

# A Hopf $C^*$ -algebra associated with an action of $SU_q(1,1)$ on a two-parameter quantum deformation of the unit disc<sup>†</sup>

Yury CHAPOVSKY

*Institute of Mathematics of the National Ukrainian Academy of Sciences,  
Tereshchenkivska Street 3, 252004 Kiev, Ukraine*

*Submitted by A. NIKITIN*

*Received December 8, 1994*

## Abstract

We define a Hopf  $C^*$ -algebra associated with an action of the quantum group  $SU_q(1,1)$  on a two-parameter quantum deformation of the unit disc, which has a left comodule structure over this Hopf  $C^*$ -algebra.

**Mathematics Subject Classification (1991).** 81C05.

## Introduction

It was shown in [10] that the quantum group  $SU_q(1,1)$  does not exist on the  $C^*$ -algebra level. By using the idea of [6], we construct a certain algebra  $A_q$ , the  $C^*$ -completion of which,  $St(A_q)$ , is endowed with a Hopf  $C^*$ -algebra structure. We show that there exists an embedding of the algebra  $A_q$  into a localization of the algebra  $SU_q(1,1)$ . The Hopf  $C^*$ -algebra structure of  $St(A_q)$  is induced by the Hopf  $*$ -algebra structure of  $SU_q(1,1)$ . The two parameter quantization of the algebra of functions on the unit disc, considered in [5], and its particular case – the one parameter deformation [8] — have a comodule structure over  $St(A_q)$ , which can be viewed as a noncommutative deformation of the action of the group  $SU(1,1)/\mathbf{Z}_2$  on the algebra of continuous functions on the unit disc.

In Section 1 we construct the algebra  $A_q$ , give all its irreducible representations, and construct the algebra  $St(A_q)$ . We prove that there is an imbedding of  $A_q$  into a localization of  $SU_q(1,1)$ . In Section 2 we introduce a Hopf  $C^*$ -algebra structure on  $St(A_q)$ . In Section 3 we prove that the two parameter deformation of the algebra of functions on the unit disc has a left comodule structure over  $St(A_q)$ .

*Copyright © 1995 by Mathematical Ukraina Publisher.  
All rights of reproduction in any form reserved.*

---

<sup>†</sup>This work was supported in part by the Fund for Fundamental Studies at the Science and Technology State Committee of Ukraine, project 1/238 "Operator".

## 1 $C^*$ -algebra associated with $SU_q(1,1)$

Let  $A_q$  denote a unital algebra with involution over the field  $\mathbf{C}$  with generators  $x, x^*, y, y^*$  satisfying the relations

$$(1 - y^*y) - q^2(1 - yy^*) = (1 - q^2)(1 - yy^*)(1 - y^*y), \quad (1)$$

$$xx^* + (1 - q^2)yy^* = 1, \quad (2)$$

$$x^*x(1 + (q^{-2} - 1)y^*y) = 1, \quad (3)$$

$$xyx^* = q^2y, \quad (4)$$

where  $q \in \mathbf{R}$ ,  $0 < q \leq 1$ .

By a  $*$ -representation of the algebra  $A_q$  on a Hilbert space  $\mathcal{H}$ , we will understand a mapping  $\pi : A_q \rightarrow L(\mathcal{H})$  to the algebra of linear operators on  $\mathcal{H}$  such that  $\pi(x^*) = \pi(x)^*$  and  $\pi(y^*) = \pi(y)^*$ . For a representation  $\pi$ , we denote  $X = \pi(x)$  and  $Y = \pi(y)$ .

**Lemma 1 (cf. [4])** *Let  $\pi : A_q \rightarrow L(\mathcal{H})$  be a  $*$ -representation of the algebra  $A_q$ . Then  $\text{Ker}(Y^*Y) = \text{Ker}(YY^*)$ ,  $\text{Ker}(1 - Y^*Y) = \text{Ker}(1 - YY^*)$ . If we set  $\mathcal{H}' = \text{Ker}(Y^*Y)$ ,  $\mathcal{H}'' = \text{Ker}(1 - Y^*Y)$ , then the Hilbert spaces  $\mathcal{H}'$  and  $\mathcal{H}''$  are invariant with respect to  $A_q$ .*

**Proof.** The equalities  $\text{Ker}(Y^*Y) = \text{Ker}(YY^*)$  and  $\text{Ker}(1 - Y^*Y) = \text{Ker}(1 - YY^*)$  immediately follow from (1). We will prove the second part. A vector  $v \in \mathcal{H}'$  if and only if  $Y^*Y(v) = Y^*Y(v) = 0$ . Hence  $YY^*(Y(v)) = Y(Y^*Y(v)) = 0$ , and so  $Y(v) \in \mathcal{H}'$ . Similarly we show that  $Y^*(v) \in \mathcal{H}'$ . It follows from (2) and (3) that  $v \in \mathcal{H}'$  if and only if  $X^*X(v) = XX^*(v) = v$ . So  $XX^*(X(v)) = X(X^*X(v)) = X(v)$  showing that  $\mathcal{H}'$  is invariant with respect to  $X$ . It can be similarly shown that  $\mathcal{H}'$  is invariant with respect to  $X^*$ .

By the same type of argument we can also show that  $\mathcal{H}''$  is invariant with respect to  $A_q$ .  $\square$

**Corollary 1** *If  $\pi : A_q \rightarrow L(\mathcal{H})$  is a  $*$ -representation of  $A_q$ , then*

$$\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}'' \oplus \mathcal{H}^\perp,$$

where  $\mathcal{H}^\perp$  is the orthogonal complement to the Hilbert space  $\mathcal{H}' \oplus \mathcal{H}''$ .

**Theorem 1** *Let  $0 < q < 1$  and  $\rho$  be an irreducible  $*$ -representation of the algebra  $A_q$ . Then  $\rho$  is unitarily equivalent to:*

a) a representation of the series of one-dimensional representations on  $\mathcal{H} = \mathbf{C}$ :

$$\begin{aligned} \rho_0^X(x) &= e^{i\chi}, & \rho_0^X(x^*) &= e^{-i\chi}, \\ \rho_0^X(y) &= \rho_1^X(y^*) = 0, & \chi &\in \mathbf{R}/2\pi\mathbf{Z}; \end{aligned} \quad (5)$$

b) a representation of the series of one-dimensional representations on  $\mathcal{H} = \mathbf{C}$ :

$$\begin{aligned} \rho_1^{\varphi\psi}(x) &= qe^{i\varphi}, & \rho_1^{\varphi\psi}(x^*) &= qe^{-i\varphi}, \\ \rho_1^{\varphi\psi}(y) &= e^{i\psi}, & \rho_1^{\varphi\psi}(y^*) &= e^{-i\psi}, \quad \varphi, \psi \in \mathbf{R}/2\pi\mathbf{Z}; \end{aligned} \quad (6)$$

c) a representation of the series of infinite dimensional representations on  $\mathcal{H} = l_2(\mathbf{Z})$  given on an orthonormal basis in  $\mathcal{H}$ ,  $\{e_n\}_{n \in \mathbf{Z}}$ , by

$$\begin{aligned}\rho_\infty^{\varkappa\tau}(x) \cdot e_n &= e^{i\varkappa q} \left( \frac{1 + \tau^2 q^{2(n+1)}}{1 + \tau^2 q^{2(n+2)}} \right)^{1/2} e_{n+2}, \\ \rho_\infty^{\varkappa\tau}(x^*) \cdot e_n &= e^{-i\varkappa q} \left( \frac{1 + \tau^2 q^{2(n-1)}}{1 + \tau^2 q^{2n}} \right)^{1/2} e_{n-2}, \\ \rho_\infty^{\varkappa\tau}(y) \cdot e_n &= \frac{1}{(1 + \tau^2 q^{2(n+1)})^{1/2}} e_{n+1}, \\ \rho_\infty^{\varkappa\tau}(y^*) \cdot e_n &= \frac{1}{(1 + \tau^2 q^{2n})^{1/2}} e_{n-1},\end{aligned}\tag{7}$$

where  $\varkappa \in \mathbf{R}/2\pi\mathbf{Z}$ ,  $\tau \in \left[ \frac{q}{1+q}, \frac{1}{1+q} \right)$ .

**Proof.** Because  $\rho$  is an irreducible representation, we use Corollary 1 and consider  $\rho$  restricted to  $\mathcal{H}'$ ,  $\mathcal{H}''$ , and  $\mathcal{H}^\perp$ . We use  $X_0, Y_0, X_1, Y_1$ , and  $X, Y$  to denote the restriction of the operators  $\rho(x), \rho(y)$  to the corresponding subspaces.

Because  $Y_0^* Y_0 = Y_0 Y_0^* = 0$ , we see that  $Y_0 = 0$ . We also have from (2) and (3) that  $X_0 X_0^* = X_0^* X_0 = I$ . This means that  $X_0$  is a unitary operator and it commutes with all the elements of  $A_q$ . Because  $\rho$  is irreducible, this implies that  $\rho$  is unitarily equivalent to (5).

Now we consider the operators  $X_1$  and  $Y_1$ . By definition of  $\mathcal{H}''$ ,  $Y_1^* Y_1 = Y_1 Y_1^* = I$ , whence  $Y_1$  is unitary. It again follows from (2) and (3) that  $X_1^* X_1 = X_1 X_1^* = q^2 I$ , so the operator  $q^{-1} X_1$  is unitary. By using (4), we see that  $X_1 Y_1 = Y_1 X_1$  and this means that the algebra  $A_q$  is commutative, whence  $\mathcal{H}''$  is one-dimensional, and this gives the representation (6).

Consider the restrictions  $X$  and  $Y$  to  $\mathcal{H}^\perp$ . Because  $\text{Ker } X = \text{Ker } Y = \emptyset$ , in the polar decomposition  $X = U|X|$  and  $Y = V|Y|$ , the operators  $U$  and  $V$  are unitary. It readily follows from (3) that

$$|X|^2 = \left( 1 + (q^{-2} - 1)|Y|^2 \right)^{-1}.\tag{8}$$

From (1) we can see that

$$V|Y|^2 V^* = \frac{q^2 |Y|^2}{1 - (1 - q^2)|Y|^2},\tag{9}$$

which, in its turn, implies that

$$V^* |Y|^2 V = \frac{|Y|^2}{q^2 + (1 - q^2)|Y|^2}.\tag{10}$$

By finding the expressions for  $XX^*$  and  $X^*X$  in terms of  $YY^*$  and  $Y^*Y$  correspondingly from (2) and (3), substituting them into (1), and then using (8), we find that

$$U|Y|^2 U^* = \frac{q^4 |Y|^2}{1 - (1 - q^2)(1 + q^2)|Y|^2}.\tag{11}$$

Now, if we use (9) to calculate  $V^2 |Y|^2 V^{*2}$  and compare with (11), we see that  $V^2 |Y|^2 V^{*2} = U|Y|^2 U^*$ , whence,  $U^* V^2 |Y| = |Y| U^* V^2$ . Rewrite (4) as  $U|X|V|Y|X|U^* = q^2 V|Y|$  and

use (8) and (10) to get

$$V \frac{|Y|}{(q^2 + (q^{-2} - q^2)|Y|^2)^{1/2}} = U^* V U \frac{|Y|}{(q^2 + (q^{-2} - q^2)|Y|^2)^{1/2}}.$$

This relation shows that  $UV = VU$ . Recalling that  $U^*V^2$  commutes with  $|Y|$ , we see that the operator  $U^*V^2$  commutes with any operator from  $\rho(A_q)$ . Because  $\rho$  was assumed to be irreducible,  $U^*V^2 = e^{-i\kappa}I$  since the operator is unitary, and consequently

$$U = e^{i\kappa}V^2 \tag{12}$$

for some  $\kappa \in \mathbf{R}/2\pi\mathbf{Z}$ .

Let  $A_{q_y}$  denote the unital subalgebra of  $A_q$  generated by  $y$  and  $y^*$  and  $\rho_y$  denote the restriction of  $\rho$  to  $A_{q_y}$ . Then the representation  $\rho$  is irreducible if and only if  $\rho_y$  is irreducible. It is clear that if  $\rho_y$  is irreducible, then  $\rho$  is irreducible. Suppose that  $\rho_y$  is reducible, i.e. there is  $Z \in L(\mathcal{H}^\perp)$ , which is not a scalar operator, and such that  $Z$  commutes with  $Y$  and  $Y^*$ . This means that  $Z$  commutes with  $Y^*Y$  and, hence, with  $|Y|$ , and also with  $V$ . But then, as it follows from (8) and (12),  $Z$  will commute with  $|X|$  and  $U$ , which means that  $Z$  commutes with  $X$  and  $X^*$ , hence  $\rho$  is reducible.

These considerations allow to consider only irreducible representations of (1) and then use (8) and (12). Using the results of [9] we finally get the representations given by (7).  $\square$

**Corollary 2** *If  $\rho$  is an irreducible  $*$ -representation of the algebra  $A_q$ , then:  $q \leq \|\rho(x)\| \leq 1$ ,  $\|\rho(y)\| \leq 1$ , and  $\|\rho(xy^*)\| \leq q$ .*

This corollary implies, in particular, that the algebra  $A_q$  is  $*$ -bounded (see, for example, [3]) and so we can define a norm on  $A_q$  by setting, for  $a \in A_q$ ,  $\|a\| = \sup_\rho(\|\rho(a)\|)$ , where  $\rho$  runs over the set of all irreducible representations of  $A_q$ . By completing  $A_q$  with respect to this norm, we get a  $C^*$ -algebra which will be denoted by  $St(A_q)$ .

Actually, the  $C^*$ -norm in  $St(A_q)$  is given by the infinite dimensional representations (7). To prove that we need the following lemma.

**Lemma 2** *Let  $X$  and  $Y$  be weighted shift operators on  $l_2(\mathbf{Z})$ , given on an orthonormal basis  $\{e_n\}_{n \in \mathbf{Z}}$  by*

$$\begin{aligned} X(e_n) &= c_1 \lambda_n e_{n+2}, & \lambda_{-\infty} &> \dots > \lambda_{-1} > \lambda_0 > \lambda_1 > \dots > \lambda_\infty, \\ Y(e_n) &= c_2 \mu_n e_{n+1}, & \mu_{-\infty} &< \dots < \mu_{-1} < \mu_0 < \mu_1 < \dots < \mu_\infty, \end{aligned}$$

with  $\lambda_n, \mu_n \in \mathbf{R}$ ,  $c_1, c_2 \in \mathbf{C}$ , and finite  $\lambda_{\pm\infty} = \lim_{n \rightarrow \pm\infty} \lambda_n$ ,  $\mu_{\pm\infty} = \lim_{n \rightarrow \pm\infty} \mu_n$ . Then there exists a sequence of vectors  $\{e'_m\}_{m \in \mathbf{Z}_-}$ ,  $\|e'_m\| = 1$ , and a sequence of vectors  $\{e''_m\}_{m \in \mathbf{Z}_+}$ ,  $\|e''_m\| = 1$ , such that

$$\begin{aligned} \lim_{m \rightarrow -\infty} \|X(e'_m) - c_1 \lambda_{-\infty} e'_m\| &= 0, \\ \lim_{m \rightarrow -\infty} \|Y(e'_m) - c_2 \mu_{-\infty} e'_m\| &= 0 \end{aligned} \tag{13}$$

and

$$\begin{aligned} \lim_{m \rightarrow \infty} \|X(e''_m) - c_1 \lambda_\infty e''_m\| &= 0, \\ \lim_{m \rightarrow \infty} \|Y(e''_m) - c_2 \mu_\infty e''_m\| &= 0. \end{aligned} \tag{14}$$

**Proof.** Since constructions of the sequences  $\{e'_m\}$  and  $\{e''_m\}$  are similar, we will only construct the sequence  $\{e''_m\}_{m \in \mathbf{Z}}$ . For each  $m > 0$ ,  $m \in \mathbf{Z}$ , set

$$e''_m = \frac{1}{\sqrt{m}} \sum_{i=m+1}^{2m} e_i.$$

Then  $\|e_m\| = 1$  for all  $m$  and

$$\begin{aligned} \|X(e''_m) - \lambda_\infty e''_m\| &= \frac{1}{\sqrt{m}} \left\| \sum_{i=m+1}^{2m} (X(e_i) - c_1 \lambda_\infty e_i) \right\| = \\ &= \frac{|c_1|}{\sqrt{m}} \left\| \lambda_{2m-1} e_{2m+1} + \lambda_{2m} e_{2m+2} - \lambda_\infty (e_{m+1} + e_{m+2}) + \sum_{i=m+3}^{2m} (\lambda_\infty - \lambda_i) e_i \right\| \leq \\ &= \frac{|c_1 \lambda_{-\infty}|}{\sqrt{m}} (\sqrt{2} + 2) + (\lambda_{2m} - \lambda_\infty), \end{aligned}$$

whence the first limit of (13) follows. A similar estimate holds for the second limit in (13).  $\square$

**Theorem 2** *The  $C^*$  norm on the  $C^*$ -algebra  $St(A_q)$  is given, for any  $a \in St(A_q)$ , by*

$$\|a\| = \sup_{\substack{\tau \in [\frac{q}{1+q}, \frac{1}{1+q}] \\ \varkappa \in S^1}} \|\rho_\infty^{\varkappa\tau}(a)\|.$$

**Proof.** We will prove that, for any element  $a$  in  $A_q$  and any fixed  $\tau$ , there exists  $\varkappa_0(\chi) \in \mathbf{R}/2\pi\mathbf{Z}$  such that  $\|\rho_\infty^{\varkappa_0\tau}(a)\| \geq |\rho_0^\chi(a)|$  and, similarly, there exists such  $\varkappa_1(\varphi) \in \mathbf{R}/2\pi\mathbf{Z}$  that  $\|\rho_\infty^{\varkappa_1\tau}(a)\| \geq |\rho_1^\varphi(a)|$ . Indeed, set

$$\begin{aligned} \lambda_n &= q \left( \frac{1 + \tau^2 q^{2(n+1)}}{1 + \tau^2 q^{2(n+2)}} \right)^{1/2}, & c_1 &= e^{i\chi}, \\ \mu_n &= \frac{1}{(1 + \tau^2 q^{2(n+1)})^{1/2}}, & c_2 &= 1, \end{aligned}$$

and  $\lambda_{-\infty} = 1$ ,  $\mu_{-\infty} = 0$ . It follows from Lemma 2 that there is a sequence  $\{e'_m\}$  such that  $\lim_{m \rightarrow \infty} \|\rho_\infty^{\chi\tau}(a)(e'_m) - \rho_0^\chi(a)e'_m\| \rightarrow 0$ , whence we get the first part of the proof. The second part is proved similarly.  $\square$

**Lemma 3** *Let  $\pi : St(A_q) \rightarrow L(\mathcal{H})$  be an irreducible representation on a Hilbert space  $\mathcal{H}$ . Let  $\mathcal{J}$  denote the commutator ideal in  $St(A_q)$ . Then either  $\pi(\mathcal{J}) = 0$  or  $\pi(\mathcal{J}) = \mathcal{K}_{\mathcal{H}}$ , where  $\mathcal{K}_{\mathcal{H}}$  is the  $C^*$ -algebra of all compact operators on  $\mathcal{H}$ .*

**Proof.** By using Theorem 1, we see that  $\pi(\mathcal{J}) = 0$  if and only if  $\pi$  is a one dimensional representation. Consider the case when  $\dim(\mathcal{H}) = \infty$ . Denote  $Y = \pi(y)$ . It readily follows from Theorem 1 that  $[Y, Y^*] \in \pi(\mathcal{J}) \cap \mathcal{K}_{\mathcal{H}}$ . We will show that  $\pi(\mathcal{J})$  is irreducible. Suppose not, i.e. there exists a nonscalar operator  $Z \in L(\mathcal{H})$ , which commutes with all the elements of  $\pi(\mathcal{J})$ . Let  $Y = V|Y|$  be the polar decomposition of  $Y$ . Because  $Z$  commutes

with  $[Y, Y^*]$ , it follows from (9) that it commutes with  $|Y\rangle$ . Relation (9) also implies that if  $Z$  commutes with  $[Y, Y^*Y]$  and  $|Y\rangle$ , then it commutes with  $V$ . This means that  $Z$  commutes with any element of  $\pi(St(A_q))$  which is a contradiction since  $\pi$  is irreducible.

Because  $\pi(\mathcal{J})$  is irreducible and contains a compact operator,  $\pi(\mathcal{J}) = \mathcal{K}_{\mathcal{H}}$ .  $\square$

**Corollary 3** *The  $C^*$ -algebra  $St(A_q)$  is a GCR- $C^*$ -algebra and, hence, a  $C^*$ -algebra of type I.*

Recall that the quantum group  $SU_q(1, 1)$  is a Hopf  $*$ -algebra generated by elements of the matrix  $T = (t_{ij})_{i,j=1,2}$  which satisfy the following relations:

$$\begin{aligned} t_{11}t_{12} &= qt_{12}t_{11}, & t_{11}t_{21} &= qt_{21}t_{11}, & t_{11}t_{22} - t_{22}t_{11} &= (q - q^{-1})t_{12}t_{21}, \\ t_{12}t_{21} &= t_{21}t_{12}, & t_{12}t_{22} &= qt_{22}t_{12}, & t_{21}t_{22} &= qt_{22}t_{21}, \\ t_{11}t_{22} - qt_{12}t_{21} &= 1. \end{aligned} \quad (15)$$

The Hopf  $*$ -algebra structure is defined by the comultiplication  $\Delta$  and counit  $\epsilon$  given by

$$\Delta(t_{ij}) = \sum_{k=1}^2 t_{ik} \otimes t_{kj}, \quad \epsilon(t_{ij}) = \delta_{ij}, \quad (16)$$

and the antipode  $S$  and involution  $*$ :

$$S \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} t_{22} & -q^{-1}t_{12} \\ -qt_{21} & t_{11} \end{pmatrix}, \quad \begin{pmatrix} t_{11}^* & t_{12}^* \\ t_{21}^* & t_{22}^* \end{pmatrix} = \begin{pmatrix} t_{22} & qt_{21} \\ q^{-1}t_{12} & t_{11} \end{pmatrix}. \quad (17)$$

It follows from (15) that

$$t_{11}t_{22} - q^2t_{22}t_{11} = 1 - q^2, \quad (18)$$

$$t_{22}t_{11} - q^{-1}t_{12}t_{21} = 1. \quad (19)$$

Now consider a unital algebra,  $SU_q(1, 1)^l$ , generated by  $t_{ij}$ ,  $i, j = 1, 2$ , and  $t_{11}^{-1}$ ,  $t_{22}^{-1}$  subject to relations (15) and

$$t_{11}t_{11}^{-1} = t_{11}^{-1}t_{11} = 1, \quad t_{22}t_{22}^{-1} = t_{22}^{-1}t_{22} = 1.$$

Let

$$x' = t_{22}^{-1}t_{11}, \quad y' = q^{-1}t_{22}^{-1}t_{12}, \quad x'^* = t_{22}t_{11}^{-1}, \quad y'^* = t_{21}t_{11}^{-1}, \quad (20)$$

and let  $SU_q(1, 1)_0^l$  denote the subalgebra of  $SU_q(1, 1)^l$  generated by  $x'$ ,  $x'^*$ ,  $y'$ ,  $y'^*$ .

**Theorem 3** *The algebra  $A_q$  is  $*$ -isomorphic to the algebra  $SU_q(1, 1)_0^l$ , namely, the mapping  $\iota : A_q \rightarrow SU_q(1, 1)_0^l$ , defined on the generators by  $\iota(x) = x'$ ,  $\iota(y) = y'$  can be extended to a well defined  $*$ -isomorphism.*

**Proof.** First of all we show that  $\iota$  is well defined, i.e.  $x', x'^*, y', y'^*$  satisfy (1) – (4). Indeed, by using (18), we get

$$x'x'^* = t_{22}^{-1}t_{11}t_{22}t_{11}^{-1} = t_{22}^{-1}((1 - q^2) + q^2t_{22}t_{11})t_{11}^{-1} = (1 - q^2)t_{22}^{-1}t_{11}^{-1} + q^2,$$

and so

$$\frac{1 - q^2}{x'x'^* - q^2} = t_{11}t_{22}. \quad (21)$$

Similarly,

$$(x'^*x')^{-1} = t_{11}^{-1}t_{22}t_{11}t_{22}^{-1} = t_{11}^{-1}(q^{-2}t_{11}t_{22} - (q^{-2} - 1))t_{22}^{-1} = q^{-2} - (q^{-2} - 1)t_{11}^{-1}t_{22}^{-1}.$$

Whence

$$\frac{(1 - q^2)x'^*x'}{x'^*x' - q^2} = t_{22}t_{11}. \quad (22)$$

By using (19), we see that

$$y'y'^* = q^{-1}t_{22}^{-1}t_{12}t_{21}t_{11}^{-1} = t_{22}^{-1}(t_{22}t_{11} - 1)t_{11}^{-1} = 1 - t_{22}^{-1}t_{11}^{-1},$$

which yields

$$(1 - y'y'^*)^{-1} = t_{11}t_{22}. \quad (23)$$

In the same way, by applying the last relation of (15), we obtain

$$y'^*y' = qt_{21}t_{11}^{-1}t_{22}^{-1}t_{12} = qt_{11}^{-1}t_{21}t_{12}t_{22}^{-1} = t_{11}^{-1}(t_{11}t_{22} - 1)t_{22}^{-1} = 1 - t_{11}^{-1}t_{22}^{-1},$$

hence

$$(1 - y'^*y')^{-1} = t_{22}t_{11}. \quad (24)$$

At this point, relation (1) readily follows if (23) and (24) are used in (18), relations (2) and (3) follow directly from (21), (23) and (22), (24), respectively. Relation (4) follows from

$$x'y'x'^* = t_{22}^{-1}t_{11}t_{12}t_{22}^{-1}t_{22}t_{11}^{-1} = t_{22}^{-1}t_{11}t_{12}t_{11}^{-1} = qt_{22}^{-1}t_{12} = q^2y'.$$

Because  $\iota$  maps elements of  $A_q$  into generators of  $SU_q(1, 1)_0^l$ , it is certainly surjective. We will show that it is also injective. It was shown in [10] that an irreducible  $*$ -representation of  $SU_q(1, 1)$  is unitarily equivalent to either a one dimensional representation

$$\pi_0(t_{11}) = e^{i\varphi}, \quad \pi_0(t_{12}) = 0, \quad \varphi \in \mathbf{R}/2\pi\mathbf{Z}, \quad (25)$$

or to an infinite dimensional representation  $\pi_\infty$  on  $l_2(\mathbf{Z})$ , given on an orthonormal basis  $\{f_n\}_{n \in \mathbf{Z}}$  by

$$\pi_\infty^b(t_{11}) \cdot f_n = \sqrt{1 + |b|^2 q^{2(n-1)}} f_{n-1}, \quad \pi_\infty^b(t_{12}) \cdot f_n = bq^n f_n, \quad b \in \mathbf{C}. \quad (26)$$

If we extend these representations to  $SU_q(1, 1)_0^l$ , then  $\iota$  defines representations of  $A_q$ , which are unitarily equivalent to (5) and (7).

Assuming now that  $\iota(a) = 0$  for some  $a \in A_q$  would mean, in particular, that  $\rho_\infty^{z\tau}(a) = 0$  and, since, as it follows from Theorem 2,  $\|a\| = 0$ ,  $a = 0$ .  $\square$

## 2 Hopf $C^*$ -algebra associated with $SU_q(1, 1)$

Let  $B, D$  be  $C^*$ -algebras of the type  $I$ . It follows from [7] that a  $C^*$ -cross-norm on the algebraic tensor product  $B \otimes_{\text{alg}} D$  is unique. In the rest of the paper, we will denote

by  $B \otimes D$  the  $C^*$  algebra obtained from  $B \otimes_{\text{alg}} D$  by completing it with respect to this norm. The following definition and Definition 1 in Section 3 are particular cases of the corresponding definitions given in [2].

**Definition 1** Let  $B$  be a  $C^*$ -algebra of the type I with identity. A Hopf  $C^*$ -algebra is a triple  $(B, \Delta, \epsilon)$ , where  $\Delta : B \rightarrow B \otimes B$  and  $\epsilon : B \rightarrow \mathbf{C}$  are homomorphisms called coproduct and counit, respectively, such that the following diagrams are commutative:

$$\begin{array}{ccc}
 B & \xrightarrow{\Delta} & B \otimes B \\
 \Delta \downarrow & & \downarrow id \otimes \Delta \\
 B \otimes B & \xrightarrow{\Delta \otimes id} & B \otimes B \otimes B
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & & B & & \\
 & \Delta \swarrow & | id & \searrow \Delta & \\
 B \otimes B & \xrightarrow{\epsilon \otimes id} & B & \xleftarrow{id \otimes \epsilon} & B \otimes B
 \end{array}
 \tag{27}$$

It will be convenient to introduce an element  $z \in A_q$  by

$$z = xy^*. \tag{28}$$

To define a comultiplication on the  $C^*$ -algebra  $St(A_q)$ , we will need the following lemmas.

**Lemma 4** Let  $z$  be given by (28). The elements  $(z \otimes y + 1 \otimes 1)$  and  $(z^* \otimes y^* + 1 \otimes 1)$  are invertible in the  $C^*$  algebra  $St(A_q) \otimes St(A_q)$ .

**Proof.** By Corollary 2,  $\|z\| \leq q$  and  $\|y\| \leq 1$ . This implies that  $\|z \otimes y\| \leq q < 1$  and thus the lemma is proved.  $\square$

**Lemma 5** The following relations hold:

$$zz^* = q^2yy^*, \quad z^*z = q^2y^*y, \tag{29}$$

$$\begin{aligned}
 (x \otimes y + y \otimes 1)(q^{-2}z \otimes y + 1 \otimes 1) &= (z \otimes y + 1 \otimes 1)(x^{*-1} \otimes y + y \otimes 1), \\
 (q^{-2}z^* \otimes y^* + 1 \otimes 1)(x^* \otimes y^* + y^* \otimes 1) &= (x^{-1} \otimes y^* + y^* \otimes 1)(z^* \otimes y^* + 1 \otimes 1),
 \end{aligned}
 \tag{30}$$

$$\begin{aligned}
 (x \otimes y + y \otimes 1)(y^* \otimes z^* + x^* \otimes x^*) &= (z \otimes y + 1 \otimes 1)(1 \otimes z^* + z^* \otimes x^*), \\
 (y \otimes z + x \otimes x)(x^* \otimes y^* + y^* \otimes 1) &= (1 \otimes z + z \otimes x)(z^* \otimes y^* + 1 \otimes 1),
 \end{aligned}
 \tag{31}$$

$$\begin{aligned}
 (z \otimes y + 1 \otimes 1)(z^* \otimes y^* + 1 \otimes 1) - (x \otimes y + y \otimes 1)(x^* \otimes y^* + y^* \otimes 1) &= \\
 (1 - yy^*) \otimes (1 - yy^*), &
 \end{aligned}
 \tag{32}$$

$$\begin{aligned}
 (q^{-2}z^* \otimes y^* + 1 \otimes 1)(q^{-2}z \otimes y + 1 \otimes 1) - (x^{-1} \otimes y^* + y^* \otimes 1)(x^{*-1} \otimes y + y \otimes 1) &= \\
 (1 - y^*y) \otimes (1 - y^*y), &
 \end{aligned}
 \tag{33}$$

$$\begin{aligned}
 (x \otimes x + y \otimes z)(x^* \otimes x^* + y^* \otimes z^*) + (1 - q^2)(x \otimes y + y \otimes 1)(x^* \otimes y^* + y^* \otimes 1) &= \\
 (z \otimes y + 1 \otimes 1)(z^* \otimes y^* + 1 \otimes 1), &
 \end{aligned}
 \tag{34}$$

$$\begin{aligned}
 (x \otimes x + y \otimes z)(x^* \otimes x^* + y^* \otimes z^*) + (q^{-2} - 1)(z \otimes x + 1 \otimes z)(z^* \otimes x^* + 1 \otimes z^*) &= \\
 (z \otimes y + 1 \otimes 1)(z^* \otimes y^* + 1 \otimes 1), &
 \end{aligned}
 \tag{35}$$

$$(x \otimes x + y \otimes z)(1 \otimes z^* + z^* \otimes x^*) = q^2(x \otimes y + y \otimes 1)(z^* \otimes y^* + 1 \otimes 1). \tag{36}$$



**Proof.** Let us prove the first relation of (29). It follows from (3) that

$$y^*y = (q^{-2} - 1)^{-1}(x^{-1}x^{*-1} - 1).$$

Whence, by using (28) and (2), we obtain

$$zz^* = xy^*yx^* = (q^{-2} - 1)^{-1}x(x^{-1}x^{*-1} - 1)x^* = q^2(1 - q^2)^{-1}(1 - xx^*) = q^2yy^*.$$

To prove the second relation of (29), rewrite (1) as  $y^*y = q^{-2}(1 + (q^{-2} - 1)yy^*)^{-1}yy^*$ . Then use (3) to get

$$z^*z = yx^*xy^* = y(1 + (q^{-2} - 1)y^*y)^{-1}y^* = (1 + (q^{-2} - 1)yy^*)^{-1}yy^* = q^2y^*y.$$

Now we prove the first relation of (30). Consider the difference of the left- and right-hand sides of the first relation of (30) multiplied on the right by the invertible element  $(x^* \otimes 1)$  and use (28), (4), and (2).

$$\begin{aligned} & (x \otimes y + y \otimes 1)(q^{-2}zx^* \otimes y + x^* \otimes 1) - (z \otimes y + 1 \otimes 1)(1 \otimes y + yx^* \otimes 1) = \\ & (x \otimes y + y \otimes 1)(y^* \otimes y + x^* \otimes 1) - (z \otimes y + 1 \otimes 1)(1 \otimes y + z^* \otimes 1) = \\ & xx^* \otimes y + yy^* \otimes y - zz^* \otimes y - 1 \otimes y = \\ & (1 - (1 - q^2)yy^*) \otimes y + yy^* \otimes y - q^2yy^* \otimes y - 1 \otimes y = 0. \end{aligned}$$

The second relation of (30) is obtained by taking the conjugate of the first.

The first relation of (31) follows from that of (30) by multiplying both sides from the right by the invertible element  $(x^* \otimes x^*)$  and using (28). The second relation is obtained by conjugation.

Consider (32). By using (28), (29), and (2), we get

$$\begin{aligned} & (z \otimes y + 1 \otimes 1)(z^* \otimes y^* + 1 \otimes 1) - (x \otimes y + y \otimes 1)(x^* \otimes y^* + y^* \otimes 1) = \\ & zz^* \otimes yy^* + 1 \otimes 1 - xx^* \otimes yy^* - yy^* \otimes 1 = \\ & q^2yy^* \otimes yy^* + 1 \otimes 1 - (1 - (1 - q^2)yy^*) \otimes yy^* - yy^* \otimes 1 = \\ & (1 - yy^*) \otimes (1 - yy^*). \end{aligned}$$

To prove (33), first note that (4) and (28) imply that  $y^*x^{*-1} = q^{-2}z$  and  $x^{-1}y = q^{-2}z^*$ . Use (29) and (3) to find

$$\begin{aligned} & (q^{-2}z^* \otimes y^* + 1 \otimes 1)(q^{-2}z \otimes y + 1 \otimes 1) - (x^{-1} \otimes y^* + y^* \otimes 1) \times \\ & (x^{*-1} \otimes y + y \otimes 1) = q^{-4}z^*z \otimes y^*y + 1 \otimes 1 - x^{-1}x^{*-1} \otimes y^*y - y^*y \otimes 1 = \\ & q^{-2}y^*y \otimes y^* + 1 \otimes 1 - (1 + (q^{-2} - 1)y^*y) \otimes y^*y - y^*y \otimes 1 = \\ & (1 - y^*y) \otimes (1 - y^*y). \end{aligned}$$

To see that (34) holds, we use relations (2), (4), and (28) to simplify the left-hand side. We get

$$\begin{aligned} & (z \otimes x + y \otimes z)(x^* \otimes x^* + y^* \otimes z^*) + (1 - q^2)(x \otimes y + y \otimes 1)(x^* \otimes y^* + y^* \otimes 1) = \\ & xx^* \otimes xx^* + xy^* \otimes xz^* + yx^* \otimes zx^* + yy^* \otimes zz^* + \\ & (1 - q^2)(xx^* \otimes yy^* + xy^* \otimes y + yx^* \otimes y^* + yy^* \otimes 1) = \\ & (1 - (1 - q^2)yy^*) \otimes (1 - (1 - q^2)yy^*) + q^2z \otimes y + q^2z^* \otimes y^* + q^2yy^* \otimes yy^* + \\ & (1 - q^2)((1 - (1 - q^2)yy^*) \otimes yy^* + z \otimes y + z^* \otimes y^* + yy^* \otimes 1) = \\ & 1 \otimes 1 + z \otimes y + z^* \otimes y^* + q^2yy^* \otimes yy^* = \\ & (z \otimes y + 1 \otimes 1)(z^* \otimes y^* + 1 \otimes 1). \end{aligned}$$

Now we will prove relation (35). We have

$$\begin{aligned}
& (x \otimes x + y \otimes z)(x^* \otimes x^* + y^* \otimes z^*) + (q^{-2} - 1)(z \otimes x + 1 \otimes z)(z^* \otimes x^* + 1 \otimes z^*) = \\
& \quad (xx^* + (1 - q^2)yy^*) \otimes xx^* + q^{-2}xy^* \otimes xz^* + q^{-2}yx^* \otimes zx^* + (yy^* + (q^{-2} - 1)1) \otimes \\
& \quad zz^* = 1 \otimes (1 - (1 - q^2)yy^*) + z \otimes y + z^* \otimes y^* + q^2yy^* \otimes yy^* + (1 - q^2)1 \otimes yy^* = \\
& \quad 1 \otimes 1 + z \otimes y + z^* \otimes y^* + zz^* \otimes yy^* = \\
& \quad (z \otimes y + 1 \otimes 1)(z^* \otimes y^* + 1 \otimes 1).
\end{aligned}$$

To see that (36) holds, we use (4) and (2) to get for the right-hand side

$$\begin{aligned}
& (x \otimes x + y \otimes z)(1 \otimes z^* + z^* \otimes x^*) = x \otimes xz^* + xz^* \otimes xx^* + \\
& \quad y \otimes zz^* + yz^* \otimes zx^* = q^2(x \otimes y + y \otimes xx^* + y \otimes yy^* + \\
& \quad yz^* \otimes y^*) = q^2(x \otimes y + y \otimes 1 + q^2y \otimes yy^* + yz^* \otimes y^*).
\end{aligned}$$

On the other hand,  $q^2(x \otimes y + y \otimes 1)(z^* \otimes y^* + 1 \otimes 1) = q^2(q^2y \otimes yy^* + x \otimes y + yz^* \otimes y^* + y \otimes 1)$ , whence (36) follows.  $\square$

**Lemma 6** *The elements*

$$(y^* \otimes z^* + x^* \otimes x^*), \quad (y \otimes z + x \otimes x), \quad (q^{-2}z \otimes y + 1 \otimes 1), \quad (q^{-2}z^* \otimes y^* + 1 \otimes 1) \quad (37)$$

are invertible in the  $C^*$ -algebra  $St(A_q) \otimes St(A_q)$ .

**Proof.** Let  $\pi_i : A_q \rightarrow L(\mathcal{H}_i)$ ,  $i = 1, 2$ , be  $*$ -representations of the algebra  $A_q$  on the Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . Denote by  $X_i, Y_i, Z_i$  the operators  $\pi_i(x), \pi_i(y)$ , and  $\pi_i(z)$ , respectively. We first show that the operator  $Q = (Y_1^* \otimes Z_2^* + X_1^* \otimes X_2^*)$  is invertible. Suppose the converse. Then there is a sequence of vectors  $v_n \in \mathcal{H}_1 \otimes \mathcal{H}_2$ ,  $n \in \mathbf{Z}_+$  such that  $\|v_n\| = 1$  for all  $n$  and  $\lim_{n \rightarrow \infty} \|Q(v_n)\| = 0$ . Because, by Lemma 4, the operator  $(X_1 \otimes Y_2 + I \otimes I)$  is invertible, it follows from (31) that

$$\lim_{n \rightarrow \infty} \|(I \otimes Z_2^* + Z_1^* \otimes X_2^*)(v_n)\| = 0.$$

But then, from relation (35), it follows that

$$\lim_{n \rightarrow \infty} \|(Z_1 \otimes Y_2 + I \otimes I)(Z_1^* \otimes Y_2^* + I \otimes I)(v_n)\| = 0.$$

But this is a contradiction to Lemma 4.

It follows from (4) that  $(q^{-2}z \otimes y + 1 \otimes 1)(x^* \otimes x^*) = (y^* \otimes z^* + x^* \otimes x^*)$  and, since the element  $(x^* \otimes x^*)$  is invertible, the third element in (37) is invertible.

The other two elements are adjoint to the first and the third.  $\square$

Now we are in position to define a Hopf  $*$ -algebra structure on  $St(A_q)$ . Let the comultiplication  $\Delta$ , counit  $\epsilon$  be defined on the generators of the algebra  $A_q$  as follows

$$\begin{aligned}
\Delta(x) &= (z \otimes y + 1 \otimes 1)^{-1}(x \otimes x + y \otimes z), \\
\Delta(x^*) &= (x^* \otimes x^* + y^* \otimes z^*)(z^* \otimes y^* + 1 \otimes 1)^{-1} \\
\Delta(y) &= (z \otimes y + 1 \otimes 1)^{-1}(x \otimes y + y \otimes 1),
\end{aligned} \tag{38}$$

$$\begin{aligned}
\Delta(y^*) &= (x^* \otimes y^* + y^* \otimes 1)(z^* \otimes y^* + 1 \otimes 1)^{-1}, \\
\epsilon(x) = 1, \quad \epsilon(x^*) = 1, \quad \epsilon(y) = 0, \quad \epsilon(y^*) = 0.
\end{aligned} \tag{39}$$

It follows from Lemmas 6 and 5 that

$$\begin{aligned}\Delta(y) &= (x^{*-1} \otimes y + y \otimes 1)(q^{-2}z \otimes y + 1 \otimes 1)^{-1} = \\ &= (z^* \otimes x^* + 1 \otimes z^*)(x^* \otimes x^* + y^* \otimes z^*)^{-1},\end{aligned}\tag{40}$$

$$\begin{aligned}\Delta(y^*) &= (q^{-2}z^* \otimes y^* + 1 \otimes 1)^{-1}(x^{-1} \otimes y^* + y^* \otimes 1) = \\ &= (x \otimes x + y \otimes z)^{-1}(z \otimes x + 1 \otimes z).\end{aligned}\tag{41}$$

The following lemma will be used to prove that  $\Delta$  can be extended to a well defined homomorphism.

**Lemma 7 (cf. [4])** *Let  $\pi : A_q \rightarrow L(\mathcal{H})$  be a  $*$ -representation of the algebra  $A_q$ . Denote by  $Y$  the operator  $\pi(y)$ . Then  $\text{Ran}(1 - Y^*Y) = \text{Ran}(1 - YY^*)$ . Let  $\text{Ran}(I - YY^*) = \mathcal{H}^\circ$ . Then  $\mathcal{H}^\circ$  is invariant with respect to  $A_q$ .*

**Proof.** Rewriting (1) as

$$(1 - YY^*) = (1 - Y^*Y)(1 - (1 - q^2)Y^*Y)^{-1},$$

we see that  $\text{Ran}(1 - Y^*Y) = \text{Ran}(1 - YY^*)$ . Because

$$Y(1 - Y^*Y) = (1 - YY^*)Y \quad \text{and} \quad Y^*(1 - YY^*) = (1 - Y^*Y)Y^*,$$

we see that  $\mathcal{H}^\circ$  is invariant with respect to  $Y$  and  $Y^*$ .

To see that  $\mathcal{H}^\circ$  is invariant with respect to  $X$ , we use relations (3) and (2) to get

$$\begin{aligned}X(1 - YY^*) &= (q^{-2} - 1)^{-1}X(q^{-2} - (X^*X)^{-1}) = \\ &= (q^{-2} - 1)^{-1}(q^{-2}XX^* - 1)X^{*-1} = \\ &= (q^{-2} - 1)^{-1}(q^{-2}(1 - (1 - q^2)YY^*) - 1)X^{*-1} = \\ &= (1 - YY^*)X^{*-1}.\end{aligned}$$

Similarly, applying (2) and (3), we get

$$\begin{aligned}X^*(1 - YY^*) &= X^*(1 - (1 - q^2)^{-1}(1 - XX^*)) = \\ &= (1 - q^2)^{-1}X^*(XX^* - q^2) = \\ &= (1 - q^2)^{-1}(1 - q^2(X^*X)^{-1})X^*XX^* = \\ &= (1 - q^2)^{-1}(1 - q^2(1 + (q^{-2} - 1)Y^*Y))X^*XX^* = \\ &= (1 - Y^*Y)X^*XX^*,\end{aligned}$$

which proves the invariance of  $\mathcal{H}^\circ$  with respect to  $X^*$ .

Because  $\mathcal{H}^\circ$  is invariant with respect to all the generators of the algebra  $A_q$ , it is invariant with respect to  $A_q$ .  $\square$

**Lemma 8** *Let  $\Delta, \epsilon$  be defined on the generators of the algebra  $A_q$  as in (38), (39), correspondingly. Then these mappings can be extended to well defined  $*$ -homomorphisms on the  $C^*$ -algebra  $St(A_q)$ .*

**Proof.** It is immediate that  $\epsilon$  is well defined.

We will prove that  $\Delta$  is well defined on  $A_q$ , i.e. the elements  $\Delta(x)$ ,  $\Delta(y)$  satisfy relations (1) – (4). To show that, let  $\pi_i : A_q \rightarrow L(\mathcal{H}_i)$ ,  $i = 1, 2$ , be  $*$ -representations of the algebra  $A_q$  and denote by  $X_i, Y_i, Z_i$  the operators  $\pi_i(x), \pi_i(y), \pi_i(z)$  on the corresponding spaces. By Lemma 1, each  $\mathcal{H}_i$  admits, in particular, the decomposition  $\mathcal{H}_i = \mathcal{H}_i'' \oplus \mathcal{H}_i''^\perp$ ,  $i = 1, 2$ , where  $\mathcal{H}_i''$  is the kernel of the operator  $(1 - Y_i Y_i^*)$ ,  $\mathcal{H}_i''^\perp$  is the orthogonal complement to  $\mathcal{H}_i''$ , and each subspace is invariant with respect to  $St(A_q)$ . This decomposition induces the following decomposition of the representation space  $\mathcal{H}_1 \otimes \mathcal{H}_2$ :

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{K} + \mathcal{L},$$

where  $\mathcal{K} = \mathcal{H}_1'' \otimes \mathcal{H}_2 + \mathcal{H}_1 \otimes \mathcal{H}_2''$  and  $\mathcal{L} = \mathcal{H}_1''^\perp \otimes \mathcal{H}_2''^\perp$ . We will prove that relation (1) holds on each of the spaces  $\mathcal{K}$  and  $\mathcal{L}$ .

Consider restrictions of the operators to  $\mathcal{K}$ . It follows from (32) that

$$(Z_1 \otimes Y_2 + 1 \otimes 1)(Z_1^* \otimes Y_2^* + 1 \otimes 1) = (X_1 \otimes Y_2 + Y_1 \otimes 1)(X_1^* \otimes Y_2^* + Y_1^* \otimes 1). \quad (42)$$

Hence, by using (38), we see that

$$\begin{aligned} (\pi_1 \otimes \pi_2)\Delta(yy^*) &= (Z_1 \otimes Y_2 + I \otimes I)^{-1}(X_1 \otimes Y_2 + Y_1 \otimes I) \times \\ &\quad (X_1^* \otimes Y_2^* + Y_1^* \otimes I)(Z_1^* \otimes Y_2^* + I \otimes I)^{-1} = I \otimes I \end{aligned}$$

and

$$\begin{aligned} (\pi_1 \otimes \pi_2)\Delta(y^*y) &= (X_1^* \otimes Y_2^* + Y_1^* \otimes I)(Z_1^* \otimes Y_2^* + I \otimes I)^{-1} \times \\ &\quad (Z_1 \otimes Y_2 + I \otimes I)^{-1}(X_1 \otimes Y_2 + Y_1 \otimes I) = I \otimes I \end{aligned}$$

on  $\mathcal{K}$ . On the other hand, relations (34) and (42) imply that

$$(X_1 \otimes X_2 + Y_1 \otimes Z_2)(X_1 \otimes X_2^* + Y_1^* \otimes Z_2^*) = q^2(Z_1 \otimes Y_2 + I \otimes I)(Z_1^* \otimes Y_2^* + I \otimes I), \quad (43)$$

which means that

$$\begin{aligned} (\pi_1 \otimes \pi_2)\Delta(x^*x) &= (X_1^* \otimes X_2 + Y_1^* \otimes Z_2^*)(Z_1^* \otimes Y_2^* + I \otimes I)^{-1} \times \\ &\quad (Z_1 \otimes Y_2 + I \otimes I)^{-1}(X_1 \otimes X_2 + Y_1 \otimes Z_2) = q^2 I \otimes I. \end{aligned}$$

Whence we see that relations (1) and (3) hold on  $\mathcal{K}$ .

Now we will prove that these relations hold on a dense subspace of  $\mathcal{L}$ , namely on  $\mathcal{L}^\circ = \text{Ran}(I - Y_1 Y_1^*) \otimes \text{Ran}(I - Y_2 Y_2^*)$ .

Consider relation (1) and rewrite it as

$$(1 - yy^*)^{-1} - q^2(1 - y^*y)^{-1} = 1 - q^2. \quad (44)$$

We will show that the operators  $(\pi_1 \otimes \pi_2)\Delta(yy^*)$  and  $(\pi_1 \otimes \pi_2)\Delta(y^*y)$  satisfy (44) on  $\mathcal{L}^\circ$ . Indeed, by using relation (32)

$$\begin{aligned} (\pi_1 \otimes \pi_2)\Delta(1 - yy^*) &= I \otimes I - (Z_1 \otimes Y_2 + I \otimes I)^{-1}(X_1 \otimes Y_2 + Y_1 \otimes I) \times \\ &\quad (X_1^* \otimes Y_2^* + Y_1^* \otimes I)(Z_1^* \otimes Y_2^* + I \otimes I)^{-1} = \\ &\quad (Z_1 \otimes Y_2 + I \otimes I)^{-1}((Z_1 \otimes Y_2 + I \otimes I)(Z_1^* \otimes Y_2^* + I \otimes I) - \\ &\quad (X_1 \otimes Y_2 + Y_1 \otimes I)(X_1^* \otimes Y_2^* + Y_1^* \otimes I))(Z_1^* \otimes Y_2^* + I \otimes I)^{-1} = \\ &\quad (Z_1 \otimes Y_2 + I \otimes I)^{-1}((I - Y_1 Y_1^*) \otimes (I - Y_2 Y_2^*))(Z_1^* \otimes Y_2^* + I \otimes I)^{-1}. \end{aligned}$$

Whence,

$$((\pi_2 \otimes \pi_2)\Delta(1 - yy^*))^{-1} = (Z_1^* \otimes Y_2^* + I \otimes I)((I - Y_1 Y_1^*)^{-1} \otimes (I - Y_2 Y_2^*)^{-1})(Z_1 \otimes Y_2 + I \otimes I). \quad (45)$$

Now we use (40) and (41), and then (33) to get

$$\begin{aligned} (\pi_1 \otimes \pi_2)\Delta(1 - y^*y) &= I \otimes I - (X_1^* \otimes Y_2^* + Y_1^* \otimes I)(Z_1^* \otimes Y_2^* + I \otimes I)^{-1} \times \\ &\quad (Z_1 \otimes Y_2 + I \otimes I)^{-1}(X_1 \otimes Y_2 + Y_1 \otimes I) = \\ &\quad I \otimes I - (q^{-2}Z_1^* \otimes Y_2^* + I \otimes I)^{-1}(X_1^{-1} \otimes Y_2^* + Y_1^* \otimes I) \times \\ &\quad (X_1^{*-1} \otimes Y_2 + Y_1 \otimes I)(q^{-2}Z_1 \otimes Y_2 + I \otimes I)^{-1} = \\ &\quad (q^{-2}Z_1^* \otimes Y_2^* + I \otimes I)^{-1}((q^{-2}Z_1^* \otimes Y_2^* + I \otimes I)(q^{-2}Z_1 \otimes Y_2 + I \otimes I) - \\ &\quad (X_1^{-1} \otimes Y_2^* + Y_1^* \otimes I)(X_1^{*-1} \otimes Y_2 + Y_1 \otimes I))(q^{-2}Z_1 \otimes Y_2 + I \otimes I)^{-1} = \\ &\quad (q^{-2}Z_1^* \otimes Y_2^* + I \otimes I)^{-1}((I - Y_1^* Y_1) \otimes (I - Y_2^* Y_2))(q^{-2}Z_1 \otimes Y_2 + I \otimes I)^{-1}. \end{aligned}$$

And so,

$$\begin{aligned} ((\pi_2 \otimes \pi_2)\Delta(1 - y^*y))^{-1} &= \\ (q^{-2}Z_1 \otimes Y_2 + I \otimes I)((I - Y_1^* Y_1)^{-1} \otimes (I - Y_2^* Y_2)^{-1})(q^{-2}Z_1^* \otimes Y_2^* + I \otimes I). \quad (46) \end{aligned}$$

Note, that (29) implies that  $zy^*y = yy^*z$  and  $z^*yy^* = y^*yz^*$ . We use (29) and (44) to get from (45) and (46) that

$$\begin{aligned} ((\pi_1 \otimes \pi_2)\Delta(1 - yy^*))^{-1} - q^2((\pi_1 \otimes \pi_2)\Delta(1 - y^*y))^{-1} &= \\ (Z_1^* \otimes Y_2^* + I \otimes I)((I - Y_1 Y_1^*)^{-1} \otimes (I - Y_2 Y_2^*)^{-1})(Z_1 \otimes Y_2 + I \otimes I) - \\ q^2(q^{-2}Z_1 \otimes Y_2 + I \otimes I)((I - Y_1^* Y_1)^{-1} \otimes (I - Y_2^* Y_2)^{-1})(q^{-2}Z_1^* \otimes Y_2^* + I \otimes I) &= \\ ((I - Y_1^* Y_1)^{-1} \otimes (I - Y_2^* Y_2)^{-1})(Z_1^* Z_1 \otimes Y_2^* Y_2) + (I - Y_1 Y_1^*)^{-1} \otimes (I - Y_2 Y_2^*)^{-1} - \\ q^{-2}((I - Y_1 Y_1^*)^{-1} \otimes (I - Y_2 Y_2^*)^{-1})(Z_1 Z_1^* \otimes Y_2 Y_2^*) - q^2(I - Y_1 Y_1^*)^{-1} \otimes \\ (I - Y_2^* Y_2)^{-1} &= q^2(-I + (1 - Y_1^* Y_1)^{-1}) \otimes (-I + (I - Y_2^* Y_2)^{-1}) - \\ q^2(I - Y_1^* Y_1)^{-1} \otimes (I - (I - Y_2^* Y_2)^{-1}) + (I - Y_1 Y_1^*) \otimes (I - Y_2 Y_2^*)^{-1} - \\ (-I + (I - Y_1 Y_1^*)^{-1}) \otimes (-I + (I + Y_2 Y_2^*)^{-1}) &= \\ ((I - Y_1 Y_1^*)^{-1} - q^2(I - Y_1^* Y_1)^{-1}) \otimes I + I \otimes ((I - Y_2 Y_2^*)^{-1} - q^2(I - Y_2^* Y_2)^{-1}) - \\ (1 - q^2)I \otimes I &= (1 - q^2)I \otimes I. \end{aligned}$$

To prove that  $\Delta(x^*x)$  and  $\Delta(y^*y)$  satisfy relation (2), we use (40), (41), and (35) to get

$$\begin{aligned} \Delta(1 + (q^{-2} - 1)y^*y) &= 1 \otimes 1 + \\ (q^{-2} - 1)(x^* \otimes y^* + y^* \otimes 1)(z^* \otimes y^* + 1 \otimes 1)^{-1}(z \otimes y + 1 \otimes 1)(x \otimes y + y \otimes 1) &= \\ 1 \otimes 1 + (q^{-2} - 1)(x \otimes x + y \otimes z)^{-1}(z \otimes x + 1 \otimes z) \times \\ (z^* \otimes x^* + 1 \otimes z^*)(x^* \otimes x^* + y^* \otimes z^*)^{-1} &= \\ (x \otimes x + y \otimes z)^{-1}[(x \otimes x + y \otimes z)(x^* \otimes x^* + y^* \otimes z^*) + \\ (q^{-2} - 1)(z \otimes x + 1 \otimes z)(z^* \otimes x^* \otimes 1 \otimes z^*)](x^* \otimes x^* + y^* \otimes z^*)^{-1} &= \\ (x \otimes x + y \otimes z)^{-1}(z \otimes y + 1 \otimes 1)(z^* \otimes y^* + 1 \otimes 1)(x^* \otimes x^* + y^* \otimes z^*)^{-1} &= \\ (\Delta(x^*x))^{-1}. \end{aligned}$$

It is easy to see that  $\Delta(xx^*)$  and  $\Delta(yy^*)$  satisfy (3). Indeed, because

$$\begin{aligned}\Delta(xx^*) &= (z \otimes y + 1 \otimes 1)^{-1}(x \otimes x + y \otimes z)(x^* \otimes x^* + y^* \otimes z^*)(z^* \otimes y^* + 1 \otimes 1)^{-1}, \\ \Delta(yy^*) &= (z \otimes y + 1 \otimes 1)^{-1}(x \otimes y + y \otimes 1)(x^* \otimes y^* + y^* \otimes 1)(z^* \otimes y^* + 1 \otimes 1)^{-1},\end{aligned}$$

it immediately follows from (34) that (3) holds.

Now consider relation (4). It readily follows from the first relation of (31) that

$$(z \otimes y + 1 \otimes 1)^{-1}(x \otimes y + y \otimes 1)(x^* \otimes x^* + y^* \otimes z^*) = (1 \otimes z^* + z^* \otimes x^*).$$

By using this relation together with (36), we get

$$\begin{aligned}\Delta(xy x^*) &= (z \otimes y + 1 \otimes 1)^{-1}(x \otimes x + y \otimes z)(z \otimes y + 1 \otimes 1)^{-1} \times \\ &\quad (x \otimes y + y \otimes 1)(x^* \otimes x^* + y^* \otimes z^*)(z^* \otimes y^* + 1 \otimes 1)^{-1} = \\ &\quad (z \otimes y + 1 \otimes 1)^{-1}(x \otimes x + y \otimes z)(1 \otimes z^* + z^* \otimes x^*)(z^* \otimes y^* + 1 \otimes 1)^{-1} = \\ &\quad q^2(z \otimes y + 1 \otimes 1)^{-1}(x \otimes y + y \otimes 1) = q^2\Delta(y).\end{aligned}$$

Because  $A_q$  is dense in  $St(A_q)$ , we extend  $\Delta$  to  $St(A_q)$  by continuity.  $\square$

We can now prove the main theorem.

**Theorem 4** *The mappings  $\Delta$  and  $\epsilon$  define a Hopf  $C^*$ -algebra structure on  $St(A_q)$ .*

**Proof.** By using Lemma 8, it is sufficient that the diagrams in (27) be commutative on the generators of the algebra  $A_q$ .

We first prove commutativity of the first diagram.

By using (38), we have that

$$\begin{aligned}(id \otimes \Delta) \circ \Delta(y) &= (id \otimes \Delta)((z \otimes y + 1 \otimes 1)^{-1}(x \otimes y + y \otimes 1)) = \\ &\quad \left\{ z \otimes ((z \otimes y + 1 \otimes 1)^{-1}(x \otimes y + y \otimes 1)) + 1 \otimes 1 \otimes 1 \right\}^{-1} \times \\ &\quad \left\{ x \otimes ((z \otimes y + 1 \otimes 1)^{-1}(x \otimes y + y \otimes 1)) + y \otimes 1 \otimes 1 \right\} = \\ &\quad \{ z \otimes x \otimes y + z \otimes y \otimes 1 + 1 \otimes z \otimes y + 1 \otimes 1 \otimes 1 \}^{-1} \times \\ &\quad \{ x \otimes x \otimes y + x \otimes y \otimes 1 + y \otimes z \otimes y + y \otimes 1 \otimes 1 \}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\Delta \otimes id) \circ \Delta(y) &= (\Delta \otimes id)((z \otimes y + 1 \otimes 1)^{-1}(x \otimes y + y \otimes 1)) = \\ &\quad \{ ((z \otimes y + 1 \otimes 1)^{-1}(z \otimes x + 1 \otimes z)) \otimes y + 1 \otimes 1 \otimes 1 \}^{-1} \times \\ &\quad \{ ((z \otimes y + 1 \otimes 1)^{-1}(x \otimes x + y \otimes z)) \otimes y + ((z \otimes y + 1 \otimes 1)^{-1}(x \otimes y + y \otimes 1)) \otimes 1 \} = \\ &\quad \{ z \otimes x \otimes y + z \otimes y \otimes 1 + 1 \otimes z \otimes y + 1 \otimes 1 \otimes 1 \}^{-1} \times \\ &\quad \{ x \otimes x \otimes y + x \otimes y \otimes 1 + y \otimes z \otimes y + y \otimes 1 \otimes 1 \}.\end{aligned}$$

Similar calculations can be performed to show that

$$\begin{aligned}(id \otimes \Delta) \circ \Delta(x) &= (\Delta \otimes id) \circ \Delta(x) = \\ &\quad (z \otimes x \otimes y + z \otimes y \otimes 1 + 1 \otimes z \otimes y + 1 \otimes 1 \otimes 1)^{-1} \times \\ &\quad (x \otimes x \otimes x + x \otimes y \otimes z + y \otimes z \otimes x + y \otimes 1 \otimes z).\end{aligned}$$

Because  $\Delta \circ * = (* \otimes *) \circ \Delta$ , we see that the first diagram of (27) is also commutative for the elements  $x^*$  and  $y^*$ .

Commutativity of the second diagram follows immediately from the definition of  $\epsilon$ , (39), and (38).  $\square$

REMARK. The definition of a Hopf algebra as well as a Hopf  $*$ -algebra  $H$  ([1, 10]) also includes an antipode  $S$  which is an antihomomorphism  $S : H \rightarrow H$  satisfying the following relation

$$\mu \circ (S \otimes id) \circ \Delta = \mu \circ (id \otimes S) \circ \Delta = \varsigma \circ \epsilon, \quad (47)$$

where  $\mu : H \otimes H \rightarrow H$  is the multiplication,  $\mu(h_1 \otimes h_2) = h_1 h_2$  for  $h_1, h_2 \in H$ ,  $\varsigma : \mathbf{C} \rightarrow H$  is the imbedding  $\varsigma(c) = c1$ ,  $c \in \mathbf{C}$ .

In our case, it is also possible to define an antipode on the algebra  $A_q$  by defining it on the generators to be

$$S(x) = x^*, \quad S(x^*) = x, \quad S(y) = -q^{-2}z^*, \quad S(y^*) = -z.$$

It is easy to prove that  $S$  can be extended on the whole  $A_q$  to a well-defined antihomomorphism. If  $\pi_i : St(A_q) \rightarrow L(\mathcal{H}_i)$  are  $*$ -representations of the  $C^*$ -algebra  $St(A_q)$ , then the operators  $(\pi_1 \otimes \pi_2)(id \otimes S) \circ \Delta(a)$  and  $(\pi_1 \otimes \pi_2)(S \otimes id) \circ \Delta(a)$ , where  $a$  is one of the generators  $x, x^*, y$ , or  $y^*$ , although unbounded, are defined on  $Ran(I - YY^*) \otimes Ran(I - YY^*)$  and relation (47) holds on this subspace invariant with respect to  $A_q \otimes_{\text{alg}} A_q$ .

### 3 Comodule structure on the two-parameter deformation of the unit disc

For  $0 < q \leq 1$  and  $0 \leq \mu < 1$ ,  $(q, \mu) \neq (1, 0)$ , we denote by  $St(C_{\mu, q})$  the universal enveloping  $C^*$ -algebra of the unital  $*$ -algebra over  $\mathbf{C}$ ,  $C_{\mu, q}$ , generated by two elements  $w, w^*$  that satisfy the following relation

$$q^{-1}(1 - w^*w) - q(1 - ww^*) = \mu(1 - ww^*)(1 - w^*w), \quad (48)$$

and call it a two-parameter quantum deformation of the unit disc. This  $C^*$ -algebra was studied in [5]. In particular, it was shown that  $St(C_{\mu, q})$  is a  $C^*$ -algebra of type I,  $\|w\| \leq 1$ , and the quantized universal enveloping algebra  $U_q(sl(2))$  acts on  $St(C_{\mu, q})$ .

The purpose of this section is to show that there is a coaction of  $St(A_q)$  on the two-parameter quantum deformation of the unit disc.

**Definition 1** *A left coaction of a type I unital Hopf  $C^*$ -algebra  $(B, \Delta, \epsilon)$  on a type I unital  $C^*$ -algebra  $D$  is a  $*$ -homomorphism  $\delta : D \rightarrow B \otimes D$  such that the following diagrams are commutative:*

$$\begin{array}{ccc} D & \xrightarrow{\delta} & B \otimes D \\ \delta \downarrow & & \downarrow id \otimes \delta \\ B \otimes D & \xrightarrow{\Delta \otimes id} & B \otimes B \otimes D \end{array} \quad \begin{array}{ccc} & & D \\ & \delta \swarrow & \downarrow id \\ B \otimes D & \xrightarrow{\epsilon \otimes id} & D \end{array} \quad (49)$$

To define a coaction on  $St(C_{\mu,q})$ , we will need some auxiliary results.

**Lemma 9** *Let  $w, w^*$  be the generators of  $C_{\mu,q}$  and  $z$  be given by (28). The elements of the  $C^*$ -algebra  $St(A_q) \otimes St(C_{\mu,q})$ ,*

$$(z \otimes w + 1 \otimes 1), \quad (z^* \otimes w^* + 1 \otimes 1),$$

are invertible.

**Proof.** Since  $\|w\| \leq 1$  and  $\|z\| \leq q$ ,  $\|z \otimes w\| \leq q < 1$ .  $\square$

**Lemma 10** *The following relations hold in the  $C^*$ -algebra  $St(A_q) \otimes St(C_{\mu,q})$ :*

$$(x \otimes w + y \otimes 1)(q^{-2}z \otimes w + 1 \otimes 1) = (z \otimes w + 1 \otimes 1)(x^{*-1} \otimes w + y \otimes 1), \quad (50)$$

$$(q^{-2}z^* \otimes w^* + 1 \otimes 1)(x^* \otimes w^* + y^* \otimes 1) = (x^{-1} \otimes w^* + y^* \otimes 1)(z^* \otimes w^* + 1 \otimes 1), \quad (51)$$

$$(z \otimes w + 1 \otimes 1)(z^* \otimes w^* + 1 \otimes 1) - (x \otimes w + y \otimes 1)(x^* \otimes w^* + y^* \otimes 1) = \\ (1 - yy^*) \otimes (1 - ww^*), \quad (52)$$

$$(q^{-2}z^* \otimes w^* + 1 \otimes 1)(q^{-2}z \otimes w + 1 \otimes 1) - (x^{-1} \otimes w^* + y^* \otimes 1)(x^{*-1} \otimes w + y \otimes 1) = \\ (1 - y^*y) \otimes (1 - w^*w). \quad (53)$$

**Proof.** One can prove all these identities in the way similar to the proof of Lemma 5.  $\square$

**Lemma 11** *Let  $w, w^*$  be the generators of  $C_{\mu,q}$  and  $z$  be given by (28). The elements of the  $C^*$ -algebra  $St(A_q) \otimes St(C_{\mu,q})$ ,*

$$(q^{-2}z \otimes w + 1 \otimes 1), \quad (q^{-2}z^* \otimes w^* + 1 \otimes 1),$$

are invertible.

**Proof.** Let  $\pi_1 : St(A_q) \rightarrow L(\mathcal{H}_1)$  and  $\pi_2 : St(C_{\mu,q}) \rightarrow L(\mathcal{H}_2)$  be  $*$ -representations, and  $X = \pi_1(x)$ ,  $Y = \pi_1(y)$ ,  $Z = \pi_1(z)$ , and  $W = \pi_2(w)$ . Suppose that the element  $(q^{-2}Z \otimes W + I \otimes I)$  is not invertible, i.e. there is a sequence of vectors  $v_n \in \mathcal{H}_1 \otimes \mathcal{H}_2$ ,  $\|v_n\| = 1$ , such that

$$\lim_{n \rightarrow \infty} \|(q^{-2}Z \otimes W + I \otimes I)(v_n)\| = 0, \\ \lim_{n \rightarrow \infty} \|(q^{-2}Z^* \otimes W^* + I \otimes I)(v_n)\| = 0. \quad (54)$$

From the estimates

$$\|(q^{-4}ZZ^* \otimes WW^* - I \otimes I)(v_n)\| = \\ \|q^{-4}ZZ^* \otimes WW^* + q^{-2}Z \otimes W - q^{-2}z \otimes W - I \otimes I)(v_n)\| \leq \\ q^{-2}\|(Z \otimes W)\| \|(q^{-2}Z^* \otimes W^* + I \otimes I)(v_n)\| + \|(q^{-2}Z \otimes W + I \otimes I)(v_n)\|$$

and (54) it follows that

$$\lim_{n \rightarrow \infty} \|(q^{-4}ZZ^* \otimes WW^* - I \otimes I)(v_n)\| = 0. \quad (55)$$



A similar estimate leads to

$$\lim_{n \rightarrow \infty} \|(q^{-4}Z^*Z \otimes W^*W - I \otimes I)(v_n)\| = 0. \quad (56)$$

By using (50) and Lemma 9, we see that (54) implies that

$$\lim_{n \rightarrow \infty} \|(X^{*-1} \otimes W + Y \otimes I)(v_n)\| = 0. \quad (57)$$

Because  $q^{-2}Z = Y^*X^{*-1}$ , we also have

$$\begin{aligned} \|(Y^*Y \otimes I - I \otimes I)(v_n)\| &= \\ \|(Y^* \otimes I + Y^*X^{*-1} \otimes W - q^{-2}Z \otimes W - I \otimes I)(v_n)\| &\leq \\ \|Y^*Y \otimes I\| \|(X^{*-1} \otimes W + Y \otimes I)(v_n)\| &+ \|(q^{-2}Z \otimes W + I \otimes I)(v_n)\|. \end{aligned}$$

This estimate, together with (57) and (54), implies that

$$\lim_{n \rightarrow \infty} \|(Y^*Y \otimes I - I \otimes I)(v_n)\| = 0. \quad (58)$$

This equality, together with (1), implies that

$$\lim_{n \rightarrow \infty} \|(YY^* \otimes I - I \otimes I)(v_n)\| = 0. \quad (59)$$

Recalling that  $ZZ^* = q^2YY^*$  and  $Z^*Z = q^2Y^*Y$  and using (55) and (56), we see that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|(I \otimes WW^* - q^2I \otimes I)(v_n)\| &= 0, \\ \lim_{n \rightarrow \infty} \|(I \otimes W^*W - q^2I \otimes I)(v_n)\| &= 0. \end{aligned} \quad (60)$$

But now (60) implies that

$$\lim_{n \rightarrow \infty} \|q^{-1}I \otimes (I - W^*W)(v_n) - qI \otimes (I - WW^*)(v_n)\| = q^{-1}(1 - q^2)^2 \quad (61)$$

whereas

$$\lim_{n \rightarrow \infty} \mu(I \otimes (I - WW^*)(I - W^*W))(v_n)\| = \mu(1 - q^2)^2. \quad (62)$$

Relations (61) and (62) lead to a contradiction because, by (48),  $\mu(1 - q^2)^2 = q^{-1}(1 - q^2)^2$ , which is impossible since  $\mu < 1$  and  $q \leq 1$ .  $\square$

Let us now define a left coaction of the  $C^*$ -algebra  $St(A_q)$  on the  $C^*$ -algebra  $St(C_{\mu,q})$ . We set  $\delta$  on the generators of  $C_{\mu,q}$  to be

$$\begin{aligned} \delta(w) &= (z \otimes w + 1 \otimes 1)^{-1}(x \otimes w + y \otimes 1), \\ \delta(w^*) &= (x^* \otimes w^* + y^* \otimes 1)(z^* \otimes w^* + 1 \otimes 1)^{-1}. \end{aligned} \quad (63)$$

By using (50) and (51), we can rewrite these expressions as

$$\begin{aligned} \delta(w) &= (x^{*-1} \otimes w + y \otimes 1)(q^{-2}z \otimes w + 1 \otimes 1)^{-1}, \\ \delta(w^*) &= (q^{-2}z^* \otimes w^* + 1 \otimes 1)^{-1}(x^{-1} \otimes w^* + y^* \otimes 1). \end{aligned} \quad (64)$$

Because of Lemmas 9 and 11, these definitions make sense.

**Lemma 12** *Let  $\delta$  be defined on the generators of the algebra  $C_{\mu,q}$  as in (63). Then this mapping can be extended to a well defined  $*$ -homomorphism on the  $C^*$ -algebra  $St(C_{\mu,q})$ .*

**Proof.** Let  $\pi_1 : St(A_q) \rightarrow L(\mathcal{H}_1)$  and  $\pi_2 : C_{\mu,q} \rightarrow L(\mathcal{H}_2)$  be  $*$ -representations. Denote  $X = \pi_1(x)$ ,  $Y = \pi_1(y)$ ,  $Z = \pi_1(z)$ , and  $W = \pi_2(w)$ . It can be shown ([4]) that  $Ker(I - WW^*) = Ker(I - W^*W)$  and  $Ran(I - WW^*) = Ran(I - W^*W)$ , moreover these subspaces are invariant with respect to  $St(C_{\mu,q})$  and  $C_{q,\mu}$ , correspondingly. As in the proof of Lemma 8, we set  $\mathcal{H}_1'' = Ker(I - YY^*)$ ,  $\mathcal{H}_2'' = Ker(I - WW^*)$ , and let  $\mathcal{H}_i''^\perp$ ,  $i = 1, 2$ , be the orthogonal complement to the space  $\mathcal{H}_i''$ . If  $\mathcal{K} = \mathcal{H}_1'' \otimes \mathcal{H}_2 + \mathcal{H}_1 \otimes \mathcal{H}_2''$ , then, by using identities (52) and (53), we can show in the same way as in the proof of Lemma 8 that (48) holds on  $\mathcal{K}$ . We will show that (48) holds on  $\mathcal{L} = \mathcal{H}_1''^\perp \otimes \mathcal{H}_2''^\perp$ . To do that, we again consider relation (48) in the form

$$q^{-1}(1 - ww^*)^{-1} - q(1 - w^*w)^{-1} = \mu \quad (65)$$

and prove that it holds on the dense subspace of  $\mathcal{L}$ ,  $\mathcal{L}^\circ = Ran(I - Y^*Y) \otimes Ran(I - W^*W)$ , for the operators  $(\pi_1 \otimes \pi_2)\delta(ww^*)$  and  $(\pi_1 \otimes \pi_2)\delta(w^*w)$ .

By using relations (52) and (53), as in the proof of Lemma 8, we obtain

$$\begin{aligned} & ((\pi_1 \otimes \pi_2)\delta(1 - ww^*))^{-1} = \\ & (Z^* \otimes W^* + I \otimes I)((I - YY^*)^{-1} \otimes (I - WW^*)^{-1})(Z \otimes W + I \otimes I), \\ & ((\pi_1 \otimes \pi_2)\delta(1 - w^*w))^{-1} = \\ & (q^{-2}Z \otimes W + I \otimes I)((I - Y^*Y)^{-1} \otimes (I - W^*W)^{-1})(q^{-2}Z^* \otimes W^* + I \otimes I). \end{aligned}$$

But then

$$\begin{aligned} & q^{-1}((\pi_1 \otimes \pi_2)\delta(1 - ww^*))^{-1} - q((\pi_1 \otimes \pi_2)\delta(1 - w^*w))^{-1} = \\ & q^{-1}(Z^* \otimes W^* + I \otimes I)((I - YY^*)^{-1} \otimes (I - WW^*)^{-1})(Z \otimes W + I \otimes I) - \\ & q(q^{-2}Z \otimes W + I \otimes I)((I - Y^*Y)^{-1} \otimes (I - W^*W)^{-1})(q^{-2}Z^* \otimes W^* + I \otimes I) = \\ & q^{-1}\{((I - Y^*Y)^{-1} \otimes (I - W^*W)^{-1})(Z^*Z \otimes W^*W + Z^* \otimes W^*) + \\ & ((I - YY^*)^{-1} \otimes (I - WW^*)^{-1})(Z \otimes W + I \otimes I)\} - \\ & q\{((I - YY^*)^{-1} \otimes (I - WW^*)^{-1})(q^{-4}ZZ^* \otimes WW^* + q^{-2}Z \otimes W) + \\ & ((I - Y^*Y)^{-1} \otimes (I - W^*W)^{-1})(q^{-2}Z^* \otimes W^* + I \otimes I)\} = \\ & q((I - Y^*Y)^{-1} \otimes (I - W^*W)^{-1})(Y^*Y \otimes W^*W - I \otimes I) + \\ & q^{-1}((I - YY^*)^{-1} \otimes (I - WW^*)^{-1})(I \otimes I - YY^* \otimes WW^*) = \\ & -q((I - Y^*Y)^{-1} \otimes (I - W^*W)^{-1})(Y^*Y \otimes (I - W^*W) + (I - Y^*Y) \otimes I) + \\ & q^{-1}((I - YY^*)^{-1} \otimes (I - WW^*)^{-1})(YY^* \otimes (I - WW^*) + (I - YY^*) \otimes I) = \\ & -q\{(I - Y^*Y)^{-1} \otimes I - I \otimes I + I \otimes (I - W^*W)^{-1}\} + \\ & q^{-1}\{(I - YY^*)^{-1} \otimes I + I \otimes I + I \otimes (I - WW^*)^{-1}\} = \\ & (q^{-1} - q)I \otimes I + (q^{-1} - q)I \otimes I + \mu I \otimes I = \mu I \otimes I. \end{aligned}$$

This shows that the mapping  $\delta$  can be extended to a well defined  $*$ -homomorphism on  $C_{\mu,q}$ . Now we can extend it by continuity to the whole  $St(C_{\mu,q})$ .  $\square$

One can prove the following theorem in the same way as we proved Theorem 4.

**Theorem 5** *The mapping  $\delta$  defines a left comodule structure on  $St(C_{\mu,q})$  over the Hopf  $C^*$ -algebra  $St(A_q)$ .*

*I would like to express my gratitude to Yu.S. Samoilenko and L.I. Vainerman for support and many useful discussions.*

## References

- [1] Abe E., Hopf algebras, *Cambridge Tracts in Math.*, Cambridge University Press, 1980, V.74.
- [2] Baaq S. and Skandalis G., Unitaires multiplicatifs et dualité pour les produits croisés de  $C^*$ -algèbres, *Ann. Sci. Ecole Norm. Sup., 4 série*, 1993, V.26, 425–488.
- [3] Helemskiĭ A.Ya., Banach and polynormed algebras: general theory, representations, homology, Nauka, Moscow, 1989. (In Russian.)
- [4] Klimek S. and Lesniewski A., Quantum Riemann surfaces I. The unit disc, *Commun. Math. Phys.*, 1992, V.146, 103–122.
- [5] Klimek S. and Lesniewski A., A two-parameter quantum deformation of the unit disc, *Journal of Functional Analysis*, 1993, V.115, 1–23.
- [6] Masuda T., Mimachi K., Nakagami Y., Noumi M., Saburi Y. and Ueno K., Unitary representations of the quantum group  $SU_q(1, 1)$ , II, Preprint, 1990.
- [7] Takesaki M., On the cross-norm of the direct product of  $C^*$ -algebras, *Tôhoku Math. J.*, 1964, V.42, 111–122.
- [8] Vaksman L.L. and Soibelman Ya.S., Algebra of functions on quantum group  $SU(n + 1)$  and odd dimensional spheres, *Algebra and Analysis*, 1990, V.2(5), 101–120. (In Russian.)
- [9] Vaysleb E.Ye. and Samoilenko Yu.S., Representations of operator relations by unbounded operators and multi-dimensional dynamical systems, *Ukrain. Mat. Zh.*, 1990, V.42, 1011–1019.
- [10] Woronowicz S.L., Unbounded elements affiliated with  $C^*$ -algebras and noncompact quantum groups. *Commun. Math. Phys.*, 1991, V.136, 399–432.