

Madelung Representation for Complex Nonlinear D'Alembert Equations in n -Dimensional Minkowski Space

N. EULER[†] and M. EULER[‡]

[†] *Technische Hochschule Darmstadt, Fachbereich Mathematik (AG.12),
Schlossgartenstr. 7, D-64289 Darmstadt, Deutschland*

[‡] *Mathematical Institute of the Ukrainian Academy of Sciences,
Tereshchenkivska Street 3, 252004 Kiev, Ukraine*

Abstract

The Madelung representation $\psi = u \exp(iv)$ is considered for the d'Alembert equation $\square_n \psi - F(|\psi|)\psi = 0$ to develop a technique for finding exact solutions. We classify the nonlinear function F for which the amplitude and phase of the d'Alembert equation are related to the solutions of the compatible d'Alembert–Hamiltonian system. The equations are studied in n -dimensional Minkowski space.

We consider the following general nonlinear d'Alembert equation

$$\square_n \psi - F(|\psi|)\psi = 0, \quad (1)$$

in n -dimensional Minkowski space $M(n-1, 1)$, where F is a real smooth function of $|\psi| \equiv \sqrt{\psi^* \psi}$ and

$$\square_n \equiv \frac{\partial^2}{\partial x_0^2} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2} .$$

Here ψ^* is the complex conjugate of the complex-valued function ψ . Equation (1) plays a fundamental role in classical and quantum field theories. Exact solutions of (1), for various nonlinear functions F , are essential for the development and interpretation of the related physical theories as well as the testing of numerical schemes which are studied for such equations. Many exact solutions of (1) were obtained by the use of Lie symmetry methods (Fushchych and Serov 1983, Grundland *et al* 1984, Grundland *et al* 1987, Grundland and Tuszynski 1987, Fushchych and Yehorchenko 1989, Fushchych *et al* 1993) and conditional symmetry methods (Fushchych *et al* 1993). Note that most methods used for constructing exact solutions of multidimensional partial differential equations involves an Ansatz (or trial solution) or a transformation (local or nonlocal) which should either reduce the equation to a more 'solvable' form, transform the equation (nonlocally) to itself (auto-Bäcklund transformation), or keep the equation form invariant (symmetry transformation).

In this paper we relate the real functions u (amplitude) and v (phase), under the Madelung representation

$$\psi(x) = u(x) \exp(iv(x)), \tag{2}$$

to the d'Alembert-Hamiltonian system

$$\square_n w = F_1(w), \quad (\nabla_n w)^2 = F_2(w). \tag{3}$$

Here

$$(\nabla_n a)(\nabla_n b) \equiv \frac{\partial a}{\partial x_0} \frac{\partial b}{\partial x_0} - \sum_{j=1}^{n-1} \frac{\partial a}{\partial x_j} \frac{\partial b}{\partial x_j}.$$

In Fushchych *et al* 1991, the necessary compatibility condition of system (3) is given. In this paper we restrict ourselves to the following compatible n -dimensional system

$$\square_n w = \frac{\lambda N}{w}, \quad (\nabla_n w)^2 = \lambda, \tag{4}$$

where $\lambda \in \{-1, 1\}$ and $N \in \{0, 1, \dots, n - 1\}$. System (4) admits the necessary condition of compatibility. Our solutions of system (4) are listed in table 1. Note that

$$\begin{aligned} \vec{a}_j \cdot \vec{x} &\equiv a_{j0}x_0 - a_{j1}x_1 - \dots - a_{j(n-1)}x_{n-1} \\ \vec{a}_j \cdot \vec{b}_k &\equiv a_{j0}b_{k0} - a_{j1}b_{k1} - \dots - a_{j(n-1)}b_{k(n-1)}, \text{ etc.} \end{aligned}$$

Using representation (2) the d'Alembert equation (1) takes the form

$$\begin{aligned} \square_n u - u(\nabla_n v)^2 - uF(u) &= 0 \\ u\square_n v + 2(\nabla_n u)(\nabla_n v) &= 0. \end{aligned} \tag{5}$$

We consider two cases: amplitude as a function of phase, i.e., $u = g(v)$, and phase as a function of amplitude, i.e., $v = g(u)$. Finally we make some comments on the treatment of system (5) with no functional relations between the phase and amplitude, as well as the application of this technique to other equations in mathematical physics. The results are presented in the form of two Propositions which give the conditions on the real function g for which the solutions of the compatible d'Alembert-Hamiltonian system (3) are related to the nonlinear d'Alembert equation (1) under the Madelung representation (2). Some examples are given to illustrate the method.

As a first case, let us assume that $u = g(v)$. We can make the following

Proposition 1: *System (5), with $u = g(v)$, takes the form of the compatible system (4), if and only if*

$$g(v) = \sigma \left(\frac{\dot{f}(v)}{f^N(v)} \right)^{1/2}, \quad f(v(x)) = w(x),$$

where

$$g\ddot{g} - 2\dot{g}^2 - g^2 - \frac{C^2}{\lambda\sigma^4}g^6 \exp\left(\frac{2}{\sigma^2} \int^v g^2(\xi)d\xi\right) F(g) = 0$$

for $N = 1$, and

$$g\ddot{g} - 2\dot{g}^2 - g^2 - \frac{1}{\lambda\sigma^4}g^6 \left(\frac{1-N}{\sigma^2} \int^v g^2(\xi)d\xi + C \right)^{2N/(1-N)} F(g) = 0$$

for $N \neq 1$. Here f, g are real smooth functions of v , $\dot{g} \equiv dg/dv$, $\ddot{g} \equiv d^2g/dv^2$, and $\sigma \in \mathcal{R} \setminus \{0\}$.

Proof: For $u = g(v)$ system (5) takes the form

$$\square_n v = \frac{-2g\dot{g}F(g)}{g\ddot{g} - 2\dot{g}^2 - g^2} \equiv F_1(v) \quad (6)$$

$$(\nabla_n v)^2 = \frac{g^2 F(g)}{g\ddot{g} - 2\dot{g}^2 - g^2} \equiv F_2(v). \quad (7)$$

Here $\dot{g} \equiv dg/dv$, etc. We now (locally) transform the compatible system (4) with the transformation

$$w(x) = f(v(x)), \quad (8)$$

where f is a smooth real function. This leads to the following system

$$\square_n v = \frac{\lambda N}{f(v)\dot{f}(v)} - \frac{\lambda \ddot{f}(v)}{\dot{f}^3(v)} \quad (9)$$

$$(\nabla_n v)^2 = \frac{\lambda}{\dot{f}^2(v)}. \quad (10)$$

Note that system (9), (10) admits the necessary condition of compatibility (Fushchych *et al* 1991). We can now combine system (9), (10) with system (6), (7) which leads to an expression of g in terms of f , namely

$$g(v) = \sigma \left(\frac{\dot{f}(v)}{f^N(v)} \right)^{1/2}, \quad (11)$$

where σ is an arbitrary real constant. The conditions on g can now be obtained by either relating (9) to (6), or (10) to (7). This leads to two cases: For the first case

$N = 1$: The expression for f takes the form

$$f(v) = C \exp \left(\frac{1}{\sigma^2} \int^v g^2(\xi)d\xi \right) \quad (12)$$

(C is an arbitrary real constant) so that the condition on g is

$$g\ddot{g} - 2\dot{g}^2 - g^2 - \frac{C^2}{\lambda\sigma^4}g^6 \exp \left(\frac{2}{\sigma^2} \int^v g^2(\xi)d\xi \right) F(g) = 0. \quad (13)$$

For the second case

$N \neq 1$: Here f takes on the form

$$f(v) = \left(\frac{1-N}{\sigma^2} \int^v g^2(\xi)d\xi + C \right)^{1/(1-N)} \quad (14)$$

(C is an arbitrary real constant) with the following condition on g

$$g\ddot{g} - 2\dot{g}^2 - g^2 - \frac{1}{\lambda\sigma^4}g^6 \left(\frac{1-N}{\sigma^2} \int^v g^2(\xi)d\xi + C \right)^{2N/(1-N)} F(g) = 0. \tag{15}$$

□

We can now make the following

Corollary 1: *Explicit solutions of (1) are given by*

$$\psi(x) = g(v(x)) \exp[iv(x)],$$

where g has to satisfy (13) for $N = 1$ (f given by (12)), and (15) for $N \neq 1$ (f given by (14)), whereby f has to be an invertible function of v , i.e., $v(x) = f^{-1}(w(x))$, and

$$\square_n w = \frac{\lambda N}{w}, \quad (\nabla_n w)^2 = \lambda.$$

Solutions for w are listed in Table 1.

We now give some examples of d'Alembert equations and their amplitude-phase-solutions under the assumption

$$g(v) = v^\beta.$$

Consider $N = 1, \beta = -1/2$. From Proposition 1, (1) takes the following form:

$$\square_n \psi + \frac{\lambda\sigma^4}{C^2} \left(\frac{1}{4}|\psi|^{4/\sigma^2} - |\psi|^{4(1-\sigma^2)/\sigma^2} \right) \psi = 0.$$

By Corollary 1, solutions of the above d'Alembert equation can be given in the form

$$\psi(x) = \left(\frac{1}{C}w(x) \right)^{-1/(2\sigma^2)} \exp \left[i \left(\frac{1}{C}w(x) \right)^{\sigma^2} \right],$$

where w are the solutions (given in table 1) of the compatible system

$$\square_n w = \frac{\lambda}{w}, \quad (\nabla_n w)^2 = \lambda.$$

Note some special cases of σ :

$$\begin{aligned} \sigma^2 = 1 : \quad & \square_n \psi - \frac{\lambda}{C^2} \left(1 - \frac{1}{4}|\psi|^4 \right) \psi = 0 \\ \sigma^2 = \frac{4}{5} : \quad & \square_n \psi - \left(\frac{4}{5} \right)^2 \frac{\lambda}{C^2} \left(|\psi| - \frac{1}{4}|\psi|^5 \right) \psi = 0 \\ \sigma^2 = \frac{2}{3} : \quad & \square_n \psi - \left(\frac{2}{3} \right)^2 \frac{\lambda}{C^2} \left(|\psi|^2 - \frac{1}{4}|\psi|^6 \right) \psi = 0. \end{aligned}$$

As a second example we consider $N = 2, \beta = -1$, so that

$$\square_n \psi + \frac{1}{\sigma^8} \left(|\psi|^2 + \frac{1}{\lambda \sigma^4} |\psi|^6 \right) \psi = 0$$

$$\psi(x) = (C - w(x)) \sigma^2 \exp \left[i \frac{1}{(C - w(x)) \sigma^2} \right],$$

where w are solutions (listed in table 1) of the compatible system

$$\square_n w = \frac{2\lambda}{w}, \quad (\nabla_n w)^2 = \lambda.$$

Let us now assume phase as a function of amplitude, i.e., $v = g(u)$. We can make the following

Proposition 2: *System (5), with $v = g(v)$, takes the form of the compatible system (4), if and only if*

$$\dot{g}(u) = \frac{\sigma}{u^2} \frac{\dot{f}(u)}{f^N(u)} \quad f(u(x)) = w(x),$$

where

$$u\ddot{g} + 2\dot{g} + u^2\dot{g}^3 + \frac{u^4 C^2}{\lambda \sigma^2} \dot{g}^3 \exp \left(\frac{1}{\sigma} \int^u \xi^2 \frac{dg(\xi)}{d\xi} d\xi \right) F(u) = 0$$

for $N = 1$, and

$$u\ddot{g} + 2\dot{g} + u^2\dot{g}^3 + \frac{u^4}{\lambda \sigma^2} \left(C - \frac{N-1}{\sigma} \int^u \xi^2 \frac{dg(\xi)}{d\xi} d\xi \right)^{2N/(1-N)} F(u) = 0$$

for $N \neq 1$. Here f, g are real smooth functions of $u, \dot{g} \equiv dg/du, \ddot{g} \equiv d^2g/du^2$, and $\sigma \in \mathcal{R} \setminus \{0\}$.

Proof: For $v = g(u)$, system (5) takes the form

$$\square_n u = \frac{(u^2\ddot{g} + 2u\dot{g})F(u)}{u\ddot{g} + 2\dot{g} + u^2\dot{g}^3} \equiv F_1(u) \tag{16}$$

$$(\nabla_n u)^2 = \frac{-u^2\dot{g}F(u)}{u\ddot{g} + 2\dot{g} + u^2\dot{g}^3} \equiv F_2(u). \tag{17}$$

Here $\dot{g} \equiv dg/du$, etc. By (locally) transforming the compatible system (4) with

$$w(x) = f(u(x)), \tag{18}$$

we obtain the compatible system

$$\square_n u = \frac{\lambda N}{f(u)\dot{f}(u)} - \frac{\lambda \ddot{f}(u)}{\dot{f}^3(u)} \tag{19}$$

$$(\nabla_n u)^2 = \frac{\lambda}{\dot{f}^2(u)}. \tag{20}$$

Combining this with system (16) and (17), it follows that

$$\dot{g}(u) = \frac{\sigma}{u^2} \frac{\dot{f}(u)}{f^N(u)}, \tag{21}$$

where $\sigma \neq 0$ is a real constants and f is a real smooth function. For the condition on g we have to destinguish between two cases:

$N = 1$: Here f takes on the form

$$f(u) = C \exp\left(\frac{1}{\sigma} \int^u \xi^2 \frac{dg(\xi)}{d\xi} d\xi\right) \tag{22}$$

(C is an arbitrary real constant) so the condition on g follows:

$$u\ddot{g} + 2\dot{g} + u^2\dot{g}^3 + \frac{u^4 C^2}{\lambda\sigma^2} \dot{g}^3 \exp\left(\frac{1}{\sigma} \int^u \xi^2 \frac{dg(\xi)}{d\xi} d\xi\right) F(u) = 0. \tag{23}$$

$N \neq 1$: f is given by

$$f(u) = \left(C - \frac{N-1}{\sigma} \int^u \xi^2 \frac{dg(\xi)}{d\xi} d\xi\right)^{1/(1-N)} \tag{24}$$

(C is an arbitrary real constant) with the following condition on g :

$$u\ddot{g} + 2\dot{g} + u^2\dot{g}^3 + \frac{u^4}{\lambda\sigma^2} \left(C - \frac{N-1}{\sigma} \int^u \xi^2 \frac{dg(\xi)}{d\xi} d\xi\right)^{2N/(1-N)} F(u) = 0. \tag{25}$$

□

We can now make the following

Corollary 2: *Explicit solutions of (1) are given by*

$$\psi(x) = u(x) \exp[i g(u(x))],$$

where g has to satisfy (23) for $N = 1$ (f given by (22)), and (25) for $N \neq 1$ (f given by (24)), whereby f has to be an invertible function of u , i.e., $u(x) = f^{-1}(w(x))$, and

$$\square_n w = \frac{\lambda N}{w}, \quad (\nabla_n w)^2 = \lambda.$$

Solutions for w are listed in table 1.

Let us consider, for example,

$$g(u) = u^\beta.$$

With $N = 1$, $\beta \neq -2$, the following nonlinear d'Alembert equation follows from Proposition 2:

$$\square_n \psi + \frac{\lambda\sigma^2}{C^2} |\psi|^{-2} \left(1 + \frac{\beta+1}{\beta^2} |\psi|^{-2\beta}\right) \exp\left[\frac{-\beta}{\sigma(\beta+2)} |\psi|^{\beta+2}\right] \psi = 0.$$

Solutions take the form

$$\begin{aligned} \psi(x) &= u(x) \exp [iu(x)^\beta] \\ u(x) &= \left(\frac{\sigma(\beta + 2)}{\beta} \ln \left| \frac{w(x)}{C} \right| \right)^{1/(\beta+2)}, \end{aligned}$$

where w is a solution (listed in table 1) of the compatible system

$$\square_n w = \frac{\lambda}{w}, \quad (\nabla_n w)^2 = \lambda.$$

Note that, if $\beta = -1$ the equation takes the form:

$$\square_n \psi + \frac{\lambda \sigma^2}{C^2} |\psi|^{-2} \exp \left[\frac{1}{\sigma} |\psi| \right] \psi = 0.$$

With $N \neq 1$, $\beta \neq -2$, the following nonlinear d'Alembert equation follows from Proposition 2:

$$\square_n \psi + \lambda \sigma^2 \beta^2 |\psi|^{-2} \left(1 + \frac{\beta + 1}{\beta^2} |\psi|^{-4\beta} \right) \left(C - \frac{(N - 1)\beta}{\sigma(\beta + 2)} |\psi|^{\beta+2} \right)^{2N/(N-1)} \psi = 0$$

with solutions

$$\begin{aligned} \psi(x) &= u(x) \exp [iu(x)^\beta] \\ u(x) &= \left(\frac{\sigma(\beta + 2)}{(N - 1)\beta} \left(C - w(x)^{1-N} \right) \right)^{1/(\beta+2)}, \end{aligned}$$

where w is a solution (listed in table 1) of the compatible system

$$\square_n w = \frac{\lambda N}{w}, \quad (\nabla_n w)^2 = \lambda.$$

Note that, if $\beta = -1$ and $C = 0$ the equation has the form

$$\square_n \psi + \alpha |\psi|^{2/(N-1)} \psi = 0,$$

where $\alpha \equiv \lambda \sigma^{2/(N-1)} (N - 1)^{2N/(N-1)}$.

Let us return to system (5). With the assumption

$$\square_n v = p(v), \quad (\nabla_n v)^2 = \lambda \tag{26}$$

(p is an arbitrary real smooth function) system (5) reduces to

$$(\nabla_n u)(\nabla_n v) + \frac{1}{2} u p(v) = 0 \tag{27}$$

$$\square_n u - \lambda u - F(u)u = 0. \tag{28}$$

For a given function p and a compatible solution v of (26) the Lagrange system for (27), i.e.,

$$\frac{dx_0}{\partial v / \partial x_0} = - \frac{dx_j}{\partial v / \partial x_j} = - \frac{2du}{u p(v)} \tag{29}$$

($j = 1, \dots, n - 1$) can be integrated to obtain an Ansatz for u of the form

$$u(x) = f_1(x)\varphi[\omega_1(x), \dots, \omega_{n-1}(x)] + f_2(x). \tag{30}$$

Here $f_1, f_2, \omega_1, \dots, \omega_{n-1}$ are defined by the first integrals of (29). Note that Ansatz (30) will reduce (28), at least, in the case where the infinitesimal generator, related to the Lagrange system (29), is a Lie symmetry generator or a Q -conditional symmetry generator (see Euler and Steeb 1992, Fushchych *et al* 1993 and Euler *et al* 1994 for details on Lie symmetries and Q -conditional symmetries). In this way the Lie symmetries and conditional symmetries of (1) can be related to compatibility problem of system (26). This will be the topic of a future paper.

Let us finally note that the technique demonstrated above for the d'Alembert equation can also be applied to other complex equation for which amplitude-phase-solutions are sought. However, the compatibility problem will change. For example, consider the n -dimensional nonlinear Schrödinger equation

$$i \frac{\partial \psi}{\partial x_0} + \Delta_{n-1} \psi - F(|\psi|)\psi = 0,$$

where $\Delta_{n-1} \equiv \sum_{j=1}^{n-1} \partial^2 / \partial x_j^2$. Let us use the same assumptions as in Proposition 1, i.e., $\psi = g(v) \exp[iv]$. The condition on g is then given by

$$\ddot{g} + \frac{f_1(v)}{f_2(v)} \dot{g} - \frac{f_2(v) + f_3(v)}{f_2(v)} g - \frac{g}{f_2(v)} F(g) = 0$$

($\dot{g} \equiv dg/dv, f_2 \neq 0$), where

$$\frac{\dot{g}(v)}{g(v)} = -\frac{f_1(v)}{2f_2(v) + f_3(v)}$$

and f_j, v is obtained from the compatible solutions of

$$\Delta_{n-1} v = f_1(v), \quad (\nabla_{n-1} v)^2 = f_2(v), \quad \frac{\partial v}{\partial x_0} = f_3(v).$$

Results are represented in this talk obtained in collaboration with Peter Basarab-Horwath and Wilhelm Fushchych.

References

- [1] Euler N., Köhler A. and Fushchych W.I., *Physica Scripta*, 1994, V.49, 518–524.
- [2] Euler N. and Steeb W.-H., *Continuous Symmetries, Lie Algebras and Differential Equations*, Mannheim, B. I. Wissenschaftsverlag, 1992.
- [3] Grundland M.A., Harnad J. and Winternitz P., *J. Math. Phys.*, 1984, V.25, 791–805.
- [4] Grundland M.A. and Tuszynski J.A. *J. Phys. A: Math. Gen.*, 1987, V.20, 6243–6258.
- [5] Grundland M.A., Tuszynski J.A. and Winternitz P., *Phys. Lett. A*, 1987, V.7, 340–344.
- [6] Fushchych W.I. and Serov N.I., *J. Phys. A: Math. Gen.*, 1983, V.16, 3645–3658.
- [7] Fushchych W.I., Shtelen W. and Serov M., *Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics*, Dordrecht, Kluwer, 1993.
- [8] Fushchych W.I. and Yegochenko I.A., *J. Phys. A: Math. Gen.*, 1989, V.22, 2643–2652.
- [9] Fushchych W.I., Zhdanov R.Z. and Yegochenko I. A., *Mathematical Analysis and Applications*, 1991, V.161, 352–360.