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Group Analysis of Nonlinear Heat-Conduction Problem for a Semi-Infinite Body

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Abstract

The transformation group theoretic approach is applied to present an analysis of the nonlinear unsteady heat conduction problem in a semi–infinite body. The application of one–parameter group reduces the number of independent variables by one, and consequently the governing partial differential equation with the boundary and initial conditions to an ordinary differential equation with the appropriate corresponding boundary conditions. The ordinary differential equation is solved analytically for some special forms of the thermal parameters. The general analysis developed in this study corresponds to thermal parameters that has different forms with coordinates and time.

1 Introduction

This paper considers the problem of temperature distribution in a material having coordinates and time dependent thermal properties. These problems have numerous applications in various branches of science and engineering, specially with the problems associated with nuclear power plants and with vehicles designed to travel in the upper atmosphere and outer space. That is why this type of problems has received considerable attention throughout the history of heat conduction and the literature of the topic is very rich. One will find an attractive discussion of the subject in [8, 12] and [16].

A heat conduction problem becomes nonlinear either due to nonlinearity of the differential equation or boundary conditions or both. Since there is no general theory available for solution of nonlinear partial differential equations, the analysis of such problems becomes difficult and each problem should be treated individually.

A variety of approximate and numerical methods are available for solution of these problems. A commonly used numerical scheme for solution of partial differential equations is the finite–difference method, which is described in [19]. The Monte Carlo method, which based on probability sampling techniques, has been considered in [11]. In 1969, Emery and Carson [9] considered the finite–element method to solve heat conduction problems in solids. The primary advantages of the finite–element over the finite–difference method are that the irregular boundaries can be handled easily and the size of the finite element can be varied readily over the region. Recently the boundary element method has been considered extensively by Aral and Tang [5], where the numerical solution of continuum problems is performed with a reduction of dimensionality of the problem.

Various approximate methods of analysis have been developed to solve heat conduction problems. Some of such methods are the integral methods that have been first used by Goodman [10], to solve one–dimensional transient heat conduction, whereas Sfeir [20] considered the case of two–dimensional steady conduction.

The mathematical technique used in the present analysis is the parameter–group transformation. The group method, as a class of methods which lead to reduction of the number of independent variables, was first introduced by Birkhoff [7] in 1948, where he made use of one–parameter transformation groups. In 1952, Morgan [14] presented a theory which has led to improvements over earlier similarity methods. In 1990 and 1991, Abd–el–Malek, et al [1, 2, 3, 6] have applied the group methods analysis, intensively, tostudy some problems in free–convective laminar boundary layer flow on a nonisothermal bodies. Detailed calculations can be found in Ames [4] and Ovsiannikov [15].

Although this review is not comprehensive, it is clear that all these investigations are limited to studies of similarity solutions since the similarity variables can give great physical insight with minimal efforts. In [21] one finds vast summary tables of the variable and boundary conditions ensuring similarity problems.

In this work we present a general procedure for applying one–parametric group transformation to the partial differential equation of heat conduction in solids, and the boundary and inital conditions. Under the transformation, the partial differential equation is reduced to an ordinary differential equation with appropriate boundary conditions. The equation is then solved analytically for some forms of the thermal parameters.

2 Mathematical formulation

The fundamental equation for the temperature distribution in a stationary, homogenous, isotropic solid material, thermal conductivity, K, and the volumetric specific heat, S, vary with coordinates and time, is derived by the usual process of equating the net heat inflow over the boundary of volume to the product of the temperature rise and the heat capacity. Assuming no heat is generated inside the solid, the governing equation is given by:

$$\nabla \cdot \Big(K \nabla T \Big) = S \frac{\partial T}{\partial t}, \tag{2.1}$$

where T is the temperature at time t of a point in the material, which coordinates are (x, y, z).

Let us consider the case of a semi–infinite metal with a constant flux of heat onto its surface. Assume the inital temperature is zero and the temperature vanishes as $x \to \infty$.

If this plane face is located at $x = 0$, equation (2.1) reduces to:

$$\frac{\partial}{\partial x}\left(K \frac{\partial T}{\partial x} \right) = S \frac{\partial T}{\partial t}, \quad x > 0, \quad t > 0 \tag{2.2}$$

with the following conditions
(1) the boundary conditions:

$$(i) \quad -K \frac{\partial T}{\partial x} = 1, \quad \text{at} \quad x = 0, \quad t > 0 \tag{2.3}$$

$$(ii) \lim_{x \to \infty} T(x, t) = 0, \quad t > 0, \tag{2.4}$$

(2) the initial condition:

$$T(x,0)=0,\quad x>0. \tag{2.5}$$

3 Solution of the problem

The method of solution depends on the application of one–parameter group transformation to the partial differential equation (2.2). Under this transformation the two independent variables will be reduced by one and we obtain the differential equation in only one independent variable, which is the similarity variable.

3.1 The group systematic formulation

The procedure is initiated with the group G, a class of transformation of one–parameter "a" of the form

$$G:\overline{Q}=C^{Q}(a)Q+E^{Q}(a), \tag{3.1}$$

where Q stands for t,x,T,K,S and the C's are real–valued and at least differentiable in the real argument "a".

3.2 The invariance analysis

To transform the differential equation, transformations of the derivatives are obtained from G via chain–rule operations:

$$\begin{aligned}\overline{Q}_{\overline{j}}&=\left(C^{Q}/C^{j}\right)Q_{j}, && j=t,x\\ \overline{Q}_{\overline{ij}}&=\left(C^{Q}/C^{i}C^{j}\right)Q_{ij}, && i=t,x,\ j=t,x\end{aligned} \tag{3.2}$$

where Q stands for T,K and S.
Equation (2.2) is said to be invariantly transformed whenever

$$\overline{S}\ \overline{T}_{\overline{t}}-\overline{K}\ \overline{T}_{\overline{xx}}-\overline{K}_{\overline{x}}\overline{T}_{\overline{x}}=H(a)\Big[ST_{t}-KT_{xx}-K_{x}T_{x}\Big], \tag{3.3}$$

for some function $H(a)$ which may be a constant.

Substitution from equations (3.1) into equation (3.3) for the independent variables, the functions and their partial derivatives yields

$$\frac{C^{s}C^{T}}{C^{t}}ST_{t}-\frac{C^{K}C^{T}}{(C^{x})^{2}}KT_{xx}-\frac{C^{K}C^{T}}{(C^{x})^{2}}K_{x}T_{x}+R=H(a)\Big[ST_{t}-KT_{xx}-K_{x}T_{x}\Big], \tag{3.4}$$

where

$$R=\frac{C^{T}E^{K}}{(C^{x})^{2}}T_{xx}-\frac{C^{T}E^{S}}{C^{t}}T_{t}.$$

The invariance of (3.4) implies $R\equiv 0$. This is satisfied by putting

$$E^{K}=E^{S}=0 \tag{3.5}$$

and

$$\frac{C^S C^T}{C^t} = \frac{C^K C^T}{(C^x)^2} = H(a). \tag{3.6}$$

Moreover, the boundary conditions (2.3) and (2.4) and initial condition (2.5) are also invariant in form, imply that

$$C^T = 1, \quad E^T = 0, \quad C^K C^x = 1. \tag{3.7}$$

Combining equations (3.6) and invoking the result (3.7), we get

$$\frac{C^K}{C^S} = \frac{(C^x)^2}{C^t}. \tag{3.8}$$

Finally, we get the one–parameter group G which transforms invariantly the differential equation (2.2) and the boundary and the initial conditions (2.3–2.5) The group G is of the form

$$G : \begin{cases} \bar{t} = C^t t + E^t \\ \overline{x} = C^x x + E^x \\ \overline{T} = T \\ \overline{K} = \dfrac{C^S (C^x)^2}{C^t} K \\ \overline{S} = C^S S \end{cases} \tag{3.9}$$

3.3 The complete set of absolute invariants

Our aim is to make use of group methods to represent the problem in the form of an ordinary differential equation (similarity representation) in a single independent variable (similarity variable). Then we have to proceed in our analysis to obtain a complete set of absolute invariants. In addition to the absolute invariant of the independent variable, there, are three absolute invariants of the dependent variables T, K and S.

If $\eta \equiv \eta(t, x)$ is the absolute invariant of the independent variables, then

$$g_j(t, x, T, K, S) = F_j\Big[\eta(t, x)\Big], \quad j = 1, 2, 3. \tag{3.10}$$

are the three absolute invariants corresponding to T, K and S. The application of a basic theorem in group theory, see [13], states that:
a function $g(t, x, T, K, S)$ is an absolute invariant of a one–parameter group if it satisfies the following first–order linear differential equation

$$\sum_{i=1}^{5} (\alpha_i Q_i + \beta_i) \frac{\partial g}{\partial Q_i} = 0, \tag{3.11}$$

where Q_i stands for t, x, T, K and S, and

$$\begin{aligned} \alpha_i &= \frac{\partial C^{Q_i}}{\partial a}(a^0), \quad i = 1, 2, \ldots, 5 \\ \beta_i &= \frac{\partial E^{Q_i}}{\partial a}(a^0), \quad i = 1, 2, \ldots, 5 \end{aligned} \tag{3.12}$$

where a^0 denotes the value of "a" which yields identity element of the group. Without loss of generality we can assume that $E^t = E^x = 0$, then

$$\beta_i = 0, \quad i = 1, 2, \ldots, 5 \tag{3.13}$$

At first, we seek the absolute invariant of the independent variables. Owing to equation (3.11), $\eta(t, x)$ is an absolute invariant if it satisfies first–order partial differential equation

$$\alpha_1 x \eta_x + \alpha_2 t \eta_t = 0, \tag{3.14}$$

which has a solution in the form

$$\eta(t, x) = x^{\alpha} t^{\beta}, \tag{3.15}$$

where $\alpha = \alpha_1$, $\beta = -\alpha_2$.

The second step is to obtain absolute invariants of the dependent variables T, K and S. Now, from the transformations (3.9) T is itself an absolute invariant. Thus

$$g_1(t, x; T) = F_1(\eta) = T(\eta). \tag{3.16}$$

Now, equations

$$\alpha_1 x \frac{\partial g_2}{\partial x} + \alpha_2 t \frac{\partial g_2}{\partial t} + \alpha_4 K \frac{\partial g_2}{\partial K} = 0, \tag{3.17}$$

$$\alpha_1 x \frac{\partial g_3}{\partial x} + \alpha_2 t \frac{\partial g_3}{\partial t} + \alpha_4 S \frac{\partial g_2}{\partial S} = 0, \tag{3.18}$$

may be solved to get the other two absolute invariants

$$g_2(t, x; K) = \Phi_1 \left[\frac{K}{\Psi(t, x)} \right] = F_2(\eta), \tag{3.19}$$

$$g_3(t, x; K) = \Phi_2 \left[\frac{S}{\omega(t, x)} \right] = F_3(\eta), \tag{3.20}$$

where $\Psi(t, x)$ and $\omega(t, x)$ are functions to be determined. Without loss of generality, the functions Φ's in (3.19) and (3.20) are selected to be the identity functions. Then we can express the functions K and S in terms of the absolute invariants $F_2(\eta)$ and $F_3(\eta)$, respectively, in the form

$$K = \Psi(t, x) F_2(\eta), \tag{3.21}$$

$$S = \omega(t, x) F_3(\eta). \tag{3.22}$$

4 The reduction to ordinary differential equation

As the general analysis proceeds, the established forms of the dependent and independent absolute invariants are used to obtain ordinary differential equation. Generaly, the absolute invariant $\eta(t, x)$ has the form given in (3.15).

Substituting from (3.16), (3.21) and (3.22) into equation (2.2) yields

$$\Big[\omega\eta_t\Big]F_3T' = \Big[\Psi\eta_{xx}\Big]F_2T' + \Big[\Psi\eta_x^2\Big]F_2T'' + \Big[\Psi_x\eta_x\Big]F_1T' + \Big[\Psi\eta_x^2\Big](F_2)'T', \tag{4.1}$$

where the primes refer to differentiation with respect to η.

For (4.1) to reduce to on expression in single independent variable η, it is nesessary that the coefficients be constants or functions of η alone. Thus

$$\omega\eta_t = C_1, \tag{4.2}$$

$$\Psi\eta_{xx} = C_2, \tag{4.3}$$

$$\Psi(\eta_x)^2 = C_3, \tag{4.4}$$

$$\Psi_x\eta_x = C_4. \tag{4.5}$$

It follows, then, that (4.1) may be rewritten as

$$(C_3F_2)T'' = \Big[C - 1F_3 - C_2F_2 - C_4F_2 - C_3(F_2)'\Big]T'. \tag{4.6}$$

The boundary condition (2.3), under the similarity variable (3.15), is transformed to the boundary condition

$$T'(0) = -1. \tag{4.7}$$

For the following forms of $\Psi(t,x)$ and $F_2(t,x:K)$

$$\begin{aligned} &\Psi = x^{-(n+1)\alpha+1}\ t^{-(n+1)\beta}, \\ &F_2 = \eta^n, \end{aligned} \tag{4.8}$$

where $n = \ldots, -2, -1, 0, 1, 2, \ldots$, the only possible form, in our case, for $K(t,x)$, using (3.21), is

$$K(t,x) = x^{1-\alpha}\ t^{-\beta}, \quad \alpha > 0, \quad \beta < 0. \tag{4.9}$$

The second boundary condition (2.4) and the initial condition (2.5), under the similarity variable (3.15), coalesce into the condition

$$T(\infty) = 0. \tag{4.10}$$

Hence the differential equation (2.2) with the conditions (2.3–2.5) will be replaced by the differential equation (4.6) with the conditions (4.7) and (4.10).

5 Analytical solution for different forms of thermal parameters

Differential equation (4.6) is intractable, and apparently can only be solved by approximate or numerical methods. We restricted ourself to find the exact solution for some possible forms of the thermal parameters K and S, and for a wide range of the similarity variable

$\eta = x^{\alpha}\, t^{\beta}$, where α is a real positive constant and β is a real negative constant.
For $F_3(t, x; S)$ to be function of η, the possible form for ω is

$$\omega(t, x) = \frac{t}{x}. \tag{5.1}$$

Hence, for η, K and ω given in (3.15), (4.9) and (5.1) respectively, we get the analytic solution for the following cases:
Case (1): for the following forms of thermal parameter S, in terms of ω and η:

(1) $S = -\frac{\alpha^2}{\beta}\omega,$

(2) $S = -\frac{\alpha^2}{\beta}\omega\eta,$

(3) $S = -\frac{\alpha^2}{\beta}\omega\eta^2 + \frac{\alpha(\alpha+2)}{\beta}\frac{\omega}{\eta},$

(4) $S = -\frac{\alpha^2}{\beta}\omega\sqrt{\eta}.$

The analytic solution for $T(\eta)$ is

$$T(\eta) = e^{-\eta}.$$

Case (2): for the following forms of thermal parameter S, in terms of ω and η:

(1) $S = -\frac{2\alpha^2}{\beta}\omega\eta$

(2) $S = -\frac{2\alpha^2}{\beta}\omega\eta^2$

(3) $S = -\frac{2\alpha^2}{\beta}\omega\eta^3 + \frac{\alpha(\alpha+2)}{\beta}\frac{\omega}{\eta}$

(4) $S = -\frac{2\alpha^2}{\beta}\omega\eta^{3/2}.$

The analytic solution for $T(\eta)$ is

$$T(\eta) = \frac{\sqrt{\pi}}{2}\,\text{erfc}\,(\eta).$$

Case (3): for the following forms of thermal parameter S, in terms of ω and η:

(1) $S = -\frac{3\alpha^2}{2\beta}\omega\sqrt{\eta}$

(2) $S = -\frac{3\alpha^2}{2\beta}\omega\eta^{3/2}$

(3) $S = -\frac{3\alpha^2}{2\beta}\omega\eta^{5/2} + \frac{\alpha(\alpha+2)}{\beta}\frac{\omega}{\eta}$

(4) $S = -\frac{3\alpha^2}{2\beta}\omega\eta.$

The analytic solution for $T(\eta)$ is

$$T(\eta) = -\int_0^{\eta} \exp(-r^{3/2})dr + \Gamma\left(\frac{5}{3}\right).$$

Case (4): for the following forms of thermal parameter S, in terms of ω and η:

(1) $S = -\frac{3\alpha^2}{\beta}\omega\eta^2$

(2) $S = -\frac{3\alpha^2}{\beta}\omega\eta^3$

(3) $S = -\frac{3\alpha^2}{\beta}\omega\eta^4 + \frac{\alpha(\alpha+2)}{\beta}\frac{\omega}{\eta}$

(4) $S = -\frac{3\alpha^2}{\beta}\omega\eta^{5/2}$.

The analytic solution for $T(\eta)$ is

$$T(\eta) = -\int_0^\eta \exp(-r^3)dr + \Gamma\left(\frac{4}{3}\right).$$

Case (5): for the following forms of thermal parameter S, in terms of ω and η:

(1) $S = \frac{\alpha^2}{\beta}\omega\frac{1-2\eta-2\eta^2}{1+\eta}$

(2) $S = \frac{\alpha^2}{\beta}\omega\frac{\eta(1-2\eta-2\eta^2)}{1+\eta}$

(3) $S = \frac{\alpha^2}{\beta}\omega\frac{\eta^2(1-2\eta-2\eta^2)}{1+\eta} + \frac{\alpha(\alpha+2)}{\beta}\frac{\omega}{\eta}$

(4) $S = \frac{\alpha^2}{\beta}\omega\frac{\sqrt{\eta}(1-2\eta-2\eta^2)}{1+\eta}$.

The analytic solution for $T(\eta)$ is

$$T(\eta) = \frac{1}{2}\left[\sqrt{\pi}\,\text{erfc}\,(\eta) + \exp(-\eta^2)\right].$$

Case (6): for the following forms of thermal parameter S, in terms of ω and η:

(1) $S = -\frac{\alpha^2}{\beta}\omega\frac{\eta}{1+\eta}$

(2) $S = -\frac{\alpha^2}{\beta}\omega\frac{\eta^2}{1+\eta}$

(3) $S = -\frac{\alpha^2}{\beta}\omega\frac{\eta^3}{1+\eta} + \frac{\alpha(\alpha+2)}{\beta}\frac{\omega}{1+\eta}$

(4) $S = -\frac{\alpha^2}{\beta}\omega\frac{\eta^{3/2}}{1+\eta}$.

The analytic solution for $T(\eta)$ is

$$T(\eta) = (2+\eta)\exp(-\eta).$$

Case (7): for the following forms of thermal parameter S, in terms of ω and η:

(1) $S = \frac{\alpha^2}{\beta}\omega\frac{1-\eta}{1+\eta}$

(2) $S = \frac{\alpha^2}{\beta}\omega\frac{\eta(1-\eta)}{1+\eta}$

(3) $S = \frac{\alpha^2}{\beta}\omega\frac{\eta^2(1-\eta)}{1+\eta} + \frac{\alpha(\alpha+2)}{\beta}\omega\eta$

(4) $S = \frac{\alpha^2}{\beta}\omega\frac{\sqrt{\eta}(1-\eta)}{1+\eta}$.

The analytic solution for $T(\eta)$ is

$$T(\eta) = (\eta^2 + 4\eta + 3)\exp(-\eta).$$

6 Conclusion

The most widely applicable method for determining analytic solution of partial differential equation that utilizes the underlying group structure has been applied to the classical problem of a semi–infinite metal with a constant flux of heat onto its surface and zero temperature at the inital time. Moreover, for large value of distance. We found analytic solution, successfully, for wide range of possible forms of the thermal parameters.

For other possible forms of the thermal parameters, where the obtained ordinary differential equation can not be solved analytically, a numerical solution can be obtained, using the fourth–order Runge–Kutta scheme and the gradient method.

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