Nonlinear Mathematical Physics 1995, V.2, N 3–4, 416–417.

Galilei Invariance of the Fokker–Planck Equation with Nonlinearity

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Consider equation

$$\frac{\partial\rho}{\partial t} = -\sum_{k=1}^{n}\frac{\partial}{\partial x_k}(A_k\rho) + \frac{1}{2}\sum_{i,k=1}^{n}\frac{\partial^2}{\partial x_i\partial x_k}(B_{ik}\rho) + F(\rho), \tag{1}$$

where $\rho(t, x_1, \ldots, x_n), A_k(t, x_1, \ldots, x_n), B_{ik}(t, x_1, \ldots, x_n), F(\rho)$ are smooth real functions. Equation (1) is multidimensional nonlinear Fokker–Planck equation. A new method in investigation of symmetry of such equations was proposed by W.I. Fushchych [1]. The idea consists in complementing (1) with equations for functions A_k and B_{ik}. That is, we add to (1) some system of equations for A_k and B_{ik}, (1), turns out then to be a nonlinear system (even if $F(\rho) = 0$). Such an extended system can have a nontrivial symmetry, as we shall see below.

1. First we consider the case $B_{ik} = B \cdot \delta^{ik}, B = \text{const} > 0$. We suggest also that coefficient functions A_k are to be partial derivatives φ_k of some function $\varphi(t, x_1, \ldots, x_n)$ and that they satisfy Euler's equation for ideal liquid

$$\frac{\partial A_k}{\partial t} + A_l\frac{\partial A_k}{\partial x_l} = F_k(\rho), \tag{2}$$

equations (1) and (2) can be written then as

$$\rho_0 + \rho_a\varphi_a + \rho\Delta\varphi - \frac{B}{2}\Delta\rho = F(\rho), \tag{3}$$

$$\varphi_0 + \frac{1}{2}\varphi_a\varphi_a = F_1(\rho). \tag{4}$$

Theorem 1. *The system of equations (3), (4) is invariant under the algebra*

1) $AG(1,n) = \langle P_0, P_a, J_{ab}, G_a, Q\rangle$, *where* $P_0 = \dfrac{\partial}{\partial x_a}$, $J_{ab} = x_aP_b - x_bP_a$, $G_a = x_0P_a + x_aQ$, $Q = \dfrac{\partial}{\partial\varphi}$, *for arbitrary* F, F^1*;*

2) $AG_1(1,n) = \langle AG(1,n), D\rangle$, *where* $D = 2x_0P_0 + x_aP_a - \dfrac{2}{k}I$, $I = \rho\partial_\rho$, *if* $F = \lambda\rho^{k+1}$, $F_1 = \lambda_1\rho^k$*;*

3) $AG_2(1,n) = \langle AG_1(1,n), A\rangle$ *where* $A = x_0^2P_0 + x_0x_aP_a + \dfrac{\vec{x}^2}{2}Q - nx_0I$, *if* $F = \lambda\rho^{\frac{2}{n}+1}$, $F_1 = \lambda_1\rho^{2/n}$*;*

4) $AG_3(1,n) = \langle AG_1(1,n), B\rangle$ *where* $B = I + \lambda_1 x_0 Q$, *if* $F_1 = \lambda_1 ln\rho$, $F = 0$;
5) $AG_4(1,n) = \langle AG(1,n), B\rangle$ *where* $B = I + \lambda_1 x_0 Q$, *if* $F_1 = \lambda_1 ln\rho, F = 0$;
6) $AG_5(1,n) = \langle AG_2(1,n), I\rangle$, *if* $F = F_1 = 0$.

We have also found a condition which allows to further enlarge symmetry of the system of equations (3),(4). It has the form $\Delta\rho = 0$. Without formulating a corresponding theorem, we simply mention that the following operators and their linear combinations (with I also) emerge:

$$Q_1 = x_a P_a + 2\varphi Q;\ Q_2 = x_0 P_0 - \varphi Q;\ C_0 = \exp(\lambda x_0)\cdot I. \tag{5}$$

2. Let us consider now the case of one–dimensional FPE with a nonconstant diffusion coefficient. It has the form

$$\frac{\partial\rho}{\partial x_0} + \frac{\partial}{\partial x_1}(A\rho) - \frac{1}{2}\frac{\partial^2}{\partial x_1^2}(B\rho) = F(\rho). \tag{6}$$

We shall require that functions $A(x_0, x_1) \equiv v$ and $B(x_0, x_1) \equiv 2u$ are to satisfy the following differential conditions:

$$v_0 + vv_1 = F^1(u, v) \tag{7}$$

$$u_0 + u_1 v + v_1 = F^2(u, v) \tag{8}$$

We shall formulate the theorem for the cases only when symmetry operators of system (6)–(8) form a basis of the Galilei algebra; the corresponding Galilei operators have the form $G = x_0\partial_1 + \partial_u$ and $G^{(1)} = \exp(\gamma x_0)(\partial_1 + \gamma\partial_v)$.

Theorem 2. *The system of equations (6)–(8) is invariant under the following algebras:*
1) $AG(1,1) = \langle P_0, P_1, G\rangle$, *for arbitrary* F *and* $F_v^1 = F_v^2 = 0$, *where* $F_v^1 = \dfrac{\partial F^1}{\partial v}$;
2) $AG^{(1)}(1,1) = \langle AG(1,1), \exp(bx_0)I\rangle$, *for* $F = \rho(bln\rho + d)$, $F_v^1 = F_v^2 = 0, d = \text{const}$;
3) $AG^{(2)}(1,1) = \langle AG^{(1)}(1,1), D^\lambda\rangle$, *where the operator* D^λ *has the form* $D^\lambda = 2x_0P_0 + (x_1 + \frac{3\lambda}{2}x_0^2)P_1 + (3\lambda x_0 - v)\partial_v$, *for* $f = 0$, $F^1 = \lambda = \text{const}$, $F^2 = 0$;
4) $AG^{(3)}(1,1) = \langle AG(1,1), D^\lambda + kI\rangle$, *for* $F = \lambda_1\rho^{1-\frac{2}{k}}$, $F^1 = \lambda$, $F^2 = 0$ ($\lambda = \text{const}$, $k \neq 0$);
5) $AG_1(1,1) = \langle P_0, P_1, G^{(1)}\rangle$ *for arbitrary* $F, F_v^1 = \gamma$;
6) $AG_1^{(1)}(1,1) = \langle AG_1(1,1), \exp(bx_0)I\rangle$, *for* $F = \rho(bln\rho + d), F_v^1 = \gamma$, $F^2 = 0$.

References

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