x,u)$ and consider $\tilde{f}_k(x,u)$ $(k=0,\dots,n)$ with the same property as $\omega(x,u)$, we obtain a natural generalization for some class of solutions of the investigated equations. We have obtained necessary and sufficient conditions (1) and (3) to have polynomial solutions (with arbitrary n) in $\omega(x,u)$.
3. If we consider in formulae (11) and (13) $h(u)$, $h(\omega,x)$, $f(x)$, $g(x)$ and $\omega(x,u)$ as arbitrary (sufficiently) smooth functions, we obtain a natural generalization of the ansatzes which are found with the help of the subgroup structure of the group $P(1,4)$. For these types of ansatzes we have obtained necessary and sufficient conditions for them to reduce the equations (1) and (3) to linear ODEs.

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