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Internal Gravity Waves with Free Upper Surface Over an Obstacle

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Abstract

In this paper we discuss a theoretical model for both the free-surface and interfacial profiles of progressive nonlinear waves which result from introducing an obstacle of finite height, in the form of a ramp of gentle slope, attached to the bottom below the flow of a stratified, ideal, two-layer fluid. The derived equations are solved by using a nonlinear perturbation method. The effect of the height of the ramp, also some flow parameters, such as the ratios of depths and densities of the two fluids, have been studied and illustrated.

1 Introduction

Over the past decades there has been a great deal of interest in the study of finite-amplitude effects in internal wave systems. Recently, Kevorkian and Yu [1], in 1989, studied the behaviour of shallow water waves excited by a small amplitude bottom disturbance in the presence of a uniform incoming flow. This paper describes a theoretical model to investigate the behaviour of nonlinear free-surface and interfacial waves when passing over an obstacle in the form of a ramp of gentle slope. Our primary motivation for the present investigation is to calculate the share of both the free-surface and interfacial profiles, and to discuss the influence of both geometrical and flow parameters of the profiles. In section 2 we extended the mathematical technique applied by Helal & Molines [2] in determining the nonlinear free-surface and interfacial waves in a tank with the flat horizontal bottom and generalized the problem applied by Boutros et al. [3] in determining the interfacial waves with the rigid upper boundary over irregular topography. Nonlinear pertrubation method is used, leading, in sections 3 and 4, to two expressions for both free-surface and interfacial waves that are derived in the form of expansions in powers of ε^2, where ε is a small parameter that provides a measure of weakness of the dispersion.

Finally, in section 5 we have illustrated and discussed the effect of the density ratio, R, the thickness ratio, H, and the ramp height, L. A comparison has been made between the second and fourth order approximations for the free-surface and interfacial profiles showing that the error, difference between them, is of order 10^{-7} and 10^{-6}, respectively.

2 Formulation of the problem

Two-dimensional irrotational motion is considered of a stably stratified two-layer inviscid, incompressible, fluid with the bottom surface in the form of a ramp inclined by a small angle a. We assume that the flow field due to the wave motion remains irrotational. A

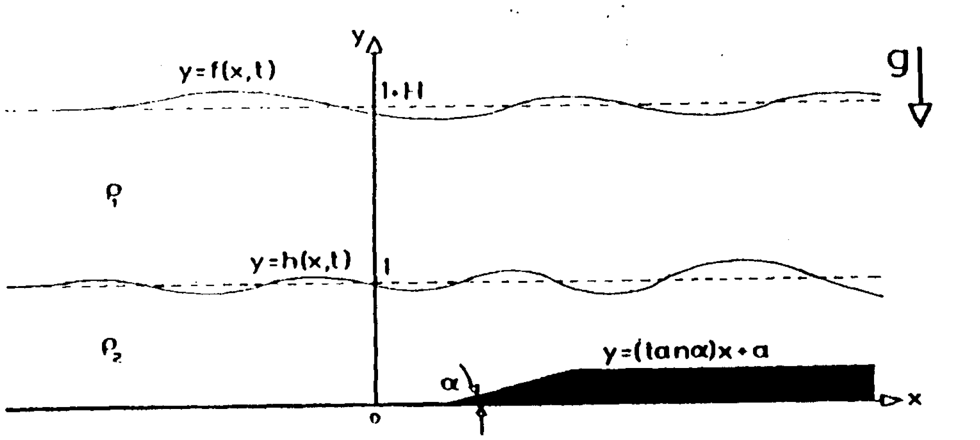

Figure 1. Geometrical configuration of gravity waves over a ramp in nondimensional variables

The fluid flows into the channel in the region left of the bottom slope region with uniform velocity U^* and a gravity wave is created on the interface of the two fluids. The Y^* coordinate is measured vertically upwards and X^* perpendicular to this direction to the right. The heights of the undistrubed lower and upper surfaces are H_2^* and $H_2^* + H_1^*$, respectively. The lower and upper surfaces disturbances from uniform conditions are given by $Y^* = h^*(X^*, \tau^*)$ and $Y^* = f^*(X^*, \tau^*)$, respectively. The component of gravity, vertically downwards, is g, and $Y^* = W^*(X^*)$ is the bed of the channel. The equations of motion are thus the Euler equations together with the continuty equation. All variables are nondimensionalized by using the characteristic length H_2^* and time $(g/H_2^*)^{-1/2}$, and accordingly

$$U = U^* / [gH_2^*]^{1/2} \quad \text{and} \quad \phi^{(i)} = \phi^{*^{(i)}} / (H_2^* [gH_2^*]^{1/2}) \tag{2.1}$$

where velocity potentials of upper and lower layers are denoted by $\phi^{*^{(1)}}$, $\phi^{*^{(2)}}$, respectively, τ^* is the time, $\rho^{(1)}$ and $\rho^{(2)}$ are densities of the upper and lower fluids, respectively.

Moreover, assuming that the fluids are in the undisturbed uniform state up/down stream at infinity, we impose the following boundary conditions with respect to X^*

$$\phi_{X^*}^{*(i)} = U^* \quad (i = 1, 2) \qquad \text{as} \quad X^* \to \pm\infty. \tag{2.2}$$

An essential step which makes our problem easier in handling is to define an appropriate **stretching** of the horizontal coordinate while leaving the vertical coordinate unchanged due to the fact that the horizontal dimensions are much greater that the vertical ones, thus we define

$$x = \varepsilon X, \quad y = Y, \quad t = \varepsilon \tau, \tag{2.3}$$

where ε is a small parameter. Thus the basic equations for this system can be written as

$$\varepsilon^2 \phi^{(1)}_{xx} + \phi^{(1)}_{yy} = 0, \quad h < y < f, \quad -\infty < x < \infty, \tag{2.4.1}$$

$$\varepsilon^2 \phi^{(2)}_{xx} + \phi^{(2)}_{yy} = 0, \quad W < y < h, \quad -\infty < x < \infty \tag{2.4.2}$$

with conditions

$$\left.\begin{aligned} &\phi^{(1)}_y = \varepsilon f_t + \varepsilon^2 \phi^{(1)}_x f_x \\ &\varepsilon \phi^{(1)}_t + \tfrac{1}{2}\left[\varepsilon^2 (\phi^{(1)}_x)^2 + (\phi^{(1)}_y)^2\right] + f - H - 1 = 0 \end{aligned}\right\} \text{ at } y = f, \tag{2.4.3}$$

$$\left.\begin{aligned} &\phi^{(i)}_y = \varepsilon h_t + \varepsilon^2 \phi^{(i)}_x h_x \qquad (i = 1, 2) \\ &R\{\varepsilon \phi^{(1)}_t + \tfrac{1}{2}\left[\varepsilon^2 (\phi^{(1)}_x)^2 + (\phi^{(1)}_y)^2\right] + h - 1\} = \\ &\quad \{\varepsilon \phi^{(2)}_t + \tfrac{1}{2}\left[\varepsilon^2 (\phi^{(2)}_x)^2 + (\phi^{(2)}_y)^2\right] + h - 1\} \end{aligned}\right\} \text{ at } y = h, \tag{2.4.4}$$

$$\phi^{(2)}_y = \varepsilon^2 \phi^{(2)}_x W_x \quad \text{at} \quad y = W(x), \tag{2.4.5}$$

$$\varepsilon \phi^{(i)}_x = 1 \quad (i = 1, 2) \quad \text{as} \quad x \to \pm\infty, \tag{2.4.6}$$

where the density ratio $R = \rho^{(1)}/\rho^{(2)}$ (less that unity) and the thickness ratio H are two characteristic parameters of the system, and $W(x)$ has the form

$$W(x) = \alpha x + a, \tag{2.5}$$

where

$$(a,\ \alpha) = \begin{cases} (0,\ 0) & x \le x_0 \\ (-\alpha x_0,\ \alpha) & x_0 \le x \le x_L \\ (L,\ 0) & x > x_L \end{cases} \tag{2.6}$$

and L is the ramp height.

Since we consider weakly nonlinear waves, we expand the dependent variables as power series in the same parameter ε around the undisturbed uniform state, following Helal and Molines [2], we get

$$\left.\begin{aligned} &\phi^{(i)} = \sum_{n=0}^{\infty} \varepsilon^{2n+1} G^{(i)}_{2n+1}(x, y, t) \qquad (i = 1, 2) \\ &f = \sum_{n=0}^{\infty} \varepsilon^{2n} f_{2n}(x, y, t) \\ &h = \sum_{n=0}^{\infty} \varepsilon^{2n} h_{2n}(x, y, t) \end{aligned}\right\} \tag{2.7}$$

with $f_0 = 1 + H, \quad h_0 = 1$.

The scale parameter ε, which is assumed to be small, provides a measure of weakness of the dispersion.

The boundary conditions on the free surface, equations (2.4.3), and on the interface, equations (2.4.4), are expanded as a Taylor expansion of the type

$$[V]_{y=y_0+\varepsilon^2 A} = [V]_{y_0} + \varepsilon^2 A\,[V_y]_{y_0} + \frac{(\varepsilon^2 A)^2}{2!}\,[V_{yy}]_{y_0} + \cdots \tag{2.8}$$

When (2.3), (2.5), using the expansion (2.6), are inserted into equations (2.4) and powers of ε are sorted out, a sequence of "cell" problems emerges, from which the unknown profiles, f and h, can be determined.

3 Orders of approximations

3.1 The first-order approximation:

Equations of the first-order approximation finally give, for $i = 1, 2$,

$$G_1^{(i)} = B^{(i)}(x,t), \tag{3.1.1}$$

where $B^{(i)}(x,t)$ are unknown functions to be determined.

3.2 The second-order approximation:

From the equations obtained from the second-order approximation, we conclude that

$$B_x^{(i)} = 0 \quad (i = 1,2) \quad \text{as} \quad x \to \pm\infty, \tag{3.2.1}$$

$$f_2(x,t) = -B_t^{(1)} \tag{3.2.2}$$

and

$$h_2(x,t) = \frac{1}{1-R}\,[RB_t^{(1)} - B_t^{(2)}]. \tag{3.2.3}$$

3.3 The third- and fourth order approximations:

Equations of the third- and fourth order approximations, finally give, for $i = 1, 2$,

$$G_3^{(i)} = -\frac{1}{2}\,y^2 B_{xx}^{(i)} + yC^{(i)}(x,t) + D^{(i)}(x,t) \tag{3.3.1}$$

where $C^{(i)}(x,t)$ and $D^{(i)}(x,t)$ are arbitrary functions that satisfy the following boundary conditions:

$$C_x^{(i)} = 0 \quad (i = 1,2) \quad \text{as} \quad x \to \pm\infty, \tag{3.3.2}$$

$$C^{(2)}(x,t) = (WB_x^{(2)})_x \quad \text{at} \quad y = W(x), \tag{3.3.3}$$

$$D_x^{(i)} = 0 \quad (i = 1,2) \quad \text{as} \quad x \to \pm\infty. \tag{3.3.4}$$

Substituting equation (3.3.1) in the boundary conditions obtained from the third- and fourth-order approximations we obtain

$$(H+1)\,B^{(1)}_{xx} - C^{(1)} - B^{(1)}_{tt} = 0, \tag{3.3.5}$$

and for $i = 1, 2$

$$B^{(i)}_{xx} - C^{(i)} + \frac{1}{1-R}\,(RB^{(1)}_{tt} - B^{(2)}_{tt}) = 0. \tag{3.3.6}$$

From equations (3.3.3), (3.3.5), and (3.3.6) we get

$$\Box_1\, B^{(1)} = B^{(2)}_{tt}, \tag{3.3.7}$$

$$\Box_2\, B^{(2)} = RB^{(1)}_{tt}, \tag{3.3.8}$$

where $\Box_1$, $\Box_2$ are the differential operators

$$\Box_1 \equiv -H\,(1-R)\,\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2}, \tag{3.3.9}$$

$$\Box_2 \equiv -(1-R)\;(1-W)\,\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} + (1-R)\,\frac{\partial W}{\partial x}\;\frac{\partial}{\partial x}. \tag{3.3.10}$$

From equations (3.3.7)–(3.3.10) we can get, after getting rid of $B^{(1)}$ and substituting for $W\,(x)$, the following differential equation for the unknown function $B^{(2)}$

$$-H\,(1-R)\;(1-a-\alpha x)\,B^{(2)}_{xxxx} + (H+1-a-\alpha x)\,B^{(2)}_{xxtt} - B^{(2)}_{tttt} - \alpha B^{(2)}_{xtt} + 3H\alpha\,(1-R)\,B^{(2)}_{xxx} = 0 \tag{3.3.11}$$

and for $f_4(x,t)$ and $h_4(x,t)$ we can get the following relations

$$f_4\,(x,t) = \frac{(H+1)^2}{2}\,B^{(1)}_{xxt} - (H+1)\,C^{(1)}_t - D^{(1)}_t - \frac{1}{2}\,(B^{(1)}_x)^2 \tag{3.3.12}$$

and

$$h_4\,(x,t) = \frac{1}{1-R}\,\Big\{R\,[-\frac{1}{2}\,B^{(1)}_{xxt} + C^{(1)}_t + D^{(1)}_t + \frac{1}{2}\,(B^{(1)}_x)^2] + \frac{1}{2}\,B^{(2)}_{xxt} - C^{(2)}_t - D^{(2)}_t - \frac{1}{2}\,(B^{(2)}_x)^2\Big\}. \tag{3.3.13}$$

3.4 The fifth- and sixth order approximations:

Equations of the fifth- and sixth order approximations lead to, for $i = 1, 2$,

$$G^{(i)}_5 = \frac{y^4}{24}\,B^{(i)}_{xxxx} - \frac{y^3}{6}\,C^{(i)}_{xx}\,(x,t) - \frac{y^2}{2}\,D^{(i)}_{xx}\,(x,t) + yE^{(i)}\,(x,t) + F^{(i)}\,(x,t), \tag{3.4.1}$$

where $E^{(i)}\,(x,t)$ and $F^{(i)}\,(x,t)$ are arbitrary functions that satisfy the following conditions:

$$E^{(1)}_x = 0 \qquad \text{as} \qquad x \to \pm\infty, \tag{3.4.2}$$

$$E^{(2)}(x,t) = \Big(-\frac{W^3}{3!}B^{(2)}_{xxx} + \frac{W^2}{2!}C^{(2)}_x(x,t) + WD^{(2)}_x\Big)_x, \tag{3.4.3}$$

$$F^{(i)}_x = 0 \qquad (i = 1,2) \qquad \text{as} \qquad x \to \pm\infty. \tag{3.4.4}$$

Introducing equations (3.2.2.)–(3.4.1) in the boundary conditions, we have the following relations:

$$\frac{(H+1)^3}{3!}B^{(1)}_{xxxx} - \frac{(H+1)^2}{2!}C^{(1)}_{xx} - (H+1)\,Dxx^{(1)} + E^{(1)} + B^{(1)}_t B^{(1)}_{xx} -$$

$$\frac{(H+1)^2}{2!}B^{(1)}_{xxtt} + (H+1)\,C^{(1)}_{tt} + D^{(1)}_{tt} + 2B^{(1)}_x B^{(1)}_{xt} = 0 \tag{3.4.5}$$

and for $i = 1,2$

$$\frac{1}{3!}B^{(i)}_{xxxx} - \frac{1}{2!}C^{(i)}_{xx} - D^{(i)}_{xx} + E^{(i)} + \frac{1}{1-R}\Big[(B^{(2)}_t - RB^{(1)}_t)\,B^{(i)}_{xx} +$$

$$(B^{(2)}_{xt} - RB^{(1)}_{xt})\,B^{(i)}_x - \frac{1}{2}B^{(2)}_{xxtt} + C^{(2)}_{tt} + D^{(2)}_{tt} -$$

$$R\Big(-\frac{1}{2}B^{(1)}_{xxtt} + C^{(1)}_{tt} + D^{(1)}_{tt}\Big) + B^{(2)}_x B^{(2)}_{xt} - RB^{(1)}_x B^{(1)}_{xt}\Big] = 0. \tag{3.4.6}$$

As it will be seen later on, there is no need to calculate $f_6(x,t)$ and $h_6(x,t)$ due to the fact that the error, difference between the second- and fourth-order approximations is of order 10^{-6} for the interfacial wave profile and 10^{-7} for the free-surface profile.

Thus, the problem is now reduced to solving equations (3.3.3), (3.4.5), and (3.4.6) for $B^{(1)}$, $B^{(2)}$, $C^{(1)}$ and $C^{(2)}$ and next equations (3.4.3), (3.4.5) and (3.4.6) for $D^{(1)}$, $D^{(2)}$, $E^{(1)}$ and $E^{(2)}$.

4 Case of a progressive wave

It must be remarked that our procedure is valid as long as $a \gg \varepsilon^2$, otherwise a two-parameter analysis has to be carried out. Moreover, we shall invoke the smallness of a and write perturbation expansions for $B^{(i)}$, $i = 1,2$, in the form

$$B^{(i)} = B^{(i)}_0 + \alpha B^{(i)}_1 + \alpha^2 B^{(i)}_2 + \cdots \tag{4.1}$$

Substituting (4.1) in (3.3.11) and equating coefficients of $a^{(j)}$, $j = 0,1,2,...$ we get the following system of differential equations

$$\Box B^{(2)}_1 = \Lambda B^{(2)}_{j-1}, \quad B^{(2)}_{-1} = 0 \tag{4.2}$$

where $\Box$, Λ are two differential operators defined as

$$\Box \equiv \Big(\psi_0\frac{\partial^2}{\partial x} + \beta_1\frac{\partial^2}{\partial t}\Big)\Big(\psi_0\frac{\partial^2}{\partial x} + \beta_2\frac{\partial^2}{\partial t}\Big) \tag{4.3}$$

and

$$\psi_0 = 2H(a-1)(1-R), \tag{4.4}$$

$$\beta_1 = (H+1-a) + [(H+1-a)^2 + 2\psi_0]^{1/2}, \tag{4.5}$$

$$\beta_2 = (H+1-a) - [(H+1-a)^2 + 2\psi_0]^{1/2}, \tag{4.6}$$

$$\Lambda \equiv -xH(1-R)\frac{\partial^4}{\partial x^4} + x\frac{\partial^4}{\partial x^2 \partial t^2} - 3H(1-R)\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial x \partial t^2}. \tag{4.7}$$

Equation (4.2), for $j = 0$, has the following general solution

$$B_0^{(2)} = \sum_{i=1}^{4} T_i(\xi) \tag{4.8}$$

with

$$\xi_1 = x - y_1 t;\ \xi_2 = x + y_1 t;\ \xi_3 = x - y_2 t;\ \xi_4 = x + y_2 t, \tag{4.9}$$

and

$$y_i^2 = -\frac{\psi_0}{\beta_i}; \qquad i = 1, 2 \tag{4.10}$$

and T_i ($i = 1, 2, 3$ and 4) are arbitrary functions of the variable ξ.

Let us choose a pure progressive wave, i.e., $B^{(i)} = B^{(i)}(\xi)$ with $\xi = x - yt$, where y may take the two possible values y_1 and y_2 as defined in (4.10). Thus,

$$B^{(i)}(x,t) = B^{(i)}(x - yt) = B^{(i)}(\xi) \qquad (i = 1, 2). \tag{4.11}$$

From equations (2.5), (3.3.5), and (4.1) we get

$$C^{(1)} = \mu B_{0,xx}^{(1)} + \alpha\mu B_{1,xx}^{(1)} + \cdots \tag{4.12}$$

with

$$\mu = H + 1 - y^2. \tag{4.13}$$

Substituting (2.5) and (4.1) in (3.3.3), we get

$$C^{(2)} = aB_{0,xx}^{(2)} + \alpha[aB_{1,xx}^{(2)} + (xB_{0,x}^{(2)})_x] + \alpha^2[aB_{2,xx}^{(2)} + (xB_{1,x}^{(2)})_x] + \cdots \tag{4.14}$$

Again substituting equations (4.1), (4.12) in equation (3.3.6), we get after equating coefficients of $\alpha^0, \alpha^1, \alpha^2, \ldots$.

$$B_{0,x}^{(2)} = \lambda B_{0,x}^{(1)}, \tag{4.15}$$

$$B_{1,x}^{(2)} = \frac{x}{1-a} B_{0,x}^{(2)} + \lambda B_{1,x}^{(1)}, \tag{4.16}$$

where

$$\lambda = \frac{\gamma^2 - H}{1-a}. \tag{4.17}$$

The elimination of $E^{(1)}$ in equations (3.4.5) and (3.4.6) gives, for a, the following system of differential equations

$$\Big(H - \frac{\gamma^2}{1-R}\Big) D^{(1)}_{\xi\xi} + \Big(\frac{\gamma^2}{1-R}\Big) D^{(2)}_{\xi\xi} = P_1 B^{(1)}_{0,\xi\xi\xi\xi} + Q_1 B^{(1)}_{0,\xi} B^{(1)}_{0,\xi\xi}, \tag{4.18}$$

$$\Big(\frac{\gamma^2 R}{1-R}\Big) D^{(1)}_{\xi\xi} + \Big(1 - a - \frac{\gamma^2}{1-R}\Big) D^{(2)}_{\xi\xi} = P_2 B^{(1)}_{0,\xi\xi\xi\xi} + Q_2 B^{(1)}_{0,\xi} B^{(1)}_{0,\xi\xi}, \tag{4.19}$$

where

$$\begin{aligned} P_1 \quad = \quad & \frac{H(H^2+3H+3)}{6} - \frac{\mu H}{2}(H+2) + \frac{\gamma^2}{2}(H+1)(2\mu - H - 1) + \\ & \frac{\gamma^2(\lambda - 2a) + R(2\mu - 1)}{2(1-R)}, \end{aligned} \tag{4.20}$$

$$Q_1 = \frac{\gamma}{1_R}(\lambda^2 + 2\lambda - 3), \tag{4.21}$$

$$P_2 = \frac{\lambda}{6}(1 - 3a + 2a^3) + \frac{\gamma^2}{2(1-R)}[(2a-1)\lambda - R(2\mu - 1)], \tag{4.22}$$

$$Q_2 = \frac{\gamma}{1-R}[R(2\lambda + 1) - 3\lambda^2]. \tag{4.23}$$

For the nontrivial solution of $D^{(1)}_{\xi\xi}$ and $D^{(2)}_{\xi\xi}$, the following differential equation for $B^{(1)}_0$ should be satisfied:

$$M_1 B^{(1)}_{0,\xi\xi\xi\xi} + M_2 B^{(1)}_{0,\xi} B^{(1)}_{0,\xi\xi} = 0, \tag{4.24}$$

where

$$M_1 = \Big(1 - a - \frac{\gamma^2}{1-R}\Big) P_1 - \Big(\frac{\gamma^2}{1-R}\Big) P_2, \tag{4.25}$$

$$M_2 = \Big(1 - a - \frac{\gamma^2}{1-R}\Big) Q_1 - \Big(\frac{\gamma^2}{1-R}\Big) Q_2. \tag{4.26}$$

Define

$$\Gamma = B^{(1)}_{0,\xi}. \tag{4.27}$$

Thus equation (4.26), by virtue of equation (4.29), will be transformed to the Boussinesq equation

$$M_1 \Gamma_{\xi\xi\xi} + M_2 \Gamma \Gamma_\xi = 0. \tag{4.28}$$

Helal & Molines [2] mentioned that the general solution of equation (4.28) was found by Byrd and Friedmann [4] to be, in terms of the Jacobi elliptic function $\mathrm{sn}\,(u,k)$, as

$$B^{(1)}_{0,\xi} = Y_1 \Big[1 - \frac{3k^2}{k^2+1} \mathrm{sn}^2 \Big(\frac{1}{2}\Big(-\frac{3AY_1}{k^2+1}\Big)^{1/2} \xi, k^2\Big)\Big], \tag{4.29}$$

where Y_1 is the greatest of the roots of the polynomial resulting from integrating equation (4.27) twice and k is the modulus of the Jacobian elliptic function.

For small values of k, the above elliptic function could be calculated in terms of circular functions, see Milne-Thomson [5], thus we have

$$\begin{aligned} B^{(1)}_{0,\xi} = {} & Y_1\Big[1-\frac{3k^2}{k^2+1}\Big[\Big(\frac{1}{2}+\frac{k^2}{8}+\frac{k^4}{16}\Big)\Big]+\Big(\frac{k^4-64}{128}\Big)\cos 2\delta\xi- \\ & \Big(\frac{8k^2+k^4}{64}\Big)\cos 4\delta\xi-\frac{k^4}{128}\cos 6\delta\xi-\delta\xi\Big\{\Big(\frac{k^2}{2}+\frac{k^4}{8}\Big)\sin 2\delta\xi+ \\ & \frac{k^4}{16}\sin 4\delta\xi\Big\}+\delta^2\xi^2\Big\{\frac{k^4}{8}+\frac{k^4}{8}\cos 2\delta\xi\Big\}\Big], \end{aligned} \tag{4.30}$$

where

$$\delta=\frac{1}{2}\Big(-\frac{3AY_1}{k^2+1}\Big)^{1/2}. \tag{4.31}$$

Substituting in equation (4.3) for $B^{(2)}_{0,x}$ and $B^{(2)}_{0,t}$, we get the following fourth-order linear partial differential equation

$$\begin{aligned} LB^{(2)}_1 = {} & \sum_{n=1}^{3}(A_n x\sin 2n\delta\xi+A_{n+6}\cos 2n\delta\xi)+ \\ & \delta\xi\sum_{n=1}^{2}(A_{n+3}x\cos 2n\delta\xi+A_{n+10}\sin 2n\delta\xi)+ \\ & \delta^2\xi^2(A_6 x\sin 2\delta\xi+A_{13}\cos 2\delta\xi)+A_{10}, \end{aligned} \tag{4.32}$$

where the coefficients $A_1, A_2, ..., A_{13}$ are given at the end of the paper, as Appendix 1.

Solving equation (4.34) for the unknown $B^{(2)}_1$, following Miller [6], and calculating $B^{(2)}_{1,t}$ we get

$$\begin{aligned} B^{(2)}_{1,t} = {} & B^{(2)}_{0,t}+r_1t^3+(r_2+r_3x^2+r_4xt+r_5t^2)\sin 2\delta\xi+(r_6+r_7x^2+ \\ & r_8xt+r_9t^2)\sin 4\delta\xi+r_{10}\sin 6\delta\xi+(r_{11}x+r_{12}t+r_{13}x^3+ \\ & r_{14}x^2t+r_{15}xt^2+r_{16}t^3)\cos 2\delta\xi+(r_{17}x+r_{18}t)\cos 4\delta\xi+ \\ & (r_{19}x+r_{20}t)\cos 6\delta\xi. \end{aligned} \tag{4.33}$$

where the coefficients $r_1, r_2, ..., r_{20}$ are also given at the end of the paper, as Appendix 2.

Taking into consideration the value of $B^{(1)}_{0,x}$ from equation (4.31), we can get $B^{(2)}_{0,x}$ and thus, using (4.34) for $B^{(2)}_{1,t}$, we can get $B^{(1)}_{1,t}$

$$B^{(1)}_{1,t}=\frac{1}{\lambda}\Big(B^{(2)}_{1,t}-\frac{x}{1-a}B^{(2)}_{0,t}\Big). \tag{4.34}$$

In order to account for the nonlinear effects, the $O(\varepsilon^4)$ equations have to be considered as well. Thus bearing in mind the linear system of equations (4.21), the principial and secondary determinants of this system, we come to the result that

$$D^{(i)}_t=0 \qquad (i=1,2). \tag{4.35}$$

Hence, $f_4(x,t)$ and $h_4(x,t)$ may be rewritten in the simplified form

$$f_4(x,t) = \left\{\frac{1}{2}(H+1)(H+1-2\mu)\right\}\left\{B^{(1)}_{0,xxt} + \alpha B^{(1)}_{1,xxt}\right\} - \frac{1}{2}\left(B^{(1)}_{0,x} + \alpha B^{(1)}_{1,x}\right)^2 \quad (4.36)$$

and

$$\begin{aligned} h_4(x,t) \;=\; & \frac{1}{1-R}\left(\frac{1}{2}(\lambda - R) + \mu R - \lambda(\alpha x + a) + \frac{\alpha x \lambda(1-2a)}{2(1-a)}\right) B^{(1)}_{0,xxt} + \\ & \frac{1}{1-R}\left(R(\mu - \tfrac{1}{2}) + \lambda(\tfrac{1}{2} - a)\right) B^{(1)}_{1,xxt} + \\ & \frac{1}{2(1-R)}\left(R - \lambda^2 - \frac{\alpha x \lambda^2(\alpha x + 2)}{1-a}\right)(B^{(1)}_{0,x})^2 + \frac{\alpha^2(R-\alpha)}{2(1-R)}(B^{(1)}_{1,x})^2 + \\ & \frac{\alpha\lambda a}{(1-R)(a-1)} B^{(1)}_{0,xt} + \frac{\alpha}{1-R}\left(R - \lambda^2 - \frac{\alpha x \lambda^2}{1-a}\right) B^{(1)}_{0,x} B^{(1)}_{1,x}. \end{aligned} \quad (4.37)$$

Hence, h and f will take the form

$$h(x,t) = 1 + \varepsilon^2\left\{\frac{(R-\lambda)(a-1) + \lambda\alpha x}{(1-R)(a-1)} B^{(1)}_{0,t} + \frac{\alpha(R-\lambda)}{1-R} B^{(1)}_{1,t}\right\} + \varepsilon^4 h_4(x,t) + O(\varepsilon^6) \quad (4.38)$$

and

$$f(x,t) = 1 + H - \varepsilon^2(B^{(1)}_{0,t} + \alpha B^{(1)}_{1,t}) + \varepsilon^4 f_4(x,t) + O(\varepsilon^6), \quad (4.39)$$

where $f_4(x,t)$ and $h_4(x,t)$ are given by (4.36) and (4.37), respectively, and $B^{(1)}_{0,t}$ and $B^{(1)}_{1,t}$ are given by (4.30) and (4.34), respectively.

5 Presentation of results and discussion

A number of terms which have been obtained seems to be a good measure for the purpose of illustrating the effect of the parameters R, H, and L. The error, difference between the fourth and second order approximations, in both the interfacial profile and the free surface for the two approximations is of order 10^{-6} for the interfacial wave, while that in calculating the free surface profile is of order 10^{-7}. Thus, we limit our calculations up to the second-order approximation.

In figure 2, we illustrate the effect of the density ratio, R, on the wave profiles at the interfacial and free surfaces. As it is clear, for both waves the less the density ratio, the higher will be the wave. An important remark needed to be mentioned is that for both waves, especially in the downstream region, the period of oscillation is much longer for the case when the two fluids are of very nearly equal density than that of significantly different densities. This is due to the fact that the presence of the upper fluid has the effect of decreasing the velocity of propagation of the wave, which consequently causes the decrease of the potential energy of a given deformation of the interface as well as the increase of inertia. This result comes in good agreement with Lamb [7].

Figure 3 shows different wave profiles $h(x,t)$ and $f(x,t)$ for different values of the thickness ratio, H. For the interfacial wave profile, and for the free surface as H increases,

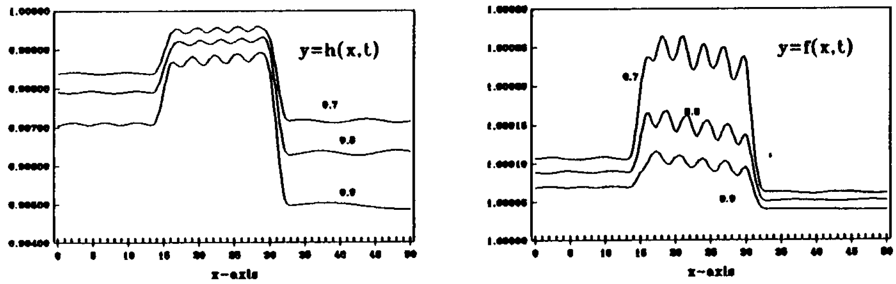

Figure 2. Effect of the density ratio, R, on the interfacial and free surfaces for $L = 0.25, H = 0.6$, $\alpha = 0.015625$ rad, $t = 50$, $x_0 = 10$ and $x_L = 31$

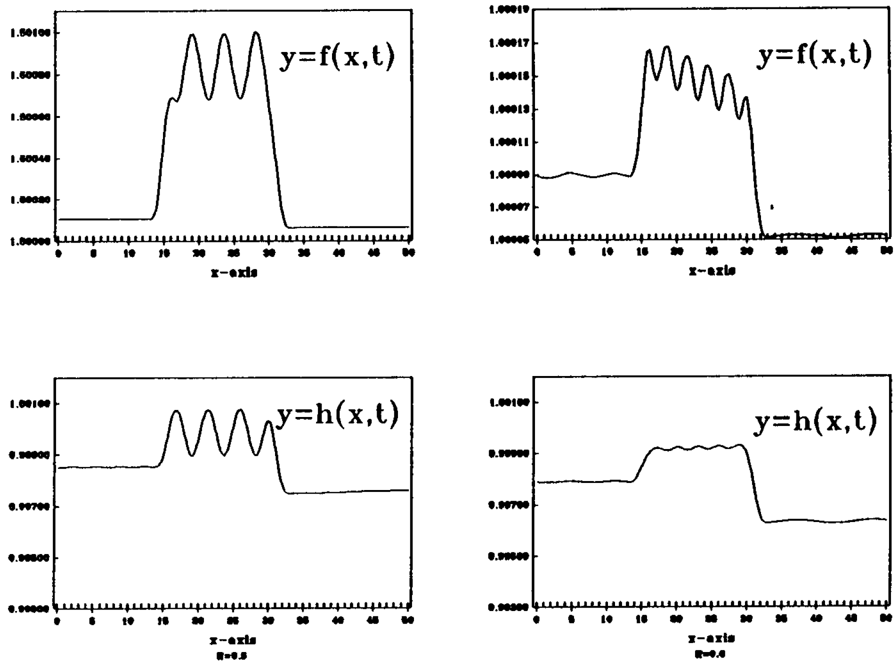

Figure 3. Effect of the thickness ratio, H, on the interfacial and free surfaces for $L = 0.25$, $R = 0.8$, $\alpha = 0.015625$ rad, $t = 50.0$ and $x_0 = 10$ and $x_L = 31$

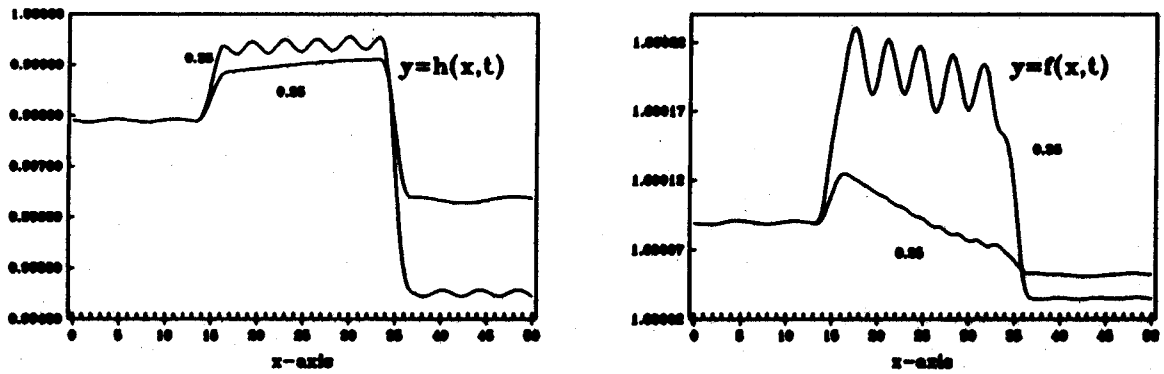

Figure 4. Effect of the ramp height, L, on the interfacial and free surfaces for $H = 0.6$, $R = 0.8$, $\alpha = 0.015625$ rad, $t = 50$, $x_0 = 10$ and $x_L = 31$

there will be a significant drop in the wave profile, and an increase in the amplitude of the wave along the ramp interval will take place.

In figure 4, we study the effect of changing the ramp height, L. For the interfacial wave, as L increases, a kind of violent disturbance in the wave profile appears, starting by a sudden increase in the profile and ending by a steep decrease at the beginning of the downstream interval. This phenomenon is also true for the free wave.

Appendix 1

$A_1 = W_1(-4 + 6k^2 + 0.0625k^4)$, $A_2 = W_1(2k^4 - 8k^2)$,
$A_3 = W_1(-1.685k^4)$, $A_4 = W_1(4k^2 - 2k^4)$,
$A_5 = 4W_1k^4$, $A_6 = W_1 1k^4$,
$A_7 = W_2(2 - 2k^2 - 0.563k^4)$, $A_8 = W_2(2k^2 - 0.25k^4)$,
$A_9 = 0.563W_2k^4$, $A_{10} = 0.25W_2k^4$,
$A_{11} = W_2(2k^2 - 0.5k^4)$, $A_{12} = W_2k^4$,
$A_{13} = -0.25W_2k^4$, $A_{14} = H(1 - R)(a - 1)$,

where
$W_1 = (y^2 - H(1 - R))(-3Y_1k^2\delta^3)(k^2 + 1)^{-1}$ and
$W_2 = (y^2 - H(1 - R))(-3Y_1k^2\delta^2)(k^2 + 1)^{-1}$,

$A_{15} = H + 1 - a$, $A_{16} = (4y\delta)^{-1}(2A_1 - A_4 - A_6)$,
$A_{17} = (2y)^{-1}\delta A_{12}$, $A_{18} = (2y)^{-1}(A_{11} - A_{13})$,
$A_{19} = (16y\delta)^{-1}(4A_2 - A_5)$, $A_{20} = (4\delta)^{-1}A_{12}$,
$A_{21} = (6y\delta)^{-1}A_3$, $A_{22} = (4y\delta)^{-1}(A_{13} - 2A_7 - A_{11})$,
$A_{23} = -(6y)^{-1}(A_4 + A_6)$, $A_{24} = -(2y)^{-1}\delta A_{13}$,
$A_{25} = -(16y\delta)^{-1}(4A_8 + A_{12})$, $A_{26} = -(4y)A_5$,
$A_{27} = -(6y\delta)^{-1}A_9$, $A_{28} = (A_{15} - 2y^2)^{-1}$,
$A_{29} = -2y^{-1}A_{14}A_{28}$, $A_{30} = -2A_{28}$,
$A_{31} = A_{15}A_{28}$, $A_{32} = -y^{-1}A_{15}A_{28}$,
$A_{33} = yA_{15}A_{28}$, $A_{34} = (2y)^{-1}A_{14}A_{28}$,
$A_{35} = 3yA_{28}$, $A_{36} = (2y)^{-1}A_{15}A_{28}$,
$A_{37} = (8yH(a - 1)(1 - R)^{-1}A_{28}$, $A_{38} = (y^2A_{15} + 2A_{14})A_{28}$,
$A_{39} = (2y)^{-1}(6A_{14} + y^2A_{15})A_{28}$, $A_{40} = 2A_{15}A_{28}$,
$A_{41} = (4y\delta)^{-1}(2A_1 - A_4 - A_6 + 2\delta[A_{11} - A_{13})$, $A_{42} = -\delta A_6$,
$A_{43} = 0.5y\delta A_6$, $A_{44} = 0.5(A_{13} - A_{11})$,
$A_{45} = (16y\delta)^{-1}(4A_2 - A_5 + 4\delta A_{12})$, $A_{46} = -0.25A_{12}$,
$A_{47} = -(2y)^{-1}(A_4 + A_6 + \delta A_{13})$, $A_{48} = 0.25(A_4 + A_6 + 2\delta A_{13})$,
$A_{49} = -0.25y\delta A_{13}$, $A_{50} = 0.25A_5$,
$A_{51} = 3(A_{38})^2A_{35}$, $A_{52} = (A_{38})^2A_{40}$,
$A_{53} = 6A_{38}(A_{30}A_{39} + A_{32}A_{40})$, $A_{54} = 6A_{38}A_{30}A_{35}$,
$A_{55} = 6A_{38}(A_{30}A_{40} + A_{32}A_{35})$, $A_{56} = -3(A_{35})^2A_{38}$,
$A_{57} = -3A_{38}((A_{40}^2 + 2A_{39}A_{35})$, $A_{58} = 2A_{35}A_{40}A_{38}$,

$A_{59} = A_{29} + 2A_{31}A_{38} - 2A_{39}A_{40} + 3(A_{38})^2A_{32} + A_{57},$
$A_{60} = A_{30} - (A_{35})^2,$
$A_{61} = A_{31} + 2A_{32}A_{38} - (A_{40})^2 - 2A_{35}A_{39} + 3A_{30}(A_{38})^2 + A_{58},$
$A_{62} = A_{32} + 2A_{32}A_{38} - 2(A_{35})A_{40} + A_{56},$
$A_{63} = 2(A_{38}A_{36} + A_{32}A_{40} + A_{32}A_{39}) - A_{40} - 6A_{39}A_{35}A_{40} + A_{53},$
$A_{64} = A_{36} + 2(A_{31}A_{35} + A_{32}A_{40}) - 3A_{35}(A_{35}A_{39}A_{40})^2 + A_{55},$
$A_{65} = 2(A_{30}A_{40} + A_{32}A_{35}) + A_{54} - 3A_{40}(A_{35})^2,$
$A_{66} = A_{39} + 2A_{38}A_{40} + A_{51},$ $A_{67} = A_{40} + 2A_{38}A_{35},$

$A_{68} = 2A_{38}A_{39} + A_{52},$
$A_{69} = A_{25} + (16\delta)^{-1}(2A_{61}A_{26} + A_{62}A_{50}),$
$A_{70} = A_{50} + 2A_{38}A_{26},$ $A_{71} = 0.5A_{38}(A_{50} + 2A_{26}A_{38}),$
$A_{72} = (4\delta)\omega^{-1}(A_{35}A_{50} + 2A_{26}A_{67}),$ $A_{73} = (4\delta)^{-1}(2A_{26}A_{66} + A_{50}A_{67}),$

$A_{74} = A_{22} + (2\delta^2)^{-1}(A_{49}A_{60} + A_{47}A_{61} + 2A_{48}A_{62}),$
$A_{75} = A_{48} + 2A_{38}A_{47},$ $A_{76} = A_{49} + 0.5A_{38}(A_{48} + 2A_{38}A_{37}),$
$A_{77} = (2\delta)^{-1}(A_{35}A_{48} + 2A_{67}A_{47}),$ $A_{78} = \delta^1(A_{35}A_{49} + 2A_{67}A_{48} + A_{66}A_{47}),$

$A_{79} = (4\delta)^{-1}(A_{35}A_{46} + A_{67}A_{47}),$
$A_{80} = A_{41} + (2\delta^2)^{-1}(A_{43}A_{60} + 3A_{17}A_{61} + A_{62}A_{42}),$
$A_{81} = A_{42} + 3A_{38}A_{17},$ $A_{82} = A_{43} + A_{38}A_{42} + 3A_{17}(A_{38})^2,$

$A_{83} = A_{44} + A_{38}A_{41} + (2\delta^2)^{-1}(A_{42}A_{61} + A_{43}A_{62}),$
$A_{84} = 0.333(A_{38}A_{43} + A_{42}(A_{38})^2) + A_{17}(A_{38})^3,$
$A_{85} = (4\delta^3)^{-1}(3A_{17}A_{63} + A_{42}A_{64} + A_{43}A_{65}) + (2\delta)^{-1}(A_{35}A_{44} + A_{41}A_{67}),$
$A_{86} = \delta^1(3A_{17}A_{66} + A_{35}A_{43} + A_{42}A_{67}),$
$A_{87} = (2\delta)^{-1}(A_{42}A_{66} + A_{43}A_{67} + 3A_{17}A_{68}),$ $A_{88} = (2\delta)^{-1}(A_{35}A_{42} + 3A_{17}A_{67}).$

Appendix 2

$r_1 = -(24H(a-1)(1-R))^{-1}A_{10}$
$r_2 = (8\delta^3)^{-1}A_{37}(2y\delta(A_{74} - A_{85}) - A_{78} - A_{83})$
$r_3 = (8\delta^3)^{-1}A_{37}(2y\delta(A_{47} - A_{88}) - A_{81})$
$r_4 = (4\delta^3)^{-1}A_{37}(y\delta(A_{75} - A_{86}) - A_{82})$
$r_5 = (8\delta^3)^{-1}A_{37}(2y\delta(A_{76} - A_{87}) - 3A_{84})$
$r_6 = (64\delta^3)^{-1}A_{37}(4y\delta(A_{69} - A_{79}) - A_{73} - A_{46} - A_{38}A_{45})$
$r_7 = (16\delta^2 y)^{-1}A_{37}A_{26}$ $r_8 = (16\delta^2)^{-1}yA_{37}A_{70}$
$r_9 = (16\delta^2)^{-1}yA_{37}A_{71}$
$r_{10} = (216\delta^3)^{-1}A_{37}(y(6\delta A_{27} - A_{67}) - A_{38}A_{21})$
$r_{11} = (8\delta^3)^{-1}A_{37}(2y\delta A_{77} + A_{75} - A_{86}) + (4\delta^2)^{-1}A_{37}A_{80}$
$r_{12} = (4\delta^3)^{-1}A_{37}(y\delta(A_{78} + A_{83}) + A_{76} - A_{87})$
$r_{13} = (4\delta^2)^{-1}yA_{37}A_{88}$
$r_{14} = (4\delta^2)^{-1}yA_{37}A_{81}$
$r_{15} = (4\delta^2)^{-1}yA_{37}A_{82}$

$r_{16} = (4\delta^2)^{-1} y A_{37} A_{84}$
$r_{17} = (64\delta^3)^{-1} A_{37}(A_{70} + 4y\delta(A_{72} + A_{45}))$
$r_{18} = (32\delta^3)^{-1} A_{37}(A_{71} + 2y\delta(A_{73} + A_{46} + A_{38}A_{45}))$
$r_{19} = (36\delta^2)^{-1} y A_{37} A_{21}$
$r_{20} = (36\delta^2)^{-1} y A_{37} A_{38} A_{21}$

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