

# Elliptic even finite-gap potentials and spectra of the Schrödinger operator

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## Abstract

A simple method for calculating finite-gap elliptic potentials and corresponding spectra of the one-dimensional Schrödinger operator which is based on a general representation of the potentials in a form of rational functions of the Weierstrass function and trace formulae is proposed. It is shown that the 1-3-gap even elliptic potentials can be expressed only in a form of some linear combinations of the Lamé potentials with shifted arguments. The expressions for the 1-3-gap even elliptic potentials and corresponding spectral polynomials (the roots of which are spectrum boundaries) are obtained.

## 1 Introduction

A spectral theory of linear differential operators (in particular, the one-dimensional Schrödinger operators), which is connected with a reconstructing of periodic coefficients of these operators in terms of finite-gap spectral data, is based on the study of analytic properties of the Baker–Akhiezer function defined on the corresponding Riemann surfaces [1-5]. A sufficiently general algebraic geometric approach of this theory permits to obtain general expressions for finite-gap linear differential operators and corresponding spectra that allows to obtain solutions of hierarchies of integrable nonlinear equations in partial derivatives. These results were obtained in terms of  $n$ -dimensional theta functions [6] containing implicit parameters, a calculating of which is a special problem.

Explicit expressions for finite-gap elliptic linear differential operators and solutions of hierarchies of nonlinear partial differential equations were obtained using the reduction [7, 8] of  $n$ -dimensional theta functions to the one-dimensional Jacobi theta functions [9]. In the mentioned papers and in the papers [4, 11] the elliptic differential operators (specifically, the one-dimensional Schrödinger operator) and corresponding spectra were studied by means of the elliptic ansatz for the corresponding Baker-Akhiezer functions.

In the paper [12] in the framework of the algebraic geometric approach, the finite-gap elliptic solutions were found using an ansatz for the spectral  $n$ -sheeted Riemann surfaces. Introducing a representation of even elliptic potentials as a linear combination of the Lamé potentials with shifted arguments, there were obtained elliptic even potentials and spectra of the Schrödinger equation and corresponding solutions of the KdV equation for the case of the 1–2 gaps.

However, there were not considered relations of the above mentioned Lamé potentials with shifted arguments in the general formula for even elliptic potentials and a computation of corresponding spectra for an arbitrary number  $g$  of spectral gaps. It was also reasonable to build a more simple method for the calculating of elliptic potentials and spectra by means of a simple algorithm on the basis of an unified system of equations.

In this paper a simple general method is proposed to calculate the finite-gap elliptic potentials and spectra of the Schrödinger operators based on the representation of elliptic potentials by rational functions of the elliptic Weierstrass function [9] and an usage of so-called trace formulae [2, 13]. It is shown that substitution of these rational functions into the trace formulae yields, under the condition of vanishing the principal parts at poles, a system of algebraic equations on potential parameters and the symmetrized products of spectrum boundaries. The compatibility conditions of these equations bring up to possible potentials and spectral polynomials such that the set of their roots is the set of boundary points of a one-dimensional spectrum. It is shown that even elliptic potentials of the Schrödinger operator are described by only two types of the linear combinations of the Lamé potentials with shifted arguments. Using a simple algorithm of the proposed method and the well-known program package “Mathematica” for analytic calculations [14], we have calculated even 1–3-gap elliptic potentials and corresponding spectra. These results are compared with previous ones.

## 2 Equations for potential parameters and spectral polynomial coefficients

Finite-gap eigenfunctions of the one-dimensional Schrödinger operator  $H = -d^2/dx^2 + U(x)$  which satisfy the equation  $H\Psi(x, E) = E\Psi(x, E)$  have the form  $\Psi(x, E) = \sqrt{\chi_R(x, E)} \exp(\int^x dx \chi_R(x, E))$ , where  $\chi_R$  is the real function that can be represented as the asymptotic series

$$\chi_R(x, E) = \sqrt{E} \left( 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}} \chi_{2n+1}(x) E^{-(n+1)} \right) \tag{1}$$

(further we shall omit the argument of the functions  $\chi_n$  for simplicity). Coefficients of (1) satisfy the well-known [2] recurrent relation

$$\chi_{n+1} = \frac{d}{dx} \chi_n + \sum_{k=1}^{n-1} \chi_k \chi_{n-k}, \quad \chi_1 = -U(x). \tag{2}$$

According to this equation, the functions  $\chi_{2n+1}$  can be represented in the form

$$\chi_{2g+1} = -U^{(2g)} + (-1)^{g+1} \text{const } U^{(g+1)} + \dots, \tag{3}$$

where superscripts in parentheses mean order of a derivative,  $(\dots)$  means a polynomial on the potential with its derivatives containing lower powers of  $U$  than the first two terms in the right-hand side of (3). In the case of  $g$ -gap spectra (which have  $g$  gaps and  $2g + 1$  boundaries  $\{E_i\}$ ), the real function  $\chi_R$  is described by the expression [2]

$$\begin{aligned}\chi_R(x, E) &= \frac{\sqrt{P(E)}}{Q(E, x)}, \\ P(E) &= \prod_{n=1}^{2g+1} (E - E_n), \quad Q(E, x) = \prod_{n=1}^g (E - \mu_n(x)),\end{aligned}\tag{4}$$

which after removal of parentheses transforms into the form

$$\chi_R(x, E) = \frac{\sqrt{\sum_{n=0}^{2g+1} a_n E^{-n}}}{\sum_{n=0}^g b_n E^{-n}}, \quad a_0 = 1, \quad b_0 = 1.\tag{5}$$

Here  $a_n$  and  $b_n$  are symmetrized products of spectral boundaries  $E_j$  and  $\mu$ -functions of the  $n$ th order, respectively:

$$a_n = (-1)^n \sum_{j_1, \neq j_2, \dots, \neq j_n}^{2g+1} \prod_{i=1}^n E_{j_i}, \quad b_n = (-1)^n \sum_{j_1, \neq j_2, \dots, \neq j_n}^g \prod_{i=1}^n \mu(x)_{j_i},$$

The expression (5), as in the case (1), can be represented as the asymptotic series

$$\chi_R \sim \sqrt{E} \left( 1 + \sum_{n=1}^{\infty} A_n E^{-n} \right),\tag{6}$$

with coefficients

$$A_n = \frac{1}{n!} \frac{d^n}{dz^n} \sqrt{\frac{\sum_{n=0}^{2g+1} a_n z^n}{\sum_{n=0}^g b_n z^n}} \Big|_{(z=0)}.\tag{7}$$

Comparing coefficients at equal powers of the independent variable  $E^{-1}$  in expressions (1) and (6) we obtain trace formulae in the form

$$A_{n+1} = \frac{(-1)^n}{2^{2n+1}} \chi_{2n+1},\tag{8}$$

which in view of equation (7) and equation (2) are relations between symmetrized products  $b_n$ ,  $a_n$  and polynomials of the  $n$ th order on the potential  $U(x)$  with its derivatives. In accordance with the definition of equation (7),  $b_n = 0$ , if  $n \geq g$ . The first  $g$  equations (8) are a system of algebraic equations that defines values  $(b_1, \dots, b_g)$  as polynomials on the potential and its derivatives. If we substitute them into the left-hand side of equation (8), it will lead for  $n \geq g$  to the system of equations, which we look for. A substitution into this system of equations of the general expression for the elliptic potential in a form

of the rational function of the Weierstrass function leads to equations both for potential parameters and for symmetrized products of boundaries (which are coefficients of a spectral polynomial  $P(E)$ ).

By substituting of the derived elliptic potentials  $U(x)$  into expressions for  $b_i$  and taking into account  $P(E)$ , we obtain the explicit expressions both for finite-gap eigenfunctions and zone spectra. The elliptic zone spectrum follows from the expression for an eigenfunction shifted on the period  $T$ ,  $\Psi(x+T, E) = \exp(kT)\Psi(x, E)$ . Taking into account (4) gives the relation

$$k = \int_0^E \frac{\langle Q(E) \rangle}{P(E)}, \quad \langle \dots \rangle = \frac{1}{T} \int_0^T Q(E, x),$$

which determines the dependence of  $E$  on the wave vector  $k$ .

The finite-gap elliptic potentials under consideration as elliptic functions are rational functions of the elliptic Weierstrass function  $\wp(z | \omega, \omega')$  ( $\omega, \omega'$  are real and imaginary half-periods) and can be written in the form [9]

$$U(z) = \left\{ \sum_{n=1}^N \alpha_n \wp^n(z) + \sum_i \sum_{n_i=1}^{M_i} \alpha_{n_i} (\wp(z) - h_i)^{-M-n_i} \right\} + \wp^{(1)} \tilde{R}(\wp), \tag{9}$$

where the expression in braces  $\{\dots\}$  and the last term in (9) correspond to even and odd parts of the elliptic potential,  $\tilde{R}(\wp)$  is a rational function of  $\wp$  with new coefficients and indices  $N', M'$ . Maximum values of the integers  $N, N', M = \max(M_i), M'$  in (9) are determined by a comparing of maximum values of exponents at independent variables in equation (8) for  $n = g$ . Hence, taking into account expression (3) gives the equalities

$$\begin{aligned} N + g = N(g + 1), \quad M + 2g &= M(g + 1), \quad M' + 2g = M'(g + 1), \\ N' + g &= \frac{3}{2}g + N'(g + 1), \end{aligned}$$

according to which  $N = 1, M = M' = 2, N' = 0$ . In a similar way, can be shown that the function  $\tilde{R}$  has no positive degree of  $\wp(z)$ . In the case  $h_i = e_i$  ( $e_1 = \wp(\omega), e_2 = \wp(\omega + \omega'), e_3 = \wp(\omega')$ ), the values  $M$  and  $M'$  satisfy the equalities  $M + g = M(g + 1)$  and  $M = M' = 1$ . Thus, the general expression for the elliptic  $g$ -gap potential (9) takes the form

$$U(z) = \left\{ \alpha \wp(z) + \sum_{i=1}^3 \frac{\Delta_i}{\wp(z) - \omega_i} + \sum_i \sum_{n_i=1}^2 \frac{\alpha_{n_i}}{(\wp(z) - h_i)^{n_i}} \right\} + \wp^{(1)} \tilde{R}(\wp). \tag{10}$$

The odd part of (10) corresponds to the odd part of the known formula for the elliptic potential

$$U(z) = 2 \sum_j^N \wp(z - z_j), \quad \sum_{i,j} \wp^{(1)}(z_i - z_j) = 0,$$

which can be rewritten in the form

$$U(z) = \alpha \wp(z) + \sum_j \left\{ \frac{6h_j^2 - \frac{1}{2}g_2}{\wp(z) - h_j} + \frac{h_j^{(1)2}}{(\wp(z) - h_j)^2} - h_j \right\} + \wp^{(1)}(z) \sum_j \frac{2h_j^{(1)}}{(\wp(z) - h_j)^2},$$

where the notation  $h_j = \wp(z_j)$  is introduced.

By unique independent parameters of the potential (10) are the parameters  $\omega$  and  $\tau = \omega'/\omega$  which are expressed by means of a relative width of the spectral gap. These spectral parameters in equation (8) are input for determination of the potential parameters and coefficients of the spectral polynomial  $P(E)$ . The coefficients of the rational expression (10) are determined from the condition of vanishing of the principal parts at poles relative to the variable  $\wp$  and  $1/\wp$  (where the argument  $z$  will be omitted for simplicity) in equation (8) with  $n \geq g$  after of a corresponding substitution of the expression (10).

### 3 Calculating even elliptic potentials and boundaries

Let us consider even elliptic potentials described by the general expression (10) in which  $\tilde{R} = 0$  for cases of 1–3-gap spectra. We calculate parameters of the elliptic potentials and corresponding spectral polynomials  $P(E)$  in the above mentioned way from equation (8).

#### 3.1 One-gap elliptic potentials

According to equation (8) for the case of one-gap elliptic potential,  $b_1 \neq 0$  and  $b_n = 0$  when  $n \geq 2$ . Taking into account the expressions

$$\begin{aligned} h_3 &= U^2(x) - U^{(2)}(x), \quad A_2 = \frac{1}{2!} \left\{ \left( a_2 - \frac{1}{4}a_1^2 \right) + 2(b_1^2 - b_2) - a_1 b_1 \right\}, \\ h_5 &= U^{(4)} - 5U^{(1)2} + 6UU^{(2)} - 2U^3, \\ A_3 &= \frac{1}{3!} \left\{ \left( \frac{3}{8}a_1^3 - \frac{3}{2}a_1 a_2 + \frac{3}{2}a_3 \right) + 3 \left( \frac{1}{4}a_1^2 - a_2 \right) b_1 + 3a_1(b_1^2 - b_2) + \right. \\ &\quad \left. (12b_1 b_2 - 4!b_3 - 6b_1^3) \right\}, \end{aligned}$$

that follow from the recurrent relations (2) and the definition (7), where

$$\begin{aligned} b_1 &= \frac{1}{2}(U(x) + a_1), \\ b_2 &= \frac{1}{8}(3U^2 - U^{(2)}) + \frac{1}{4}a_1 U + \frac{1}{2}a_2 - \frac{1}{8}a_1^2, \end{aligned}$$

we shall obtain the system of equations which can be written in the form

$$b_2 = \frac{1}{8}(3U^2 - U^{(2)}) + \frac{1}{4}a_1 U + \frac{1}{2}a_2 - \frac{1}{8}a_1^2 = 0, \quad (11)$$

$$\begin{aligned} b_3 &= -\frac{1}{32}(U^{(4)} + 10U^3 - 5U^{(1)2} - 10UU^{(2)}) - \frac{1}{16}a_1(3U^2 - U^{(2)}) + \\ &\quad \frac{1}{16}U(a_1^2 - 4a_2) + \frac{1}{2}a_3 + \frac{1}{4}a_1 a_2 - \frac{1}{16}a_1^3 = 0. \end{aligned} \quad (12)$$

Substituting the general expression (10) in equation (11) and equating to zero principal parts at poles  $\wp = h$ ,  $\wp = e_i$  and at pole  $y = \infty$ , where  $y = 1/\wp$ , we obtain the system

$$\alpha = 2; \quad \alpha_{2_i} = 2h'^2; \quad \alpha_{1_i} = 2\Omega, \quad a_1 = 0; \quad (13)$$

$$\Omega = 6h^2 - \frac{1}{2}g_2; \quad \Omega^2 = 24hh'^2 \quad (h = \wp(\varphi^{(1)})); \quad (14)$$

$$\Delta_i = 2\Lambda_i \quad (\Lambda_i = 3e_i^2 - \frac{1}{4}g_2); \quad (15)$$

$$\sum_i \frac{\Delta_i}{h - e_i} \equiv 2 \sum_i \{\wp(\varphi^{(1)} + \omega_i) - \wp(\omega_i)\} = 0; \quad (16)$$

$$\sum_i \left\{ \frac{\alpha_1}{e_i - h} + \frac{\alpha_2}{(e_i - h)^2} + \frac{\Delta_{i \neq j}}{e_j - e_i} \right\} \equiv 2 \sum_i \{\wp(\omega_i + \varphi^{(1)}) + \wp(\omega_i - \varphi^{(1)}) - 2\wp(\varphi^{(1)}) + 2\wp(\omega_i - \omega_j)\} = 0, \quad (17)$$

where  $g_2$  is an invariant of the elliptic Weierstrass function [9]. It follows from expressions (13)–(15) that  $\alpha$ - and  $h$ -parameters of the elliptic potential are determined by the spectral parameters  $\tau$ ,  $\omega$  and do not depend on the index  $i$  (which will be omitted henceforth). If  $\Delta_i \neq 0$ , then the system of equations (16)–(17) is valid only if  $\varphi^{(1)} = 0$  that corresponds to the condition  $\alpha_{1,2} = 0$ . If  $\varphi^{(1)} \neq 0$  and consequently  $\alpha_{1,2} = 0$ , then the indicated system of equations can be valid only if  $\Delta_i = 0$ . The conditions of compatibility of equations (13)–(17) can be expressed by the relations

$$\begin{aligned} 1) \alpha_1 &= \alpha_2 = 0, \Delta_i = 0; \\ 2) \Delta_i &= 0 \quad (i = 1, 2, 3), \alpha_i \neq 0, \quad (i = 1, 2), \end{aligned}$$

which determine two types of solutions of the system (11)–(12) for the desired parameters of 1-gap elliptic potentials and coefficients of spectral polynomials. Taking into account the well-known formulae of sums for  $\wp$ -functions [9], we can express potentials and spectral polynomials in the following form:

$$\begin{aligned} 1) U(z) &= 2\wp(z), \quad P(E) = E^3 - \frac{1}{4}g_2E - \frac{1}{4}g_3, \quad E_i = e_i, \quad (i = 1, 2, 3); \\ 2) U(z) &= 2\wp(z) + 2(\wp(z - \varphi^{(1)}) + \wp(z + \varphi^{(1)}) - 4\wp(\varphi^{(1)}), \\ P(E) &= E^3 + \frac{1}{4}(11g_2 - 120\wp^2(\varphi^{(1)}))E - \frac{1}{4}g_3 + \\ &\quad 4\wp(\varphi^{(1)})(12\wp^2(\varphi^{(1)}) - g_2) - 4\wp^{(1)2}(\varphi^{(1)}), \\ E_j &= \frac{10\wp(\varphi^{(1)})e_j + 6e_j^2 - 4\wp^2(\varphi^{(1)}) - g_2}{2(\wp(\varphi^{(1)}) - e_j)}, \\ \varphi_i^{(1)} &= \frac{2}{3}\omega_i, \quad (i = 1, \dots, 4), \quad \omega_4 = \omega_1 - \omega_3, \end{aligned}$$

where  $z = ix + \omega$  means a complex variable.

In view of the known relations [9]

$$\wp(z \mid \frac{\omega}{2}, \omega') = \wp(z) + \wp(z + \omega_1) - e_1, \quad (18)$$

$$\wp(z \mid \omega, \frac{\omega'}{2}) = \wp(z) + \wp(z + \omega_3) - e_3, \quad (19)$$

$$\wp(z \mid \omega, \frac{\omega + \omega'}{2}) = \wp(z) + \wp(z + \omega_2) - e_2, \quad (20)$$

$$\wp(z) + \sum_{i=1}^3 \wp(z + \omega_i) = 4\wp(2z), \quad (21)$$

the elliptic potentials

$$U(z) = 2\wp(z) + 2\wp(z + \omega_j)$$

and

$$U(z) = 2\wp(z) + 2 \sum_{j=1}^3 \wp(z + \omega_j)$$

together with corresponding spectral polynomials

$$\begin{aligned} P(E) &= E^3 - (3e_j^2 - 2\Lambda_j)E + 2e_j(e_j^2 + 4\Lambda_j), \quad E_1 = -2e_j, \\ E_{2,3} &= e_j \pm 2\sqrt{\Lambda_j} \end{aligned}$$

and

$$P(E) = E^3 - 4g_2E - 16g_3, \quad E_i = 4e_i \quad (i = 1, 2, 3)$$

are also placed to the type 1. Potentials 1 and 2 coincide with the known Lamé potential [9] and the potential obtained by Smirnov in [12] by an algebraic geometric method.

### 3.2 Two-gap elliptic potentials

In the case of 2-gap spectra, the elliptic potentials and corresponding spectra are described by a system of equations of the form (8) with  $b_i |_{i \leq 2} \neq 0$  and  $b_i |_{i \geq 3} = 0$ . The first three from these equations are reduced to (12) that determines the potential parameters and coefficients  $a_i$ , ( $i = 1, \dots, 3$ ) of the spectral polynomial  $P(E)$ . The two rest equations

$$A_4 |_{b_{i>2}=0} = -\frac{1}{27}\chi_7, \tag{22}$$

$$A_5 |_{b_{i>2}=0} = -\frac{1}{29}\chi_9, \tag{23}$$

determine the coefficients  $a_4, a_5$ . Substituting the general elliptic potential formula (10) into (12) for  $b_3 = 0$  and equating to zero principal parts at poles  $\wp = h$ ,  $\wp = e_i$  and pole  $y = \infty$ , where  $y = 1/\wp$ , we obtain the system

$$\alpha = 6, \alpha_1 = 2\Omega, \alpha_2 = 2h'^2, a_1 = 0; \tag{24}$$

$$8h'^4 + 12hh'^2\Omega - \Omega^3 = 0, (\Omega = 6h^2 - \frac{1}{2}g_2, h = \wp(\varphi^{(2)})) \tag{25}$$

$$\Delta_i = 2\Lambda_i (\Lambda_i = 3e_i^2 - \frac{1}{4}g_2); \tag{26}$$

$$\sum_i \frac{\Delta_i}{h - e_i} \equiv 2 \sum_i \{\wp(\varphi^{(2)} + \omega_i) - \wp(\omega_i)\} = 0; \tag{27}$$

$$\begin{aligned} \sum_i \left\{ \frac{\alpha_1}{e_i - h} + \frac{\alpha_2}{(e_i - h)^2} + \frac{\Delta_{i \neq j}}{e_j - e_i} \right\} \equiv 2 \sum_i \{ \wp(\omega_i + \varphi^{(2)}) + \\ \wp(\omega_i - \varphi^{(2)}) - 2\wp(\varphi^{(2)}) + 2\wp(\omega_i - \omega_j) \} = 0. \end{aligned} \tag{28}$$

Equations (24)–(26) uniquely determine  $\alpha$ ,  $\Delta$ , and  $h$ -parameters of the even 2-gap elliptic potentials through the spectral parameters  $\tau$  and  $\omega$ . Equation (27) can be valid if  $\varphi^{(2)} = 0$  that corresponds to the condition  $\alpha_{1,2} = 0$ . Equation (28) can be valid if  $\omega_i = 0$

that corresponds to the condition  $\Delta_i = 0$ . The conditions for equations (24)–(28) to be compatible can be expressed by the relations

$$\begin{aligned} 1) \alpha_1 &= \alpha_2 = 0, \Delta_i = 0; \\ 2) \alpha_1 &= \alpha_2 = 0; \Delta_i = \begin{cases} \Lambda_i, & \text{if } i = j; \\ 0, & \text{if } i \neq j; \end{cases} \\ 3) \alpha_1 &= \alpha_2 = 0; \Delta_i = \begin{cases} \Lambda_i, & \text{if } i = j, k; \\ 0, & \text{if } i = l; \end{cases} \\ 4) \Delta_i &= 0, (i = 1, 2, 3), \alpha_i \neq 0, (i = 1, 2), \end{aligned}$$

determining the full system of the solutions of equations (12), (22), (23) for the parameters  $(\alpha, \alpha_1, \alpha_2)$  and  $(a_1, \dots, a_5)$ . Then, the corresponding even elliptic 2-gap potentials and the spectral polynomials can be written in the form

$$\begin{aligned} 1) U(z) &= 6\wp(z), P(E) = E^5 - \frac{21}{4}g_2E^3 - \frac{27}{4}g_3E^2 + \frac{27}{4}g_2^2E - \frac{81}{4}g_2g_3, \\ E_1 &= -\sqrt{3g_2}, E_{j=1,2,3} = -3e_j, E_5 = \sqrt{3g_2}; \\ 2) U(z) &= 6\wp(z) + 2\wp(z + \omega_i) - 2e_i, \\ P(E) &= E^5 - 7(9e_i^2 + 2\Lambda_i)E^3 + 18(3e_i^3 - 5e_i\Lambda_i)E^2 + \\ &27(36e_i^4 + 16e_i^2\Lambda_i + 3\Lambda_i^2)E - 54(36e_i^5 - 52e_i^3\Lambda_i - 9e_i\Lambda_i^2), \\ E_1 &= 6e_j, E_{2,3} = -(e_k + 2e_j) \pm 2\sqrt{(7e_j + 2e_k)(e_j - e_k)}, \\ E_{4,5} &= -(e_i + 2e_j) \pm 2\sqrt{(7e_j + 2e_i)(e_i - e_k)}; \\ 3) U(z) &= 6\wp(z) + 2\wp(z + \omega_i) + 2\wp(z + \omega_k) + 2e_k, \\ P(E) &= E^5 + (161\Lambda_j - 378e_j^2)E^3 + (531e_j\Lambda_j + 108e_j^3)E^2 + \\ &27(240\Lambda_j^2 + 1280e_j^2 - 1159e_j^2\Lambda_j)E + 27(1594e_j\Lambda_j^2 + \\ &8100e_j^5 + 120e_j^3\Lambda_j^2) \equiv (E^2 - 3e_jE + 81\Lambda_j - 180e_j^2) \\ &\times (E^3 + 3e_jE^2 + (80\Lambda_j - 189e_j^2)E + 528e_j\Lambda_j - 1215e_j^3); \\ 4) U(z) &= 6\wp(z) + 2\wp(z - \varphi^{(2)}) + 2\wp(z + \varphi^{(2)}) - 4\wp(\varphi^{(2)}), \\ P(E) &= E^5 - \frac{7}{2}(18h^2 + 7\Omega)E^3 + \frac{9}{4}(24h^3 - 25h'^2 - 34h\Omega)E^2 + \\ &\frac{27}{4}(144h^4 + 44hh'^2 + 112h^2\Omega + 23\Omega^2)E - \\ &\frac{27}{2}(144h^5 + 6h^2h'^2 - 148h^3\Omega - 41h'^2\Omega - 75h\Omega^2). \end{aligned}$$

These expressions exhaust an all system of 2-gap elliptic potentials and corresponding spectral polynomials. Taking into account the above mentioned relations (18)–(21), we refer the potentials

$$U(z) = 6\wp(z) + 6\wp(z + \omega_i) - 6e_i,$$

and

$$U(z) = 6\wp(z) + 6 \sum_{i=1}^3 \wp(z + \omega_i),$$



to the type 1. The potentials 1 and 2, 3 are the known Lamé and Trebich-Verdier potentials [11, 15], respectively, and the potential 4 coincides with that obtained by Smirnov in [12] by an algebraic geometric method.

### 3.3 Three-gap elliptic potentials

The even elliptic potentials and spectral polynomials in the case of 3-gap spectra are determined by the parameters  $(\alpha, \alpha_1, \alpha_2)$  and by the seven parameters  $(a_1, \dots, a_7)$ , respectively, which are solutions of the system of the equations

$$A_{j+1} |_{b_{i \leq 3} \neq 0} = \frac{(-1)^j}{2^{2j+1}} \chi_{2j+1} \quad (j = 1, 2) \quad (29)$$

$$A_4 |_{b_{i > 3} = 0} = -\frac{1}{2^7} \chi_7; \quad (30)$$

$$A_5 |_{b_{i > 3} = 0} = \frac{1}{2^9} \chi_9; \quad (31)$$

$$A_6 |_{b_{i > 3} = 0} = -\frac{1}{2^{11}} \chi_{11}; \quad (32)$$

$$A_7 |_{b_{i > 3} = 0} = \frac{1}{2^{13}} \chi_{13}, \quad (33)$$

the first two of which determine the symmetrized products  $(b_1, b_2, b_3)$  of the above mentioned  $\mu$ -functions.

Substituting the general formula (10) in (30) and equating to zero principal parts at poles  $\wp = h$ ,  $\wp = e_i$  and at pole  $y = \infty$ , where  $y = 1/\wp$ , we will obtain the system

$$\begin{aligned} \alpha &= 12, \alpha_1 = 6\Omega, \alpha_2 = 12h^2, \Delta_i = 12\Lambda_i, a_1 = 0; \\ \Omega^2 - 12hh^2 &= 0, (\Omega = 6h^2 - \frac{1}{2}g_2, h = \wp(\varphi^{(3)}), \end{aligned}$$

and equalities following from (27)–(28) by the formal passing  $\varphi^{(2)} \rightarrow \varphi^{(3)}$ . The conditions that these equalities are compatible can be expressed by the classifying relations

$$\begin{aligned} 1) \alpha_1 &= \alpha_2 = 0, \Delta_i = 0; \\ 2) \alpha_1 &= \alpha_2 = 0; \Delta_i = \begin{cases} \Lambda_i, & \text{if } i = j, k; \\ 0, & \text{if } i = l; \end{cases} \\ 3) \Delta_i &= 0, (i = 1, 2, 3), \alpha_i \neq 0, (i = 1, 2), \end{aligned}$$

which determine types of solutions of the system (29)–(33) both for the potential parameters and above mentioned parameters  $a_1, \dots, a_7$ . The corresponding expressions for the even 3-gap elliptic potentials and spectral polynomials can be written in the form

$$\begin{aligned} 1) U(z) &= 12\wp(z), P(E) = E^7 - \frac{63}{2}g_2E^5 - \frac{297}{2}g_3E^4 + \frac{4185}{16}g_2^2E^3 - \\ &\quad \frac{18225}{8}g_2g_3E^2 + \left(\frac{91125}{16}g_2^2 - \frac{3375}{16}g_2^3 + \right)E; \\ 2) U(z) &= 12\wp(z) + 12(\wp(z + \omega_i) + \wp(z + \omega_j)) + 12e_k; \\ P(E) &= E^7 + (721\Lambda_i - 147e_i^2)E^5 + (1531e_i\Lambda_i + 1312e_i^3)E^4 + \\ &\quad (1140\Lambda_i^2 + 846e_i^4 - 509e_i^2\Lambda_i)E^3 + (1932e_i\Lambda_i^2 + \end{aligned}$$

$$\begin{aligned}
 & 7650e_i^5 + 3201e_i^3\Lambda_i^2)E^2 + \\
 & (432\Lambda_i^2e_i^2 + 5641\Lambda_ie_i^4 - 1016\Lambda_i^3)E + \\
 & (1017\Lambda_i^3e_i - 301\Lambda_i^2e_i^3 - \Lambda_ie_i^5); \\
 3) U(z) &= 12\wp(z) + 12\{\wp(z + \varphi^{(3)}) + \wp(z - \varphi^{(3)}) - 2\wp(\varphi^{(3)})\}, \\
 P(E) &= E^7 + \frac{945}{11}(2h^2 + 3\Omega)E^5 + \frac{297}{2}(8h^3 - 27h'^2 - 30h\Omega)E^4 + \\
 & \frac{529497}{121}(h^4 + 27hh'^2 + 3h^2\Omega)E^3 + \\
 & \frac{729}{484}(208944h^5 + 1074794h^2h'^2 + 381356h^3\Omega - \\
 & 222189h'^2\Omega)E^2 + \frac{243}{1936}(7301760h^6 - 843322848h^3h'^2 + \\
 & 35596627h'^4 - 66565488h^4\Omega - 610253088hh'^2\Omega)E + \\
 & \frac{729h}{2662}(45109488h^6 - 48566951728h^3h'^2 - \\
 & 14375034553h'^4 - 65505732h^4\Omega - 27593771259hh'^2\Omega).
 \end{aligned}$$

Taking into account the relations (18)–(21), the potentials

$$U(z) = 12\wp(z) + 12\wp(z + \omega_i) - 12e_i,$$

and

$$U(z) = 12\wp(z) + 12\sum_{i=1}^3\wp(z + \omega_i),$$

can be also placed to the type 1. As distinguished from 1–2-gap elliptic potentials, the 3-gap potentials can not be expressed as a linear combination of the two Lamé potentials, one of which has an argument shifted on values  $\omega_i$ . The expressions 1–3 describe the system of the even 3-gap elliptic potentials and spectral polynomials, from which the first and the second ones can be placed to the Lamé and the Trebich-Verdier potentials. The third potential can be placed to the type of potentials obtained by Smirnov in [12]. The expression for the spectrum 1 agrees with the known Hermite result for the 3-gap Lamé potential. Another elliptic even 3-gap spectra have been obtained for the first time. Similary to the above considered 1–2-gap elliptic potentials, the 3-gap potentials can be expressed only as a linear combination of the Lamé potentials without and with shifts of arguments on half-periods  $\omega_i$  or with a shift on the value  $\varphi^{(3)}$  depending on spectral parameters  $\tau, \omega$ .

## 4 Conclusion

Thus, an usage of the representation of a rational function of the elliptic Weierstrass function for  $g$ -gap elliptic potentials in trace formulae brings up the simple general method for a calculating of arbitrary finite-gap elliptic potentials and spectral polynomials. On the basis of this method we have shown that the known general formula for  $g$ -gap even elliptic potentials in cases of 1–3-gap spectra can have only a form of linear combinations of the Lamé potentials without and with shifts of arguments on half-periods  $\omega_j$  or on the value

$\varphi^{(g)}$  determined by the corresponding above mentioned equations. We have calculated the explicit expressions for even elliptical potentials and corresponding spectra for both known 1–2-gap and unknown 3-gap cases.

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