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Similarity Reductions of the Zabolotskaya-Khokhlov Equation with a Dissipative Term

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Abstract

Similarity reductions of the Zabolotskaya-Khokhlov equation with a dissipative term to one-dimensional partial differential equations including Burgers' equation are investigated by means of Lie's method of infinitesimal transformation. Some similarity solutions of the Z-K equation are obtained.

1 Introduction

An approximation equation describing the propagation of a confined three-dimensional beam in a slightly nonlinear medium without dispersion or absorption was proposed by Zabolotskaya and Khokhlov [1]. The equation may be written as

$$u_{xt} - {u_x}^2 - uu_{xx} - u_{yy} = 0. \tag{1.1}$$

which is known as the Zabolotskaya-Khokhlov (Z-K) equation. This equation enables to analyze the beam deformation associated with the nonlinear properties of the medium. The infinitesimal symmetries and exact solutions of the Z-K equation have been investigated by many authors [2, 3].

Recently, the general derivation of the Z-K equation was given by Taniuti [4]. He has shown that multidimensional systems of nonlinear evolutional equations are reducible to the Kadomtsev-Petviashvili (K-P) equation and the Zabolotskaya-Khokhlov equation with a dissipative term in the weakly dispersive and weakly dissipative cases, respectively, by means of an extension of the reductive perturbation method to quasi one-dimensional propagation. The K-P equation and Z-K equation with dissipative term are two-dimensional extensions of the Korteweg-de Vries and Burgers' equations, respectively. The solution of obliquely interacting N traveling waves to the equation was obtained by Murakami [5].

The purpose of this paper is to investigate the similarity reductions of the Z-K equation with a dissipative term to one-dimensional partial differential equations including Burgers' equation and to represent some exact solutions of the Z-K equation.

2 Reductions of the Zabolotskaya-Khokhlov Equation with a Dissipative Term

The Z-K equation with a dissipative term may be written as

$$u_{xt} + {u_x}^2 + uu_{xx} + \lambda(u_{xxx}) + \mu(u_{yy}) = 0, \tag{2.1}$$

where λ and μ are some constants. Now, we consider a one-parameter (ε) Lie's group of infinitesimal transformation in the (x, y, t, u) space

$$\left.\begin{array}{l} x^* = x + \varepsilon\ X(x,y,t,u) + O(\varepsilon^2), \\ y^* = y + \varepsilon\ Y(x,y,t,u) + O(\varepsilon^2), \\ t^* = t + \varepsilon\ T(x,y,t,u) + O(\varepsilon^2), \\ u^* = \Theta(x,y,t) + \varepsilon\ U(x,y,t,u) + O(\varepsilon^2), \end{array}\right\} \tag{2.2a}$$

$$\left.\begin{array}{l} {u^*}_{x^*} = \Theta_x + \varepsilon\ [U_x] + O(\varepsilon^2), \\ {u^*}_{x^*x^*} = \Theta_{xx} + \varepsilon\ [U_{xx}] + O(\varepsilon^2), \\ {u^*}_{x^*t^*} = \Theta_{xt} + \varepsilon\ [U_{xt}] + O(\varepsilon^2), \\ {u^*}_{y^*y^*} = \Theta_{yy} + \varepsilon\ [U_{yy}] + O(\varepsilon^2), \\ {u^*}_{x^*x^*x^*} = \Theta_{xxx} + \varepsilon\ [U_{xxx}] + O(\varepsilon^2), \end{array}\right\} \tag{2.2b}$$

where $\Theta(x, y, t)$ is a solution of the Z-K equation (2.1) and total derivatives $[U_x]$, $[U_{xx}]$, $[U_{xt}]$, $[U_{yy}]$, and $[U_{xxx}]$ in eq.(2.2b) can be defined from eq.(2.2a) [6, 7]. Assuming that eq.(2.1) is invariant under the transformations (2.2a) and (2.2b), we get the following relation from the coefficient of the first order of ε,

$$[U_{tx}] + 2\ [U_x]\ \Theta_x + U\Theta_{xx} + [U_{xx}]\ Q + \lambda\ [U_{xxx}] + \mu\ [U_{yy}] = 0. \tag{2.3}$$

The solutions of eq.(2.3) give the infinitesimal elements (X, Y, T, U) leaving invariant eq.(2.1). ¿From eq.(2.3), we find the following infinitesimals:

$$\left.\begin{array}{l} X = \alpha x + Q'(t)y + R(t), \\ Y = \frac{3}{2}\alpha y - 2\mu Q(t), \\ T = 2\alpha t + \beta, \\ U = -\alpha u + Q''(t)y + R'(t), \end{array}\right\} \tag{2.4}$$

which are similar in form to the transformation for the Kadomtsev-Petviashvilli and two-dimensional Benjamin-Ono equations [8–10,], where $Q(t)$ and $R(t)$ are arbitrary functions of t, and $Q'(t) = dQ(t)/dt$, $Q''(t) = d^2Q(t)/dt^2$, and $R'(t) = dR(t)/dt$. Thus, the similarity variables and form are given by solving the characteristic equation [6],

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dt}{T} = \frac{du}{U}. \tag{2.5}$$

The calculated examples are represented as follows.

Case (A) $\alpha = \beta = 0$.

From the integrals of three equations, $dt/T = dx/X$, $dx/X = dy/Y$ and $du/U = dy/Y$, we obtain

$$\left\{\begin{array}{l} \rho = t, \qquad \eta = x + \dfrac{Q'(t)y^2 + 2R(t)y}{4\mu\ Q(t)}, \\ u = -\dfrac{Q''(t)y^2 + 2R'(t)y}{4\mu Q(t)} + H(\rho, \eta). \end{array}\right. \tag{2.6}$$

Substituting eq.(2.6) into eq.(2.1), we have the following equation

$$\frac{Q'(t)}{2Q(t)}H_\eta + H_{\eta\rho} + \frac{R(t)^2}{4\mu Q(t)^2}H_{\eta\eta} + H_\eta^2 + HH_{\eta\eta} + \lambda H_{\eta\eta\eta} - \frac{Q''(t)}{2Q(t)} = 0. \tag{2.7}$$

Integrating eq.(2.7) with respect to η , we have

$$\frac{Q'}{2Q}H + H_\rho + \frac{R^2}{4\mu Q^2}H_\eta + HH_\eta + \lambda H_{\eta\eta} - \frac{Q''}{2Q}\eta = S(\rho), \tag{2.8}$$

where $S(\rho)$ is an arbitrary function of ρ, moreover, by transformation,

$$\left\{\begin{array}{l} \tau' = \rho, \qquad \xi' = \dfrac{\eta}{\sqrt{Q}}, \\ H = \dfrac{1}{\sqrt{Q}}\left[G(\rho,\eta) + \dfrac{Q'}{2\sqrt{Q}}\eta - \dfrac{R^2}{4\mu Q^{3/2}}\right] \end{array}\right. \tag{2.9}$$

eq.(2.8) is rewritten as

$$Q(\tau')G_{\tau'} + \frac{Q'(\tau')}{2}G + GG_{\xi'} + \lambda G_{\xi'\xi'} = 0 \tag{2.10}$$

where the boundary condition, $G(\tau', \xi' = \infty) = 0$, has been imposed. Choosing an arbitrary function $Q(t)$ constant, $Q(t) = 1$, and $\tau = -\tau'/\lambda$, $\xi = -\xi'/\lambda$, we have Burgers' equation,

$$G_\tau + GG_\xi - G_{\xi\xi} = 0. \tag{2.11}$$

Thus the Z-K equation is reduced to Burgers' equation by the similarity transformation

$$\left\{\begin{array}{l} \tau = -\dfrac{t}{\lambda}, \qquad \xi = -\dfrac{x + R(t)y/2\mu}{\lambda} \\ u = -\dfrac{R'(t)y + R(t)^2/2}{2\mu} + G(\tau,\xi). \end{array}\right. \tag{2.12}$$

Case (B) $\alpha = 0$ and $\beta \neq 0$

Following the same way as case (A), we get the similarity variables and form as follows

$$\left\{\begin{array}{l} \tau = y + \dfrac{2\mu}{\beta}\displaystyle\int Q(t)dt, \\ \xi = \sqrt{|\mu|}\left\{x - \dfrac{Q(t)}{\beta}y - \dfrac{2\mu}{\beta^2}\displaystyle\int Q(t)^2 dt - \dfrac{1}{\beta}\displaystyle\int R(t)dt\right\}, \\ u = \dfrac{1}{\beta}\left\{Q'(t)y + \dfrac{\mu}{\beta}Q(t)^2 + R(t) - \beta\dfrac{\mu}{|\mu|}\right\} + G(\tau,\xi). \end{array}\right. \tag{2.13}$$

Substituting eq.(2.13) into eq.(2.1), we have the following equations

$$G_{\tau\tau} - G_{\xi\xi} + \frac{1}{2}(G^2)_{\xi\xi} + \lambda\sqrt{\mu}G_{\xi\xi\xi} = 0, \quad : \text{for } \mu > 0 \tag{2.14a}$$

$$G_{\tau\tau} - G_{\xi\xi} - \frac{1}{2}(G^2)_{\xi\xi} - \lambda\sqrt{|\mu|}G_{\xi\xi\xi} = 0, \quad : \text{for } \mu < 0 \tag{2.14b}$$

Case(C) $\quad \alpha \neq 0$ and $\beta \neq 0$

In the same way, we can obtain

$$\begin{aligned}
\tau = & \quad \frac{y}{(t+\frac{\beta}{2\alpha})^{3/4}} + \frac{\mu}{\alpha}\int \frac{Q(t)}{(t+\frac{\beta}{2\alpha})^{7/4}}dt, \\
\xi = & \quad \frac{x}{(t+\frac{\beta}{2\alpha})^{1/2}} - \left\{\frac{1}{2\alpha(t+\frac{\beta}{2\alpha})^{3/4}}\int \frac{Q'(t)}{(t+\frac{\beta}{2\alpha})^{3/4}}dt\right\} y \\
& - \frac{\mu}{2\alpha^2}\left\{\int \frac{Q(t)^2}{(t+\frac{\beta}{2\alpha})^{5/2}}dt + \frac{3}{8}\left(\int \frac{Q(t)}{(t+\frac{\beta}{2\alpha})^{7/4}}dt\right)^2\right\} - \frac{1}{2\alpha}\int \frac{R(t)}{(t+\frac{\beta}{2\alpha})^{3/2}}dt,
\end{aligned}$$

$$\begin{aligned}
u = & \quad \left\{\frac{Q'(t)}{2\alpha(t+\frac{\beta}{2\alpha})^2} - \frac{1}{8\alpha}\frac{1}{(t+\frac{\beta}{2\alpha})^{5/4}}\int \frac{Q'(t)}{(t+\frac{\beta}{2\alpha})^{3/4}}dt\right\} y \\
& + \frac{\mu}{4\alpha^2(t+\frac{\beta}{2\alpha})^{1/2}}\left\{\frac{Q(t)^2}{(t+\frac{\beta}{2\alpha})^{3/2}} + \int \frac{Q(t)^2}{(t+\frac{\beta}{2\alpha})^{5/2}}dt - \frac{3}{16}\left(\int \frac{Q(t)}{(t+\frac{\beta}{2\alpha})^{7/4}}dt\right)^2\right\} \\
& + \frac{R(t)}{2\alpha(t+\frac{\beta}{2\alpha})} + \frac{1}{4\alpha(t+\frac{\beta}{2\alpha})^{1/2}}\int \frac{R(t)}{(t+\frac{\beta}{2\alpha})^{3/2}}dt + \frac{1}{(t+\frac{\beta}{2\alpha})^{1/2}}G(\tau,\xi).
\end{aligned}$$

Then, eq.(2.1) becomes

$$\mu G_{\tau\tau} - G_{\xi} - \frac{3}{4}\tau G_{\tau\xi} - \frac{1}{2}\xi G_{\xi\xi} + \frac{1}{2}(G^2)_{\xi\xi} + \lambda G_{\xi\xi\xi} = 0. \tag{2.15}$$

3 Similarity Solutions to the Z-K Equation with a Dissipative Term

The solutions of Burgers' equation (2.11), eqs.(2.14) and (2.15) are transformed to the solutions of the Z-K equation by the similarity transformations. In this section, we consider the solutions of the Z-K equation which can be specifically obtained by substituting the solutions of Burgers' equation into similarity transformation (2.12).

3.1 Linearization of the Z-K Equation with a Dissipative Term

It is well known that Burgers' equation can be reduced to a linear heat equation by the Cole-Hopf transformation [11]. By using the results of the previous section, the Z-K equation (2.1) is reduced to

$$F_{\tau} - F_{\xi\xi} = 0 \tag{3.1}$$

by the transformation

$$u = B(y,t) - 2\frac{F_{\xi}(\tau,\xi)}{F(\tau,\xi)}, \tag{3.2}$$

where

$$B(y,t) = -\{R'(t)y + R(t)^2/2\}/2\mu, \tag{3.3}$$

τ and ξ are given by eq.(2.12).

Moreover, as the ordinary differential equation governing similarity solutions of Burgers' equation is any one of the 50 canonical types of the second-order differential equations having no movable critical points, which have been listed by Painleve and Ganbier[12] and that they can be reduced to the Riccati equation [12], the Z-K equation is also reduced to the Riccati equation by the similarity transformation.

3.2 Solution expressed by the Bessel function

Substituting the solution of Burgers' equation [13, 14], which is expressed by the Bessel function, into eq.(2.12), we have

$$\left\{ \begin{array}{l} u = B(y,t) - \dfrac{\gamma}{\lambda}t + \delta - \sqrt{2\gamma\zeta}\dfrac{J_{-2/3}(\frac{\sqrt{2\gamma}}{3}\zeta^{3/2})}{J_{1/3}(\frac{\sqrt{2\gamma}}{3}\zeta^{3/2})}, \\ \zeta = -\dfrac{1}{\lambda}(x + \dfrac{R(t)}{2\mu}y) - \dfrac{\gamma}{2\lambda^2}t - \dfrac{\delta}{\lambda}t, \end{array} \right. \tag{3.4}$$

where J denotes the Bessel function and γ and δ are arbitrary constants. As a special case, the similarity variable with $\gamma = 0$ and $R(t) = c = constant$ corresponds to progressive waves in the background $(-c/4\mu + \delta)$, in which the shock-wave solution is contained.

3.3 Similar-type solution

$$\left\{ \begin{array}{l} u = B(y,t) + \zeta + \dfrac{\lambda a}{t}\tanh\dfrac{a}{2}(\zeta + c'), \\ \zeta = \dfrac{x + R(t)y/2\mu}{t}, \end{array} \right. \tag{3.5}$$

which expresses the shock-wave solution moving in a nonsteady and nonuniform background, where a and c are arbitrary constants.

4 Conclusion

The similarity reduction of the Z-K equation with a dissipative term has been studied by the Lie's method. The Z-K equation is reduced to one-dimensional differential equations including Burgers' equation by the similarity transformations. Some exact solutions were obtained by substituting the solutions of Burgers' equation into the similarity transformations.

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