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Non-linear Schrödinger Equations, Separation and Symmetry

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Abstract

We investigate the symmetry properties of hierarchies of non-linear Schrödinger equations, introduced in [2], which describe non-interacting systems in which tensor product wave-functions evolve by independent evolution of the factors (the separation property). We show that there are obstructions to lifting symmetries existing at a certain number of particles to higher numbers. Such obstructions vanish for particles without internal degrees of freedom and the usual space-time symmetries. For particles with internal degrees of freedom, such as spin, these obstructions are present and their circumvention requires a choice of a new term in the equation for each particle number. A Lie-algebra approach for non-linear theories is developed.

1 Introduction

Reference [2] investigates hierarchies of non-linear Schrödinger equations focusing on the property that is called *separation* which means that tensor product functions evolve by separate evolution of the factors. Such systems are considered to be non-interacting. In this paper we continue the investigation of evolutions satisfying the separation property focusing now on the questions of symmetries and on further mathematical properties of these hierarchies in general, providing thus a series of basic results necessary for the exploration of theories of this type. In particular we develop a Lie algebra approach to infinitesimal symmetries, adequately modified to account for non-linearities. Though this paper is a continuation of [2], we've made it self contained.

It was pointed out in [2] that the theories here considered exhibit two new physical aspects not present in linear theories. One is the possibility of multi-particle terms in the evolution that vanish whenever the wave-function is a tensor product. These terms introduce truly new multi-particle effects that can only be seen in correlated systems and that are not apparent in systems of fewer number of particles. The other is the existence of two new universal physical constants with the dimension of energy which describe the effect

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that changing the overall phase of the initial data has on the evolution of a wave-function. The present paper adds to such differences by showing that symmetries existing in non-linear equations up to a certain number of particles do not in general persist at a greater number of particles. In other words, symmetries can be broken by mere particle number increase. Theorem 17 in Section 7 provides the general result allowing us to calculate some of the obstructions to extending infinitesimal symmetries to a greater number of particles. In the same section, Corollary 1 shows that one-particle infinitesimal symmetries lift to any number of particles if and only if they lift to two particles, and Corollary 2 shows that if at particle-number ℓ one introduces a new multi-particle effect of the type envisaged by these theories, and if at ℓ particles this effect does not break an infinitesimal symmetry lifted from a one-particle symmetry, then the symmetry is not broken at any higher number of particles if and only if it is not broken for $\ell + 1$ particles. Theorem 21 in Section 8 then shows that, remarkably enough, for particles without internal degrees of freedom, these obstructions at the next particle number vanish for the usual infinitesimal space-time symmetries. Theories of particles with internal degrees of freedom do not escape such obstructions, which pose a problem. In particular, for the case of spin greater than zero, one must either face the possibility that rotation invariance be broken at a certain particle number, or else one must introduce some new physical principle to systematically provide the proper rotationally covariant multi-particle equation for each number of particles.

Non-linear theories, as mentioned in [2], may in the end be found untenable. Their detailed study exposes the difficulties they must face, such as the symmetry lifting obstructions described in this paper. One can through this, even if one ultimately discards such theories, achieve a deeper understanding of why the quantum world as seen in the laboratory is linear to such a high degree.

2 General conventions

When we deal with complex functions defined on subsets of the complex plane we shall not assume any analyticity properties unless explicitly stated. Although it is traditional in such cases to write $f(z, \bar{z})$ instead of $f(z)$ we shall write just $f(z)$. Our linear spaces will be considered real even though they may be ostensibly complex, such as spaces of complex-values functions. In such cases, it is the real structure of the conventional complex space which is used. By an "operator" we shall mean a map F from some domain in a real linear space $\mathbf{V}$ with values in some other real linear space and we write $F : \mathbf{V} \to \mathbf{W}$ omitting any explicit mention of the domain. No linearity is implied by the term "operator". An operator F applied to a vector ϕ shall be denoted either by $F\phi$ or $F(\phi)$, the choice being dictated by clarity and simplicity of expression. For operators defined between spaces of complex functions we shall use the term "linear" to mean *complex-linear* and so the term "real-linear" will never be abbreviated when meant. For an operator F of the type we consider, acting on a complex-valued function ϕ, one traditionally would write $F(\phi, \bar{\phi})$ to express that complex conjugation is allowed, but we again adopt the simpler form $F(\phi)$. Given an operator $F : \mathbf{V} \to \mathbf{W}$ we shall denote by $\mathbb{D}F(\phi)$ its Fréchet derivative at ϕ. This is a real-linear map from $\mathbf{V}$ to $\mathbf{W}$ that satisfies $F(\phi+\eta) = F(\phi)+\mathbb{D}F(\phi)\cdot\eta+o(\eta)$. This of course means that the spaces must have appropriate topologies for this to be well defined. If $G : \mathbf{V} \to \mathbf{V}$ is another map we shall denote by $\mathbb{D}F \cdot G$ the operator that maps ϕ to $\mathbb{D}F(\phi) \cdot G(\phi)$. The product $F, G \mapsto \mathbb{D}F \cdot G$ is real-bilinear but not associative. However, for $\mathbf{W} = \mathbf{V}$ the commutator $[F, G] = \mathbb{D}F \cdot G - \mathbb{D}G \cdot F$ is a Lie bracket and is seen to be the usual Lie bracket of F with G considering these as vector fields on $\mathbf{V}$. Given r functions $\alpha_j : \mathbf{X}_j \to \mathbb{C}$, $j = 1, \ldots, r$ defined on some sets $\mathbf{X}_j$, we denote by $\alpha_1 \cdot \alpha_2 \cdots \alpha_r$ their tensor product defined in the usual way on the cartesian product of the domains $\mathbf{X}_1 \times \mathbf{X}_2 \times \cdots \times \mathbf{X}_r$. A particular case of this is when $\mathbf{X}_j = \mathbf{X}^{n_j}$ in which case we can interpret $\alpha_1 \cdots \alpha_r$ as being defined on $\mathbf{X}^n$ where $n = n_1 + \cdots + n_r$.

3 Hierarchies of equations and operators

Reference [2] provides the situation that motivated the present investigation. There one has a hierarchy of multi-particle evolution equations, one for each number of particles of designated species. Species merely differentiate one-particle evolutions, the particles are otherwise distinguishable. In such a context, for an n-tuple of species $s = (s_1, \ldots, s_n)$ and for an n-tuple of particle positions $x = (x_1, \ldots, x_n)$ where $x_j = (x_j^1, \ldots, x_j^d)$ are the position coordinates of the j-th particle in d-dimensional space, we have an n-particle probability amplitude $\psi^{(s)}(t, x) = \psi^{(s)}(t, x_1, \ldots, x_n)$ whose square modulus $|\psi^{(s)}(t, x)|^2$ is the joint probability density of finding at time t the particles at the corresponding positions x_j. These amplitudes obey a set of evolution equations:

$$i\hbar\partial_t\psi^{(s)} = F_s\psi^{(s)}.$$

In contrast with [2] we perform several mathematical generalizations. By allowing more components in each x_j (which we will indicate by using bold face) we introduce the possibility of internal degrees of freedom such as spin, isospin, flavor, etc., and by considering species as a particular type of internal degree of freedom, we suppress the species labels altogether. By this we achieve considerable notational simplification along with ability to deal with multi-component wave-functions. We also assume that the operators F can depend explicitly on time. Such a generalization is necessary since certain generators of infinitesimal symmetries, such as Galileian boosts, do depend explicitly on time and it is necessary to treat these along with F on equal mathematical footing. Furthermore, for theoretical studies, one may want to "switch on and off" certain terms by an explicit time dependence. Our hierarchy of equations will now be written as:

$$i\hbar\partial_t\psi^{(n)} = F_n(t)\psi^{(n)}. \tag{1}$$

where the label n indicates that we are dealing with n particles. We often drop this label, especially from the wave-function, and sometimes for clarity we place it thus: $F(t)_n$. For the sake of mathematical generality, we will not always impose all the requirements that physics may call for, maintaining the context that is mathematically natural for the level of abstraction adopted.

Definition 1 *Given a set* $\mathbf{X}$*, called the* one-particle configuration space*, a* hierarchy of multi-particle operators *is a family* F *of operators* F_n*,* $n = 1, 2, \ldots$*, where* F_n *acts on a space of functions* $\phi : \mathbf{X}^n \to \mathbb{C}$ *producing functions of the same type. By the* threshold *of a hierarchy we mean the smallest integer* c *for which* $F_c \neq 0$.

For our original context, the set $\mathbf{X}$ is $S \times \mathbb{R}^d$, where S is a set of *species* and $\mathbb{R}^d$ is the configuration space for a single particle in a d-dimensional Euclidean space. What makes

S into a set of species as opposed to some other type of internal degree of freedom is a specific assumption about the form of the operator F_n. Given $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathbf{X}^n$, let $\mathbf{x}_i = (s_i, x_i)$ with $s_i \in S$ and $x_i \in \mathbb{R}^d$. One can interpret a function $\phi(\mathbf{x})$ as a parameterized family of functions $\phi^{(s)}(x)$ considering the species labels as parameters. If we now assume

$$(F_n\phi)(\mathbf{x}) = F_s(\phi^{(s)})(x),$$

where each F_s is some operator acting on functions defined on $(\mathbb{R}^d)^n$, then we have recovered the context of [2]. When $\mathbf{X}$ is a finite set we would be dealing with quantum mechanics of a finite number of degrees of freedom, and each evolution equation would be just a system of ordinary differential equations in a finite dimensional space. In this case the use of the word "particle" may be questionable, though one could construe the equations as dealing with just the internal degrees of freedom of particles, ignoring the spatial distribution. Going in the other direction we could take $\mathbf{X} = S \times \mathbb{I} \times \mathbb{R}^d$ where S is a set of species, to be treated as explained above, and $\mathbb{I}$ parameterizes the internal degrees of freedom. Presumably $\mathbb{I}$ itself should be conveniently written as a Cartesian product $\prod_\lambda \mathbb{I}_\lambda$ over the different types of internal degrees of freedom. We would now be dealing with spatially distributed particles of different species with any number of internal degrees of freedom. We shall also admit hierarchies of operators that depend on some additional parameters (such as time), and also ones where the multi-particle functions all depend on some fixed set of additional variables (such as time). Besides hierarchies we also treat just isolated n-particle operators F_n for some values of n without these being associated to a hierarchy. Depending on the context, we shall denote by a capital roman letter F, G, $H, \ldots$ either an individual n-particle operator for fixed n, or a whole hierarchy of operators. In what follows we reserve the greek letter ψ for time dependent functions, that is, those defined on $\mathbb{R} \times \mathbf{X}^n$, and the greek letter ϕ for those defined on $\mathbf{X}^n$. We denote by $\psi(t)$ the parametrized function on $\mathbf{X}^n$ given by $\psi(t)(\mathbf{x}) = \psi(t, \mathbf{x})$. The right-hand side of (1) should of course strictly speaking be written as $F_n(t)(\psi^{(n)}(t))$.

We shall impose one condition on n-particle operators and hierarchies which reflects arbitrariness in labeling distinguishable physical particles. If π is any permutation of $\{1, \ldots, n\}$ then for any n-tuple $w = (w_1, \ldots, w_n)$ we define $\pi w = (w_{\pi(1)}, \ldots, w_{\pi(n)})$ and for any function ϕ on $\mathbf{X}^n$, we define $(\pi\phi)(\mathbf{x}) = \phi(\pi\mathbf{x})$.

Definition 2 *Let F be an n-particle operator. We say this operator satisfies the* permutation property *if, using the notation of the previous paragraph, for each permutation π:*

$$F(\pi\phi) = \pi(F(\phi)).$$

We say a hierarchy satisfies the permutation property if each n-particle operator does.

From now on we always, and implicitly, assume the permutation property as it simplifies some of the combinatorics and leads to no loss of generality for any physical application.

Our approach is *a priori*, disregarding mathematical questions of domains and existence and uniqueness of solutions to the initial value problem. It is however useful at times to refer to the actual evolution, if it exists and is unique, and we denote by $E(t_2, t_1)$ the evolution operator from t_1 to t_2. That is, $(E(t_2, t_1)\phi)(\mathbf{x}) = \psi(t_2)(\mathbf{x})$ where ψ is the unique solution of the initial value problem (1) with $\psi(t_1)(\mathbf{x}) = \phi(\mathbf{x})$. We of course have the group law for E:

$$E(t''', t'') \circ E(t'', t') \quad = \quad E(t''', t'), \tag{2}$$

$$E(t,t) \quad = \quad I. \tag{3}$$

From its definition, the evolution operator is easily shown to satisfy:

$$\hbar\frac{\partial}{\partial t'}E(t',t) \quad = \quad \bar{\imath}F(t')\circ E(t',t), \tag{4}$$

$$\hbar\frac{\partial}{\partial t}E(t',t) \quad = \quad -\mathbb{D}E(t',t)\cdot\bar{\imath}F(t) \tag{5}$$

where $\bar{\imath} = -i$. We note that in (5) the factor $\bar{\imath}$ cannot be moved to the front of the Fréchet derivative since this operator is only real-linear and not necessarily linear. Formal properties of the evolution operators are often useful heuristically even if one has not established existence and uniqueness theorems for the evolution equations.

We now review the definition of the separation property as introduced in [2]. Let H be a hierarchy of operators, and for $j = 1,\dots,r$ let ϕ_j be functions on $\mathbf{X}^{n_j}$. Let $n = n_1+n_2+\cdots+n_r$. Adopt the same notation for time-dependent functions ψ_j treating time as just a parameter.

Definition 3 *We say a hierarchy H is a* separating hierarchy *if in the notation of the previous paragraph:*

$$H_{n_1}(\phi_1)\cdot H_{n_2}(\phi_2)\cdots H_{n_r}(\phi_r) = H_n(\phi_1\cdot\phi_2\cdots\phi_r). \tag{6}$$

Definition 4 *We say a hierarchy of evolution equations (1) satisfies the* separation property *just in case using the notation of the paragraph prior to Definition 3, whenever the ψ_j are solutions of (1) for particle numbers n_j, then $\psi_1\cdot\psi_2\cdots\psi_r$ is a solution of (1) for particle number n. We shall also say that such a hierarchy of equations is a* separating hierarchy.

Note that for a separating hierarchy of evolution equations, the corresponding hierarchy of operators given by the right-hand side of (1) is not necessarily (and in general will not be) a separating hierarchy of operators. This abuse of language should not cause confusion however. For a separating hierarchy of evolution equations however, the corresponding hierarchy of *evolution operators*, if it exists, will be a separating hierarchy.

To analyze (6) we substitute $k_j\phi_j$ for ϕ_j where the k_j are complex numbers with $\prod_{j=1}^{r} k_j = 1$. The right-hand side does not change while the left-hand side becomes

$$H_{n_1}(k_1\phi_1)\cdot H_{n_2}(k_2\phi_2)\cdots H_{n_r}(k_r\phi_r) \tag{7}$$

which thus must be independent of the k_j. Suppose now that for some particle number n_0, for some function ϕ_0, and for some point $\mathbf{x}_0$, one has $H_{n_0}(\phi_0)(\mathbf{x}_0)\neq 0$. Let now $r=2$, $n_1=n_2=n_0$ and $\phi_1=\phi_2=\phi_0$, then the invariance of (7) implies $H_{n_0}(k\phi_1)H_{n_0}(k^{-1}\phi_2) = H_{n_0}(\phi_1)H_{n_0}(\phi_2)$. Now the *variables* in ϕ_1 and ϕ_2 in this equation are different, but we can substitute for both the same point $\mathbf{x}_0$ and conclude that for all k, $H_{n_0}(k\phi)(\mathbf{x}_0)\neq 0$. Applying again the invariance of (7) to the case $r=2, n_1=n, n_2=n_0$ with $\phi_2=\phi_0$ one has

$$H_n(k\phi_1) = \frac{H_{n_0}(\phi_2)(\mathbf{x}_0)}{H_{n_0}(k^{-1}\phi_2)(\mathbf{x}_0)}H_n(\phi_1) = c(k)H_n(\phi_1).$$

Thus unless all the operators in the hierarchy vanish identically, $H_n(k\phi) = c(k)H_n(\phi)$ for some complex function $c(k)$, and in particular $c(1) = 1$. Using this in (7) we see that $\prod_{j=1}^{n} c(k_j)$ must be independent of the k_j. This is an exponentiated version of a functional relation solved in [2] and based on that derivation we conclude that any locally integrable solution is of the form:

$$c(k) = e^{a \ln |k| + ib \arg k}$$

for some complex numbers a, b. We shall discuss such functions in section 4.

Definition 5 *Let a and b be complex numbers, an operator H satisfying*

$$H(k\phi) = e^{a \ln |k| + ib \arg k} H(\phi) \tag{8}$$

will be called mixed-power homogeneous *and the numbers a and b will be called respectively the* first *and* second exponential index *of H. We call the property expressed by (8)* mixed-power homogeneity*. When $a = 1$ and $b = 1$ we say the operator is* strictly homogeneous *and we call the corresponding property* strict homogeneity*.*

Theorem 1 *The operators of a separating hierarchy are mixed-power homogeneous with the same exponential indices for all operators.*

This theorem of course applies to the evolution operators of a separating hierarchy of evolution equations (1) in which case the exponential indices of $E(t', t)$ are in general functions of t' and t.

For a separating hierarchy of evolution equations (1) the corresponding operators hierarchy F satisfies the infinitesimal versions of (6) and (8). These were derived in [2] and will be simply stated here.

Definition 6 *We say of an operator F that it is* mixed-logarithmic homogeneous *with* first *and* second logarithmic index *p and q, respectively, if it satisfies*

$$F(k\phi) = kF(\phi) + k(p \ln |k| + iq \arg k)\phi. \tag{9}$$

We call the property expressed by the above equation mixed-logarithmic homogeneity*.*

When $p = 0$ and $q = p$ the operator is strictly homogeneous.

Definition 7 *We say of a hierarchy F that it is a* tensor derivation *if it satisfies*

$$\frac{F_{n_1}(\phi_1)}{\phi_1} + \cdots + \frac{F_{n_r}(\phi_r)}{\phi_r} = \frac{F_n(\phi_1 \cdots \phi_r)}{\phi_1 \cdots \phi_r}. \tag{10}$$

One can interpret (10) as Leibnitz's rule for the tensor product. Another useful way of seeing this is to multiply both sides of (10) by $\phi_1 \cdots \phi_r$:

$$F_{n_1}(\phi_1) \cdot \phi_2 \cdots \phi_r + \cdots + \phi_1 \cdots \phi_{r-1} \cdot F_{n_r}(\phi_r) = F_n(\phi_1 \cdots \phi_r). \tag{11}$$

In [2] the following two theorems are proved:

Theorem 2 *Each operator in a tensor derivation is mixed-logarithmic homogeneous with the same logarithmic indices.*

Theorem 3 *The operator hierarchy $F(t)$ of a separating hierarchy of evolution equations is a tensor derivation for all t.*

The common logarithmic indices of all the evolution operators thus constitute new universal physical constants with the dimension of energy.

Being a tensor derivation is the infinitesimal version of (6) and being mixed-logarithmic homogeneous is the infinitesimal version of (8).

We can derive the relationship between the logarithmic indices $p(t)$ and $q(t)$ of the operators $F(t)$ and the exponential indices $a(t',t)$ and $b(t',t)$ of the evolution operators $E(t',t)$. Applying $E(t',t)$ to $k\phi$, using the mixed-power homogeneity, and Equations (3–4), one arrives after a short calculation at:

$$i\hbar \tfrac{\partial}{\partial t'}\, a\,(t',t) = p(t')\,\mathrm{Re}\, a(t',t) + iq(t')\,\mathrm{Im}\, a(t',t), \tag{12}$$

$$i\hbar \tfrac{\partial}{\partial t'}\, b\,(t',t) = q(t')\,\mathrm{Re}\, b(t',t) + ip(t')\,\mathrm{Im}\, b(t',t), \tag{13}$$

$$a(t,t) = 1 = b(t,t). \tag{14}$$

Thus, given $p(t)$ and $q(t)$ one can in principle solve the above linear initial-value problem to uniquely determine $a(t',t)$ and $b(t',t)$. Reciprocally, given $a(t',t)$ and $b(t',t)$ one finds:

$$p(t) = i\hbar \frac{\partial}{\partial t'} a(t',t)\bigg|_{t'=t}, \tag{15}$$

$$q(t) = i\hbar \frac{\partial}{\partial t'} b(t',t)\bigg|_{t'=t}. \tag{16}$$

4 Mixed powers

Due to their ubiquity, the form $e^{a\ln|k| + ib\arg k}$ on the right-hand side of (8) and, changing letters, its logarithm $p\ln|k| + iq\arg k$ on the right-hand side of (9) deserve special attention.

Definition 8 *Let $z = re^{i\theta}$ be a non-zero complex number in polar form and (a,b) a pair of complex numbers. By the* mixed (a,b) power *of z we mean the number*

$$z^{(a,\,b)} = r^a e^{ib\theta} \tag{17}$$

which raises each factor in the polar decomposition to its own power.

Because θ in (17) is defined only modulo $2\pi i$ there is some ambiguity in defining the mixed power. It can be uniquely defined in any domain in which $\arg z$ is single-valued by

$$z^{(a,\,b)} = e^{a\ln|z| + ib\arg z}. \tag{18}$$

This makes the definition unique modulo the chosen branch of arg. If we choose a branch that includes 1 in its interior with $\arg 1 = 0$, one can interpret Equation (18) as defining a germ at 1 of a continuous function $f(z)$ with $f(1) = 1$. Such germs form a real algebra-like structure with the algebra sum of f and g being the product fg, the algebra product of f and g being the composition $f \circ g$ and the algebra scalar product of $r \in \mathbb{R}$ with f being the power f^r. This structure satisfies most but not all of the axioms of a real algebra. What fails is one of the distributive laws: $f \circ gh \neq (f \circ g)(f \circ h)$, and one of the scalar product laws: $f \circ g^r \neq (f \circ g)^r$. Within this structure the mixed powers of z however form a true real subalgebra and we have:

Theorem 4

$$z^{(a,b)} z^{(c,d)} = z^{(a+b,c+d)}$$
$$\left(z^{(c,d)}\right)^{(a,b)} = z^{(a,b)(c,d)}$$

where

$$(a,b)(c,d) = (a \operatorname{Re} c + ib \operatorname{Im} c,\ b \operatorname{Re} d + ia \operatorname{Im} d).$$

The proof is an easy verification using (18). We note in particular that $z^{(1,-1)} = \bar{z}$ so that the above algebra of mixed powers contains the complex conjugation. The following are also useful relations

$$(a,a)(c,d) = (ac, ad),$$
$$(a,-a)(c,d) = (a\bar{c}, -a\bar{d}).$$

From this it is easy to compute the multiplication table for the generators

$$E = (1,1), \quad B = (1,-1), \quad I = (i,i), \quad J = (i,-i).$$

Of these E is the multiplicative identity, B is the complex conjugation, I is the "exchange" of the logarithm of the modulus with minus the argument, and J is the same preceded by B. One has Table 1.

	B	I	J
B	E	-J	-I
I	J	-E	-B
J	I	-B	E

Table 1: Product law

In particular, the set $\{\pm E, \pm B, \pm I, \pm J\}$ forms a group. The commutator Lie bracket $[(a,b),(c,d)] = (a,b)(c,d) - (c,d)(a,b)$ for the last three generators (E commutes with everything) is found to be:

$$[B,I] = -2J, \tag{19}$$
$$[I,J] = -2B, \tag{20}$$
$$[J,B] = 2I. \tag{21}$$

These are relations for the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. This identification also follows from Theorem 5 below.

Theorem 5 *The association of the real-linear transformation*

$$z \mapsto (a,b) \cdot z = a \operatorname{Re} z + ib \operatorname{Im} z \tag{22}$$

to the mixed power germ $z^{(a,b)}$ *is an algebra isomorphism between the algebra of mixed powers and the algebra of real-linear endomorphisms of* $\mathbb{C}$*. One has:*

$$\begin{aligned}(a,b).((c,d)\cdot z) &= ((a,b)(c,d))\cdot z,\\ \ln z^{(a,b)} &= (a,b)\cdot \ln z,\\ (a,b)(c,d) &= ((a,b)\cdot c,\ (b,a)\cdot d).\end{aligned}$$

Furthermore, in the ordered real basis $(1,i)$ *of* $\mathbb{C}$*, the matrix of transformation (22) is*

$$\begin{pmatrix} \operatorname{Re} a & -\operatorname{Im} b \\ \operatorname{Im} a & \operatorname{Re} b \end{pmatrix}. \tag{23}$$

The proof is utterly straightforward. We also have:

Theorem 6 *Let* $p(z,a,b) = z^{(a,b)}$ *then*

$$\mathbb{D}\, p(z,a,b)\cdot(\zeta,\alpha,\beta) = z^{(a,b)}\left((\alpha,\beta)\cdot \ln z + (a,b)\cdot\left(\frac{\zeta}{z}\right)\right) \tag{24}$$

which in particular implies that for $f(z) = z^{(a,b)}$ *that the rank of* $\mathbb{D}\, f(z)$ *is two, unless* $\operatorname{Re} a\bar{b} = 0$ *in which case it is one, unless* $(a,b) = (0,0)$*.*

The proof of (24) is an easy verification using (18) once we note that $\ln|z| = \operatorname{Re}\ln z$ and $\arg z = \operatorname{Im}\ln z$. The statement about the rank follows now from (23).

5 Symmetries: General considerations

We begin discussing symmetries of a single n-particle evolution equation:

$$i\hbar\partial_t\psi = F(t)\psi, \tag{25}$$

going over some very well known ideas and results. Such a review is nevertheless appropriate due to the non-linear context.

Informally, by a *symmetry* of (25) we mean an operator V acting on functions $\psi(t,\mathbf{x})$ defined on $\mathbb{R}\times\mathbf{X}^n$ such that whenever ψ is a solution of (25) then $V\psi$ is also a solution. Symbolically:

$$(i\hbar\partial_t - F(t))\psi = 0 \Rightarrow (i\hbar\partial_t - F(t))V\psi = 0. \tag{26}$$

The immediate difficulty with this is that (26) is not an operator equation for V and merely states that V maps the solution set of (25) into itself. For *a priori* studies no knowledge of the solution set can be assumed and the usual recourse is to find some operator equation of which (26) is a consequence. This is practically possible only after having assumed some structure for the operator V, calculating $i\hbar\partial_t(V\psi)$, and in the resulting expression substituting $i\hbar\partial_t\psi$ by $F(t)\psi$ to arrive at a true operator equation relating V and $F(t)$. This also means that there is no general theory of symmetries, only various particular theories relative to a given operator equation and certain additional constraints.

If V and W are symmetries then it is clear that $V\circ W$ also is. Thus under composition the set of symmetries forms a semi-group, and the set of invertible symmetries whose inverse is also a symmetry, a group. Whether the set of symmetries obtained from a particular operator equation and set of constraints is closed under composition or inversion is another matter, though this is often the case and it is convenient that it be so.

We shall in this paper only consider symmetries of the form

$$(V\psi)(t, \mathbf{x}) = (V(t)\psi(T(t)))\,(\mathbf{x}) \tag{27}$$

where $V(t)$ is an operator that acts on functions on $\mathbf{X}^n$, and $T : \mathbb{R} \to \mathbb{R}$ is some diffeomorphism. The most common form for T is affine: $T(t) = at + b$ which includes such transformations as time translation, time inversion, and time dilation. One justification for assuming form (27) is precisely to be able to handle space-time symmetries with such time coordinate transformations.

Purely heuristically, such a form is not as restrictive as it may seem for if one can uniquely solve the initial value problem for (25), then any solution ψ can be constructed from any of its time instant values $\psi(t)$ by $\psi(t') = E(t', t)\psi(t)$. We can denote this by $\psi = S(t)\psi(t)$ where $S(t)$ is an operator that transforms functions defined on $\mathbf{X}^n$ to ones defined on $\mathbb{R} \times \mathbf{X}^n$. One can now use the left-hand side of (27) to define $(V(t)\phi)(\mathbf{x}) = (VS(T(t))\phi)(t, \mathbf{x})$. Such an argument must however be used with caution if one is trying to avoid assuming any knowledge of the evolution operator or, what amounts to the same thing, the solution set.

Our basic form for symmetries is invariant under composition and inversion and we easily show:

Theorem 7 *If V and W are both of the form (27) then so is $V \circ W$ where we have:*

$$\begin{aligned} (V \circ W)(t) &= V(t) \circ W(T_V(t)), \\ T_{V \circ W} &= T_W \circ T_V. \end{aligned}$$

Furthermore, if V is invertible, then V^{-1} is also of the form (27) and we have:

$$\begin{aligned} V^{-1}(t) &= V(T_V^{-1}(t))^{-1}, \\ T_{V^{-1}} &= T_V^{-1}. \end{aligned}$$

In terms of the evolution operator $E(t', t)$ for (25), the property of V being a symmetry is now expressed through

$$V(t') \circ E(T(t'), T(t)) = E(t', t) \circ V(t). \tag{28}$$

From this one determines by the group law for E that

$$V(t) = E(t, 0) \circ V(0) \circ E(T(0), T(t)). \tag{29}$$

Thus any symmetry has the form (29) where $V(0)$ is arbitrary (as long as it transforms proper initial data into proper initial data). While this is undoubtedly true, it trivially reduces all symmetries to the knowledge of the evolution operator which for practical purposes is quite unproductive. This is yet one more indication that one generally only obtains useful information from symmetries if these belong to a restricted class of operators.

To obtain the operator equation for a symmetry we differentiate $V(t)\psi(t)$ with respect to t and use (25) for both ψ and $V\psi$ to deduce:

$$\hbar\frac{\partial V(t)}{\partial t} = \imath F(t)\circ V(t) - T'(t)\,\mathbb{D}\,V(t)\cdot \imath F(T(t)). \tag{30}$$

Under general conditions on the solubility and uniqueness of the the initial-value problem, equation (30) is necessary and sufficient for V to be a symmetry. It is thus an appropriate formal definition of symmetry for *a priori* considerations.

Definition 9 *We say an operator V given by (27) is a (formal)* symmetry *of the evolution equation (25) if condition (30) holds.*

For $T(t) = t$ and real-linear operators, the right-hand side of (30) would be a usual commutator and this would be a familiar condition. If $T'(t) \neq 1$ then even for real-linear operators and $F(t)$ time independent, the right-hand side in general would be a deformed or "quantum" commutator. We don't explore condition (30) in its general form in any detail.

Solutions of (30) are subject to the operations of Theorem 7:

Theorem 8 *If both V and W satisfy their respective equations (30), then so does $V\circ W$. Furthermore if V is invertible, then V^{-1} also satisfies its equation (30).*

Note that this theorem does not say we can in general compose or invert solutions to equation (30) for a *fixed* function T since this function generally changes when we compose or invert.

A significant simplification occurs in our theory of symmetries when we deal with a one-parameter group of symmetries, that is, symmetries $V(r), r\in\mathbb{R}$ such that $V(r+s) = V(r)\circ V(s)$ and $V(0) = I$. One would have:

$$(V(r)\psi)(t,\mathbf{x}) = (V(t,r)(\psi(T(t,r)))\,(\mathbf{x}). \tag{31}$$

If we now write $V(r) = I + rK + o(r)$ and similarly $V(t) = I + rK(t) + o(r)$, and $T(t) = t + r\tau(t) + o(r)$, then one has for K, the *infinitesimal generator* of $V(r)$, the following form:

$$(K\psi)(t,\mathbf{x}) = (K(t)\psi(t))(\mathbf{x}) + \tau(t)(\partial_t\psi)(t,\mathbf{x}). \tag{32}$$

If now in (30) we evaluate the derivative with respect to r at $r = 0$ we get:

$$\hbar\frac{\partial K(t)}{\partial t} = [\imath F(t), K(t)] - \frac{\partial}{\partial t}(\tau(t)\imath F(t)). \tag{33}$$

The first term on the right-hand side is of course the bracket of the two operators considered as vector fields. The second term is the remnant of the deformed nature of the "bracket" on the right-hand side of (30):

Definition 10 *We say an operator K having the form (32) is an* infinitesimal symmetry *of (25) if and only if (33) holds.*

For real-linear operators the bracket reduces to the usual commutator. For non-linear theories we've seen that for two operators F, G there are in fact three notions of commutator that generalize the usual one. The first would be the "true" commutator $F\circ G - G\circ F$, the second can be obtained from it by replacing F by $I+rF$ and evaluating the derivative with respect to r at $r=0$, which gives us $F\circ G-\mathbb{D}\,G\cdot F$, and the third can be obtained by now replacing G with $I+rG$ and again evaluating the derivative with respect to r at $r=0$, which gives $[F,G]$. Of these only the third is a Lie bracket, and the second is not even anti-symmetric. Each one is the appropriate generalization of the real-linear commutator in the right context. One also has the "group" commutator $F\circ G\circ F^{-1}\circ G^{-1}$. This in fact is related to the "true" commutator through $F\circ G\circ F^{-1}\circ G^{-1}=I+F\circ(G\circ F^{-1}-F^{-1}\circ G)\circ G^{-1}$ and so no essentially new forms are introduced by this construct. One can also get deformed or "quantum" versions by starting with a deformation of the true commutator and then proceeding in the same manner, allowing also for the deformation to be subject to an expansion in a small parameter r.

The composition property of symmetries given by Theorem 7 of course has its counterpart for infinitesimal symmetries:

Theorem 9 *If K and L are two infinitesimal symmetries of (25) then so is $[K,L]$. Furthermore,*

$$[K,L](t) = [K(t),L(t)]+\tau_K(t)\frac{\partial L(t)}{\partial t}-\tau_L(t)\frac{\partial K(t)}{\partial t}, \tag{34}$$

$$\tau_{[K,L]}(t) = \tau_K(t)\tau_L'(t)-\tau_L(t)\tau_K'(t). \tag{35}$$

The proof of (34) is a straightforward calculation. To show that $[K,L]$ again satisfies (33) is an exercise in the Jacobi identity. The infinitesimal version of inversion is negation. Obviously if K satisfies (33) then so does $-K$. The right-hand side of (35) is the Lie bracket of τ_K and τ_L considered as vector fields on $\mathbb{R}$.

One can now pose two problems, one inverse to the other: for a given $F(t)$ find all operators of our type V or K, within some convenient class, that satisfy respectively (30) or (33), or inversely, given a set of such operators, determine the $F(t)$, within some convenient class, that have these transformations as symmetries. Such problems are often tractable but require considerable calculations and are now generally attacked by algebraic computation systems. See for example [1] and references therein. We shall not pursue these questions in any detail.

From a purely general perspective Equations (30) and (33) are not very enlightening. They are evolution equations for $V(t)$ and $K(t)$ respectively and given arbitrary $V(0)$ or $K(0)$ could in principle be solved to provide $V(t)$ or $K(t)$. This is a reflection of the trivialization of symmetries to the evolution operator that we've mentioned earlier. Equations (30) and (33) are always supplemented by additional conditions. Typically these are that $V(t)$ or $K(t)$ be independent of time or that they depend on time in a specified manner, that they be differential operators of a specified type, that they reflect space-time transformations, etc. Such additional restrictions generally transform equations (30) and (33) into overdetermined systems.

6 Separating symmetries

One restriction that is quite natural for some symmetries that reflect invariance under an action that can be performed by the experimenter (such as change of inertial frame) is once again that systems consisting of uncorrelated parts continue being so after the symmetry transformation, and each part transforms as if the other parts did not exist. This is a direct extension of the separation property to symmetries. Of course we now deal with a hierarchy of symmetries V_n or of infinitesimal symmetries K_n for each particle number n. One must be careful drawing conclusion from the separation property for a symmetry since the hierarchy of symmetries is composed of operators that act on function on $\mathbb{R} \times \mathbf{X}^n$ and not on $\mathbf{X}^n$. In particular, a separating symmetry is not a mixed-power homogeneous operator since solutions to (25) do not scale if $F(t)$ is not strictly homogeneous. As was shown in [2], if $F(t)$ is mixed-logarithmic homogeneous, then solutions scale by time-dependent factors $w(t)$ comprised of the solution of the equation $i\hbar\partial_t \ln w(t) = (p(t), q(t)) \cdot \ln w(t)$. A symmetry will have homogeneity properties only with respect to such time-dependent multipliers. If we write down however what the separation property means for a symmetry of the form (27), we immediately find that this implies that $V(t)$ must be a separating hierarchy.

Definition 11 *Given a hierarchy of evolution equations (1) we say a hierarchy V, respectively K, of operators is a* symmetry , *respectively* infinitesimal symmetry, *of the hierarchy of equations if each V_n is a symmetry, respectively, each K_n is an infinitesimal symmetry of the corresponding equation. Given a separating hierarchy of evolution equations we say of a hierarchy of symmetries of form (27), respectively (32), that it satisfies the* separation property, *or that it is a* separating symmetry *if, for all t, $V(t)$ is a separating hierarchy, respectively $K(t)$ is a tensor derivation.*

One then concludes that for a separating symmetry the $V_n(t)$ are mixed-power homogeneous with the same exponential indices and that for an infinitesimal separating symmetry the $K_n(t)$ are mixed-logarithmic homogeneous with the same logarithmic indices.

The homogeneity indices of symmetries and infinitesimal symmetries are related to those of the operators in the evolution equations through their own evolution equations. We have:

Theorem 10 *Let V of form (27) be a symmetry of a separating hierarchy of evolution equations and let $(a(t), b(t))$ be the exponential indices of $V(t)$ and $(p(t), q(t))$ be the logarithmic indices of $F(t)$. One has*

$$\hbar\frac{d}{dt}(a(t), b(t)) = (\bar{\imath}p(t), \bar{\imath}q(t)) \cdot (a(t), b(t)) - T'(t)(a(t), b(t)) \cdot (\bar{\imath}p(T(t)), \bar{\imath}q(T(t))). \quad (36)$$

Furthermore if now K of form (32) is an infinitesimal symmetry of the same hierarchy of evolution equations, and if now $(c(t), d(t))$ are the logarithmic indices of $K(t)$, then

$$\hbar\frac{d}{dt}(c(t), d(t)) = [(\bar{\imath}p(t), \bar{\imath}q(t)), (c(t), d(t))] - \frac{d}{dt}(\tau(t)(\bar{\imath}p(t), \bar{\imath}q(t))). \quad (37)$$

Finally, if K is the infinitesimal generator of the one-parameter group $V(r)$, then

$$(a(t, r), b(t, r)) = (1, 1) + r(c(t), d(t)) + o(r).$$

The theorem is easily proved by applying (30) and (33) to $k\phi$ and using the homogeneity properties of the operators involved. As a short-cut for (36) one can apply (28) to $k\phi$ and use (12)–(16). This of course is only legitimate if the evolution operators exist, but the formal result is true in any case. Likewise (37) follows directly from (33) and Theorem 12 below.

We shall from now on deal only with separating symmetries.

7 Lie algebra and liftings of tensor derivations

We first prove an analog of the well-known Euler's equation for homogeneous functions.

Theorem 11 *Let H be a mixed-power homogeneous operator with exponential indices (a, b), K be a mixed-logarithmic homogeneous operator with logarithmic indices (p, q), and η, any complex number, then*

$$\mathbb{D}H(\phi) \cdot \eta\phi = (a, b) \cdot \eta H(\phi), \tag{38}$$

$$\mathbb{D}K(\phi) \cdot \eta\phi = \eta K(\phi) + (p.q) \cdot \eta\, \phi. \tag{39}$$

Proof: One has

$$\begin{aligned} H((1 + r\eta)\phi) &= (1 + r\eta)^{(a,b)} H(\phi), \\ K((1 + r\eta)\phi) &= (1 + r\eta)K(\phi) + (1 + r\eta)\ln(1 + r\eta)^{(p,q)}\phi. \end{aligned}$$

Using (24) one can evaluate the derivative of these with respect to r at $r = 0$, and obtain (38) and (39), respectively. Q.E.D.

Given two infinitesimal generators F and G of one-parameter groups $V(r)$ and $W(r)$, one has

$$\lim_{n\to\infty} (V(r/n)W(r/n)V(-r/n)W(-r/n))^{n^2} = I + r^2[F, G] + o(r^4).$$

Thus, modulo the possibility of exponentiation, one can show that the Lie bracket preserves those properties of operators that have expression in the corresponding exponentiated groups and there behave appropriately under compositions. In particular it is not surprising that tensor derivations form a Lie algebra:

Theorem 12

1. *If F and G are mixed-logarithmic homogeneous operators with indices (p_F, q_F) and (p_G, q_G), respectively, then so is $[F, G]$ with logarithmic indices*

$$(p_{[F,G]}, q_{[F,G]}) = [(p_F, q_F), (p_G, q_G)]. \tag{40}$$

2. *If F and G are tensor derivations, then so is $[F, G]$ with logarithmic indices given by (40). The threshold of $[F, G]$ is greater than or equal to the maximum of the thresholds of F and G.*

Proof: To prove the first part, we first note that:

$$\mathbb{D}\, F(k\phi) \cdot G(k\phi) = \frac{d}{ds} F(k\phi + sG(k\phi))\Big|_{s\,=\,0}.$$

The term to be differentiated above is equal to:

$$F(k\phi + ks(G\phi + \ln k^{(p_G,q_G)}\phi)) =$$

$$k(F(\phi + sG\phi + s\ln k^{(p_G,q_G)}\phi) + \ln k^{(p_F,q_F)}(\phi + sG\phi + s\ln k^{(p_F,q_F)}\phi)).$$

The derivative of this with respect to s at $s = 0$ is:

$$k\left(\mathbb{D}F(\phi)\cdot G(\phi) + \mathbb{D}F(\phi)\cdot \ln k^{(p_G,q_G)}\phi + \ln k^{(p_F,q_F)}(G(\phi)+\ln k^{(p_G,q_G)}\phi)\right).$$

From this,

$$[F,G](k\phi) = k[F,G](\phi) + k(\mathbb{D}F(\phi)\cdot \ln k^{(p_G,q_G)}\phi - \ln k^{(p_G,q_G)}F(\phi)) +$$
$$k(\mathbb{D}G(\phi)\cdot \ln k^{(p_F,q_F)}\phi - \ln k^{(p_F,q_F)}G(\phi)).$$

Using the generalized Euler's formula (39) in this expression we deduce the formula for $(p_{[F,G]}, q_{[F,G]})$ and prove the first part.

For the case of tensor derivations, one has, using the notation of Definition 3:

$$\mathbb{D}F_n(\phi_1\cdots\phi_r)\cdot G_n(\phi_1\cdots\phi_r) = \frac{d}{ds}F_n(\phi_1\cdots\phi_r + sG_n(\phi_1\cdots\phi_r))|_{s=0} =$$

$$\frac{d}{ds}(F_n(\phi_1\cdots\phi_r + sG_{n_1}(\phi_1)\cdot\phi_2\cdots\phi_r + \cdots + s\phi_1\cdot\phi_2\cdots\phi_{r-1}\cdot G_{n_r}(\phi_r))|_s = 0.$$

The quantity being differentiated differs by a term of order $o(s^2)$ from

$$F_n((\phi_1 + sG_{n_1}\phi_1)\cdots(\phi_r + sG_{n_r}\phi_r)).$$

Using the fact that F is a tensor derivation, we can apply to this expression Leibnitz's rule (11), evaluate the derivative with respect to s at 0, and arrive at:

$$\sum_{j=1}^{r}\mathbb{D}F_{n_j}(\phi_j)\cdot G_{n_j}(\phi_j)\cdot\hat{\phi}_j + \sum_{j\neq k}F_{n_j}(\phi_j)\cdot G_{n_k}(\phi_k)\cdot\hat{\phi}_{jk} \tag{41}$$

where we've introduced the partial (tensor) products $\hat{\phi}_j = \prod_{i\neq j}\phi_i$ and $\hat{\phi}_{jk} = \prod_{i\neq j,k}\phi_i$, and where all the tensor products in (41) are to be interpreted as occurring in the original order of $\phi_1\cdots\phi_r$. To not be misled by the notation in (41), we mention that the first in-line dot in the first term designates an application of a Fréchet derivative to a vector, while in the second term it designates a tensor product. As the second term in (41) is symmetric under the interchange of F and G, the corresponding terms in $[F,G](\phi_1\cdots\phi_r)$ cancel and we deduce that the bracket satisfies the separation property. The statement about the thresholds is obvious. Q.E.D.

We now review the lifting properties of tensor derivations.

Let F be an operator acting on functions from $\mathbf{X}^n$ to $\mathbb{C}$ producing functions of the same type. Let now $m > n$ and $J = (j_1,\ldots,j_n)$ be an n-tuple of distinct elements of $\{1,\ldots,m\}$ in increasing order. A function $\phi(\mathbf{x}_1,\ldots,\mathbf{x}_m)$ can be construed as a parametrized family of functions $\phi_{\mathbf{y}}(\mathbf{x}_{j_1},\ldots,\mathbf{x}_{j_n})$ by taking each $\mathbf{x}_k$ for $k\notin\{j_1,\ldots,j_{n_j}\}$ as a parameter $\mathbf{y}_k$. Applying F to each member of this family one gets another parametrized family of functions $F(\phi_{\mathbf{y}})(\mathbf{x}_{j_1},\ldots,\mathbf{x}_{j_n})$ which we can reinterpret back as a function $F^J(\phi)(\mathbf{x}_1,\ldots,\mathbf{x}_m)$. This defines a new operator F^J.

Definition 12 *The operator F^J defined in the previous paragraph is called a* lifting *of F.*

The three following theorems were proved in [2].

Theorem 13 *Let F be a one-particle mixed-logarithmic homogeneous operator with logarithmic indices p and q. For $n \geq 1$ define n particle operators by*

$$F_n^{\#}\phi = \sum_{j=1}^{n} F^{(j)}\phi - (n-1)(p,q) \cdot \ln\phi\,\phi. \tag{42}$$

The resulting hierarchy $F^{\#}$ is a tensor derivation extending F (called the canonical lifting *of F).*

Theorem 14 *Let $\ell > 1$ and F be a strictly homogeneous ℓ-particle operator which vanishes on any tensor product function. For $n \geq \ell$ define n particle operators by*

$$F_n^{\#} = \sum_J F^J \tag{43}$$

where the sum runs over all $J = (j_1, \ldots, j_\ell)$ of ℓ-tuples of distinct elements of $\{1, \ldots, n\}$ in increasing order. The resulting hierarchy $F^{\#}$ is a tensor derivation of threshold ℓ extending F (called the canonical lifting *of F).*

One sees that in (42) if p and q vanish then (42) can be construed as the $\ell = 1$ case of (43). It is sometimes useful, in spite of the breach of good notational discipline, to write a single formula:

$$F_n^{\#}\phi = \sum_J F^J\phi - (n-1)(p,q) \cdot \ln\phi\,\phi \tag{44}$$

to cover both cases in a single argument with the understanding that the second term is zero for $\ell \neq 1$.

Theorem 15 *Let F be a tensor derivation. Define derivations d_jF as follows:*

$$d_1F = F_1{}^{\#},$$

and having defined $d_1F, \ldots, d_rF$, let

$$d_{r+1}F = (F - \sum_{j=1}^{r} d_jF)_{r+1}{}^{\#}.$$

One has $F = \sum_{j=1}^{\infty} d_jF$ (called the canonical decomposition *of F), the d_j are real-linear idempotents, and if d_jF is not zero, its threshold is j. Conversely, if for each j we are given a j-particle operator $F_{(j)}$ satisfying:*

1. *$F_{(1)}$ is mixed-logarithmic homogeneous.*

2. *For $j > 1$, $F_{(j)}$ is strictly homogeneous and vanishes on tensor product functions;*

then the derivation $F = \sum_{j=1}^{\infty} F_{(j)}{}^{\#}$ satisfies $d_jF = F_{(j)}{}^{\#}$.

These theorems provide us with a canonical procedure to construct tensor hierarchies by the introduction of new generators at each particle number threshold. The operator

$(d_jF)_j$ is called the *canonical generator* of F at threshold j. These uniquely define the hierarchy and are themselves objects that can be freely given subject only to conditions (1) and (2) above. In physical theories, generators at particle numbers greater than one introduce truly new effects in correlated systems that are absent for smaller number of particles.

Definition 13 *We call a j-particle operator F a* generator *if it satisfies item (1) or (2) of Theorem 15.*

The lifting properties of the Lie bracket are quite complex. Suppose we are given two one-particle generators F and G. Let $H = [F, G]$ define a third one, and let $F^\#$, $G^\#$, and $H^\#$ be the corresponding canonical liftings. It is not generally true that $H^\# = [F^\#, G^\#]$. This is a purely non-linear effect and has the consequence that if one has a set of one-particle symmetries of a one-particle evolution equation then the canonically lifted multi-particle equations are not necessarily symmetric under the canonically lifted one-particle symmetries. Since one naively would expect one-particle symmetries to be extensible to multi-particle symmetries, especially if no new multi-particle effects are introduced through new canonical generators, this question bears examining.

Definition 14 *For a pair of complex numbers (a, b) define the n-particle operator $\Lambda(a, b)$ by*

$$\Lambda(a, b)\phi = (a, b) \cdot \ln \phi\, \phi.$$

For a mixed-logarithmic homogeneous operator F with indices (a, b) let $\Lambda_F = \Lambda(a, b)$. Define then $F^\natural$ by:

$$F = F^\natural + \Lambda_F. \tag{45}$$

One easily verifies:

Theorem 16

1. *$\Lambda(a, b)$ is mixed-logarithmic homogeneous with logarithmic indices (a, b).*
2. *$[\Lambda(a, b), \Lambda(c, d)] = \Lambda([(a, b), (c, d)])$.*
3. *The canonical lifting of the one-particle $\Lambda(a, b)$ to an n-particle operator is the corresponding n-particle $\Lambda(a, b)$.*
4. *If F and G are mixed-logarithmic homogeneous operators, then $\Lambda_{[F,G]} = [\Lambda_F, \Lambda_G]$.*
5. *For a one-particle generator F with logarithmic indices (a, b), the canonical lifting $F^\#$ satisfies:*

$$F^\#_n = F^{\natural\#}_n + \Lambda(a, b).$$

Item (2) shows that $(a, b) \to \Lambda(a, b)$ is a representation of the mixed-power Lie algebra.

Theorem 17

1. *Let F be an ℓ-particle generator and G an m-particle generator with $1 \le \ell \le m$. Let $F^\#$ and $G^\#$ be their respective canonical liftings. For any particle number n with $n > m$:*

$$[F^\#_n, G^\#_n] - [F^\#_m, G]^\#_n = \sum_K \sum_{J \not\subset K} [F^{\natural J}, G^{\natural K}] \tag{46}$$

where J is an ℓ-tuples $(j_1, \ldots, j_\ell)$ of elements of $\{1, \ldots, n\}$ in increasing order, K is an m-tuple of the same type and where we write $J \not\subset K$ to mean $\{j_1, \ldots, j_\ell\} \not\subset \{k_1, \ldots, k_m\}$.

2. *The obstruction to the equality*

$$[F^\#, G^\#] = [F^\#_m, G]^\# \tag{47}$$

is the set of operators on the right hand side of (46) at particle numbers from $m+1$ to $m+\ell$. These operators are zero if and only if (47) holds.

Proof: We first note that $[F^\#_m, G] = [F^\#, G^\#]_m$ so the left-hand side is a legitimate m-particle generator (even though $F^\#_m$ itself is not) since by Theorems 12, 13 and 14, $[F^\#, G^\#]$ is a tensor derivation of threshold at least m.

To expedite the proof, we use the notational shortcut indicated by (44) and explained in that paragraph. We shall also employ set-theoretic notation such as "$J \subset K$". This is to be understood as referring to the underlying sets.

One has for n-particle operators:

$$[F^\#_n, G^\#_n] = [\sum_J F^J - (n-1)\Lambda_F, \sum_K G^K - (n-1)\Lambda_G]$$

which is

$$\sum_K \sum_{J\subset K} [F^J, G^K] + \sum_K \sum_{J\not\subset K} [F^J, G(k)] +$$
$$(n-1)\sum_J [F^J, \Lambda_G] - (n-1)\sum_K [\Lambda_F, G^K] + (n-1)^2[\Lambda_F, \Lambda_G].$$

Now

$$[F^\#_m, G]^\#_n = \sum_K \sum_{J\subset K} [F^J, G^K] - (n-1)\Lambda_{[F,G]}.$$

So, using Item 4 of Theorem 16, $[F^\#_n, G^\#_n] - [F^\#_m, G]^\#_n$ is found to be:

$$\sum_K \sum_{J\not\subset K} [F^J, G^K] + (n-1)\sum_J [F^J, \Lambda_G] - (n-1)\sum_K [\Lambda_F, G^K] + n(n-1)[\Lambda_F, \Lambda_G].$$

Substituting into this expression $F^J = F^{\natural J} + \Lambda_F$ and $G^K = G^{\natural K} + \Lambda_G$ one arrives after a short calculation at (46). To deduce the statement about the obstruction, let $n > m + \ell$ and consider the right-hand side of (46). Let Γ be the set of all pairs (J, K) of m and ℓ-tuples of elements of $\{1, \ldots, n\}$ in increasing order with $J \not\subset K$. For $\gamma = (J, K) \in \Gamma$ let $C(\gamma) = [F^J, G^K]$. For any subset Ω of Γ let $C(\Omega) = \sum_{\gamma\in\Omega} C(\gamma)$. The right-hand side of (46) is $C(\Gamma)$. Let $I_1, \ldots, I_d$ be the enumerated distinct $(m+\ell)$-tuples of elements of $\{1, \ldots, n\}$ in increasing order. Now, since the number of variables affected by any $C(\gamma)$ is at most $m + \ell$, one has $\Gamma = \bigcup_{i=1}^{d} \Gamma_i$ where Γ_i consists of those pairs (J, K) for which $J \subset I_i$ and $K \subset I_i$. One has the classic formula:

$$C(\Gamma) = \sum_{k=1}^{d} (-1)^{k+1} \sum_{i_1<i_2<\cdots<i_k} C(\Gamma_{i_1} \cap \Gamma_{i_2} \cap \cdots \cap \Gamma_{i_k}).$$

Consider now the term $C = C(\Gamma_{i_1} \cap \Gamma_{i_2} \cap \cdots \cap \Gamma_{i_k})$. This is a sum over all pairs (J, K) for which $J, K \subset I = I_{i_1} \cap I_{i_2} \cap \cdots \cap I_{i_k}$. Suppose the set of such pairs is not empty, so that the number of elements in I is some number h with $m + 1 \leq h \leq m + \ell$. When C is applied to a function ϕ on $\mathbf{X}^n$, all the variables not indexed by an element of I can be considered as mere parameters and so C is a lifting of one of the operators in the claimed obstructing set. Thus $C(\Gamma)$ vanishes if all the operators in the indicated set vanish. Q.E.D

The properties of the Lie bracket under canonical decomposition are also quite complex. One can however obviously state

$$d_j[F, G] = d_j[d_1F + \cdots + d_jF, d_1G + \cdots + d_jG]. \tag{48}$$

Concerning symmetries we begin with some general considerations. Equation (33) is equivalent to:

$$\hbar \frac{\partial d_j K(t)}{\partial t} = d_j[\bar{\imath}F(t), K(t)] - d_j \frac{\partial}{\partial t}(\tau(t)\bar{\imath}F(t)) \tag{49}$$

for $j = 1, 2, \ldots$ Let now $G_\ell(t)$ be a canonical generator of $K(t)$ at threshold ℓ. From (48) and (49) one can deduce:

$$\hbar \frac{\partial G_\ell(t)}{\partial t} = [\bar{\imath}F_\ell(t), L_\ell(t)] + [\bar{\imath}F_\ell(t), G_\ell(t)] + \frac{\partial}{\partial t}(\tau(t)\bar{\imath}F_\ell(t)) \tag{50}$$

where $L_\ell(t)$ is an operator constructed from the canonical generators of $K(t)$ at thresholds less then ℓ. Equation (50) is then an inhomogeneous linear evolution equation for $G_\ell(t)$ and so in principle can be solved once the generators at thresholds less then ℓ are known. Thus, barring other constraints, one can solve (33) iteratively, threshold by threshold. In particular, if one is given a partial hierarchy $K(t)$ with n-particle operators for $n \leq \ell$, and if each $K_n(t)$ is an infinitesimal symmetry of $F_n(t)$, then one can, again barring other constraints, extend the partial hierarchy to a full symmetry of $F(t)$ introducing new generators at thresholds above ℓ.

This general statement must be tempered by two considerations if, as is usually the case, we want symmetries satisfying additional conditions. In the first place one could at any threshold run into a true obstruction to extending the symmetry maintaining the additional conditions, and secondly, given that the extension at any threshold may not be unique, the existence of such an obstruction can depend on the choices made at lower thresholds. This means that for practical calculations, such as those done by computer algebra systems, one should either always determine the most general solution at each threshold or be ready to do some backtracking to revise decisions made at lower thresholds.

We now investigate two specific situations. Generally one would start with one-particle operators $F(t)$ and corresponding one-particle infinitesimal symmetries K. For K defined by (32) we define the canonical lifting $K^\#$ as the hierarchy of operators defined again through (32) by $K(t)^\#$ and the same function $\tau(t)$. We can now ask under what conditions is $K^\#$ a symmetry of $F(t)^\#$. Another natural question occurs if now at threshold ℓ we add new canonical generators $G(t)$ to $F(t)$. We can then ask under what conditions does $K^\#$ continue being a symmetry now of $F(t)^\# + G(t)^\#$. Theorem 17 provides simple and useful answers to both questions.

Corollary 1 *Let $F(t)$ and K be one-particle generators, and suppose K is an infinitesimal symmetry of $F(t)$. The canonical lifting $K^\#$, defined in the previous paragraph, is a* *symmetry of* $F(t)^{\#}$ *if and only if the two-particle operator* $K_2^{\#}$ *is a symmetry of* $F(t)_2^{\#}$ *and this happens if and only if the two-particle operator*

$$[F(t)^{\natural(1)}, K(t)^{\natural(2)}] + [F(t)^{\natural(2)}, K(t)^{\natural(1)}] \tag{51}$$

vanishes.

Proof: Since lifting is linear, one has from (33) for the one-particle equations that:

$$\hbar \frac{\partial K(t)^{\#}}{\partial t} = [\bar{\imath} F(t), K(t)]^{\#} - \frac{\partial}{\partial t}(\tau(t)\bar{\imath} F(t)^{\#}).$$

Thus $K^{\#}$ is a symmetry if and only if $[\bar{\imath}F(t), K(t)]^{\#} = [\bar{\imath}F(t)^{\#}, K(t)^{\#}]$ and now we apply Theorem 17.

Thus if a symmetry extends to two particles it extends to any number of particles. One notices that for two real-linear one-particle operators F and G, the two-particle operator $[F^{(1)}, G^{(2)}]$ is always zero as $F^{(1)}$ and $G^{(2)}$ act on different tensor factors and such real-linear operators commute. Thus obstruction (51) can be non-zero only in non real-linear theories. Nevertheless, for some of the usual symmetries considered, such as space-time symmetries for particles without internal degrees of freedom, the obstruction generally vanishes even in the non-linear case. We shall see this in Section 8.

Corollary 2 *Let* $F(t)$ *be a tensor derivation and* K *a symmetry with only a one-particle generator:* $K(t) = d_1 K(t)$*. Let* $G(t)$ *be a* ℓ*-particle generator with* $\ell > 1$*, then* K *is a symmetry of* $F(t) + G(t)^{\#}$ *if and only if* K_ℓ *and* $K_{\ell+1}$ *are symmetries of* $F_\ell(t) + G(t)$ *and* $F_{\ell+1}(t) + G(t)_{\ell+1}^{\#}$*, respectively, and this happens if and only if* $[G(t), K_\ell(t)] = 0$ *and the following* $(\ell+1)$*-particle operator vanishes:*

$$\sum_{j=1}^{\ell+1} [G(t)^{j^\sharp}, K(t)^{\natural(j)}] \tag{52}$$

where $j^\sharp$ *is the* ℓ*-tuple.* $(1, \ldots, j-1, j+1, \ldots, \ell+1)$.

Proof: Suppose K is a symmetry of $F(t) + G(t)^{\#}$, then at particle numbers $n \geq \ell$ one has from the fact that K is a symmetry of $F(t)$ and (33):

$$[\bar{\imath}G(t)_n^{\#}, K_n(t)] - \frac{\partial}{\partial t}(\tau(t)\bar{\imath}G(t)_n^{\#}) = 0.$$

Again, since lifting is linear, this equation at particle number ℓ then implies

$$[\bar{\imath}G(t), K_\ell(t)]_n^{\#} - \frac{\partial}{\partial t}(\tau(t)\bar{\imath}G(t)_n^{\#}) = 0$$

and once again we apply Theorem 17.

Thus, under the hypotheses of Corollary 2, if at some particle number we've added a new generator preserving the symmetry at that particle number, the symmetry is then preserved in the whole hierarchy if and only if it is preserved at the next higher particle number. As before, in the real-linear case obstruction (52) vanishes automatically.

Theorem 17 of course covers the situation more complex than the ones covered by the two corollaries above, but the corollaries take care of the most frequent cases. In the more general situations the obstructions do not necessarily vanish even in the real-linear case.

8 Space-time symmetries

In this section we shall consider particles without internal degrees of freedom. One can admit any number of species, but we suppress any indication of these, as the reintroduction of species labels is a straightforward matter as explained in the paragraph following Definition 11. Most of the literature on non-linear Schrödinger equations considers, as a simplifying assumption, only this type of particle.

By a *space-time symmetry* $V(t)$ we mean one related to an underlying transformation of space-time $\Phi : (t, x) \mapsto (T(t, x), X(t, x))$. Our interpretation of such a symmetry is that it describes the same physical system seen from a changed reference frame, said change deriving from Φ. Our first simplifying assumption is that T depends only on t as otherwise given n particle positions $x_1, \ldots, x_n$ at instant t, the transformed time instances $T(t, x_i)$ could all be different and the construction of a multi-particle wave-function at one time instant would not be a straightforward matter. We have thus opted for at most Galileian relativity, if any, and we have:

$$\Phi(t, x) = (T(t), X(t, x)). \tag{53}$$

Since the transformed function in principle describes the same physical system as seen from the transformed frame, the probability densities, which in principle are objective observable quantities, should transform accordingly. Likewise it is natural that the symmetry be separating and we continue to assume this. Finally one must make a conventional choice deciding if Φ describes the old coordinates in terms of the new or vice-versa. We opt for the old in terms of the new, and thus assume for the probability densities that:

$$|V\psi|^2(t, x_1, \ldots, x_n) = |\psi|^2(T(t), X_1, \ldots, X_n)|JX_1| \cdots |JX_n| \tag{54}$$

where we've used the abbreviation $X_i = X(t, x_i)$, and where JX_i is the Jacobian determinant evaluated at x_i of the transformation $x \mapsto X(t, x)$. For particles with internal degrees of freedom one may need to sum both sides of (54) over internal indices. This would modify substantially the rest of the argument.

From (27) we identify the right-hand side as $|V(t)\psi(T(t))|^2(x_1, \ldots, x_n)$. Equation (54) now imposes conditions on the exponential indices $(a(t), b(t))$ of $V(t)$. Substituting ψ by $k\psi$ in the equation one deduces that $\operatorname{Re} a(t) = 1$ and $\operatorname{Im} b(t) = 0$. Putting all this together we get:

Theorem 18 *Using the notation of the previous paragraph, a space-time symmetry V of type (27) associated to a space-time transformation (53) is given by:*

$$V(t)\phi = e^{i\Theta(t)(\phi)}\phi(X_1, \ldots, X_n)|JX_1|^{\frac{1}{2}} \cdots |JX_n|^{\frac{1}{2}} \tag{55}$$

where $\Theta(t)$ is a real *operator (produces real-valued functions) with the homogeneity property:*

$$\Theta(t)(k\phi) = \Theta(t)(\phi) + (\alpha(t), \beta(t)) \cdot \ln k \tag{56}$$

for some real functions $\alpha(t)$ and $\beta(t)$.

Note that what (56) says is that $\phi \mapsto \Theta(t)(\phi)\phi$ is mixed-logarithmic homogeneous with real indices.

A one parameter group $V(r)$ of space-time symmetries would be associated to a one-parameter group $\Phi(r) : (t, x) \mapsto (T(t, r), X(t, x, r))$ of space-time transformations. Setting $T(t, r) = t + r\tau(t) + o(r)$, $X(t, x, r) = x + r\xi(t, x) + o(r)$, and $\Theta(t, r) = I + r\theta(t) + o(r)$, one deduces from (55):

Theorem 19 *Using the notation of the previous paragraph, the infinitesimal generator K of a space-time symmetry has the form:*

$$K(t)(\phi) = i(\theta(t)\phi)\,\phi + \sum_{j=1}^{n}\left((\xi\cdot\nabla)^{(j)} + \frac{1}{2}(\nabla\cdot\xi)^{(j)}\right)\phi,$$

where $\theta(t)$ is a real operator such that $\phi \mapsto \theta(t)(\phi)\phi$ is mixed-logarithmic homogeneous with real logarithmic indices.

The most common form of space-time symmetries are *point symmetries* of the type generally considered for differential equations [3]. By this we mean that $(V(t)\phi)(x) = H(\phi(X(t, x)), t, x)$ for some complex function H. This amounts to saying that the transformation $(w, t, x) \mapsto (H(w, \Phi^{-1}(t, x)), \Phi^{-1}(t, x))$ maps the graph of ψ to the graph of $V\psi$; that is, the symmetry is effected by a point transformation in the space in which the graph of a solution lies. One can envisage more general transformations in which H is a differential operator [4] and though there are differential equations with such generalized symmetries we shall not pursue this here.

The homogeneity and separation property of $V(t)$ now impose a condition on the function H. In fact, from $V(t)(k\phi) = k^{(1 + i\alpha(t), \beta(t))}V(t)(\phi)$ one deduces $H(kw, t, x) = k^{(1 + i\alpha(t), \beta(t))}H(w, t, x)$ from which setting $w = 1$ and renaming k as w one gets $H(w, t, x) = w^{(1 + i\alpha(t), \beta(t))}H(1, t, x)$ and the separation property implies $H(1, t, x) = \prod_{j=1}^{n} N(t, x_j)$ for some complex function N. Comparing this with (55) and then also deducing the version for infinitesimal symmetries, one finds:

Theorem 20 *A point space-time symmetry, using the notation of the paragraph preceding Theorem 18, has the form:*

$$V(t)\phi = e^{i\left(\sum_{j=1}^{n} \upsilon^{(j)}(t)\right)}(\phi(X_1, \ldots, X_n))^{(1+i\alpha(t),\beta(t))}|JX_1|^{\frac{1}{2}} \cdots |JX_n|^{\frac{1}{2}}$$

for some real functions $\alpha(t)$ and $\beta(t)$ and $\upsilon(t, x)$. An infinitesimal point space-time symmetry has the form:

$$K(t)\phi = \sum_{j=1}^{n}\left(i\eta^{(j)} + (\xi\cdot\nabla)^{(j)} + \frac{1}{2}(\nabla\cdot\xi)^{(j)}\right)\phi + i(\gamma(t), \delta(t))\cdot \ln\phi\,\phi \tag{57}$$

for some real functions $\gamma(t)$ and $\delta(t)$ and $\eta(t, x)$.

We see from (57) that an infinitesimal point space-time symmetry is always a canonical lift from the one-particle generator.

Theorem 21 *The lifting obstructions (51) and (52) vanish for point space-time symmetries.*

Proof: We see from (57) that for the one-particle symmetry

$$K(t)^\natural = i\eta\phi + \xi \cdot \nabla\phi + \frac{1}{2}(\nabla \cdot \xi)\phi$$

which is a first order linear differential operator. Since the obstruction expressions are real-linear in $K(t)^\natural$, we can consider each term separately. To expedite the argument one can in an obvious manner consider (51) as the $\ell = 1$ case of (52), remembering that $G^\natural = G$ for a generator at particle number above one. For a strictly homogeneous one-particle operator L, consider one term in the obstruction (52):

$$[G^{\jmath^\sharp}, L^{(j)}]. \tag{58}$$

Let $\alpha(t, x)$ be a real one-particle function and consider αL. One has

$$\alpha^{(j)}\,\mathbb{D}\,G^{\jmath^\sharp}(\phi) \cdot L^{(j)}(\phi) = \mathbb{D}\,G^{\jmath^\sharp}(\phi) \cdot \alpha^{(j)} L^{(j)}(\phi)$$

since $\mathbb{D}G^{\jmath^\sharp}$ is real-linear and the position variables upon which it acts are disjoint from the position variable in $\alpha^{(j)}$ allowing us to consider this function as a parametrized real number. One therefore has $[G^{\jmath^\sharp}, \alpha^{(j)} L^{(j)}] = \alpha^{(j)}[G^{\jmath^\sharp}, L^{(j)}]$. From this, to prove the theorem one need only show that (58) vanishes for L equal to the identity I, to iI, and to a partial derivative. Now for any complex number η, $\mathbb{D}\,\eta I = \eta I$, and so Equation (38) for a strictly homogeneous operator states exactly that the Lie bracket of that operator with ηI vanishes. This takes care of the first two cases. Let now L be a partial derivative $\partial/\partial x^k$. Since $L^{(j)}$ acts on what to $G^{\jmath^\sharp}$ are just parameters, we have by the chain rule for Fréchet derivatives:

$$L^{(j)} G^{\jmath^\sharp}\phi = \mathbb{D}\,G^{\jmath^\sharp} \cdot L^{(j)}\phi.$$

Again, since $\mathbb{D}L = L$, the above equation just says that (58) vanishes. Q.E.D.

We see therefore that although there are true obstructions to lifting of symmetries in non-linear theories, point space-time symmetries in separating hierarchies for particles without internal degrees of freedom are not affected by these. This result justifies the claims made about the Galileian invariance of the modified Doebner-Goldin hierarchy at the end of [2].

Some of the cases covered by the last theorem are otherwise obvious for hierarchies of differential operators. For instance, such operators are symmetric under space translations if each operator does not depend explicitly on the spatial coordinates, and this property obviously is maintained by liftings. For other symmetries such as Galileian boosts, or for operators that are not differential, one must rely on the theorem.

The case of particles with internal degrees of freedom is different and the possible non-vanishing of the obstructions can be viewed as either new phenomena, or the necessity of further constraints upon possible theories requiring that the obstructions vanish, or the necessity of introducing new generators to maintain symmetry. This shows that the simplifying assumption of particles without internal degrees of freedom is not as innocuous a postulate as it may seem to be.

To understand the problem involved suppose we wish to construct a hierarchy of equations (1) where the particles have internal degrees of freedom, which we indicate in the usual way by putting indices on the wave-function (rather than adding components to the position variable) assuming that such objects transform according to some representation (which may be non-linear) of an appropriate symmetry group. Assume the one-particle equations are symmetric under this group and write the action of the one-particle operator as

$$F_\alpha(\phi_\cdot)$$

where α is the internal degree of freedom index and where we've indicated by the subscript "$\cdot$" on the wave-function the fact that the operator F acts on these components of ϕ to produce in the end another object with the same transformational properties under the symmetry group. The canonically lifted two-particle operator can now be written as:

$$F^{(1)}_\alpha(\phi_{\cdot\beta}) + F^{(2)}_\beta(\phi_{\alpha\cdot}).$$

Now in the first term β is just a parameter and in the second term α is. For a non-linear F the above object does not in general transform appropriately in the pair of indices (α, β) and so the canonically lifted two-particle equation is not symmetric. The only way to recover symmetry is to introduce two-particle generators either in the equations (which is the more likely) or the infinitesimal symmetry. But the same problem will then arise at three-particles and by repeating the argument one sees that in general one will need to introduce new generators at each particle number. Thus in contrast to the case of no internal degrees of freedom where one can construct symmetric hierarchies with just a finite number of generators, for theories with internal degrees of freedom one needs an infinite number. For such theories to be tenable, one must either abandon the symmetry at some particle number or introduce some principle that would systematically pick out the needed generators.

The problem is most acute for theories with spin. One imposes rotation invariance on the grounds that space itself is isotropic. In linear theories, once the one-particle equation is chosen and has the appropriate transformation property with respect to the rotation group, then the multi-particle equations are unique and automatically have the right (tensorial) transformation properties. In the non-linear case one must assert or deny rotation invariance for each particle number. To deny it at any point would call in question the very idea of space isotropy, and to assert it universally one must then make infinite choices of generators for the hierarchy of equations. This is another challenge that non-linear theories must meet.

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