# Theory of Economic Equilibrium 

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#### Abstract

New concepts of economics such as an average demand matrix of society, strategy of a firm and consumer behaviour, and others are introduced. We give sufficient conditions for technological mapping under which there exist both the Walras equlibrium state and optimal Walras equilibrium one. We obtain the set of equations which equilibrium price vector solves. The theory of interindustry economic equilibrium is developed. The model of economy with regular interests of consumers is proposed.


Key words: average demand vector of consumer, income tax matrix, redistribution profit matrix, matrix of share in firms profits, Walras equilibrium state, economy with regular interests of consumer.

## 1 The Walras equilibrium

In [1-4] we characterized a consumer by her average demand

$$
\bar{x}_{i}(p)=M \xi_{i}(p)=\left(x_{i 1}(p), \ldots, x_{i n}(p)\right)
$$

for a definite period of economy functioning. Let us introduce the notion of an average demand vector of consumers that is more convenient and pithy from economic point of view. As before [1], let $K_{i}\left(p, z_{1}, \ldots, z_{m}\right)$ be the profit function of the $i$-th consumer, then the average profit for a period of the functioning of economy is defined as follows

$$
K_{i}(p)=M K_{i}\left(p, \eta_{1}(p), \ldots, \eta_{m}(p)\right) .
$$

As to definitions and notations see $[1,3]$.
Now let us introduce the vector

$$
\begin{equation*}
\gamma_{i}(p)=\left(\gamma_{i 1}(p), \ldots, \gamma_{i n}(p)\right), \tag{1}
\end{equation*}
$$

which we call the average demand vector of the $i$-th consumer, where

$$
\begin{equation*}
\gamma_{i k}(p)=\frac{p_{k} x_{i k}(p)}{M K_{i}\left(p, \eta_{1}(p), \ldots, \eta_{m}(p)\right)} \tag{2}
\end{equation*}
$$

Find out economic sense of $\gamma_{i}(p)$. It is easy to see

$$
\bar{x}_{i}(p)=\int_{\prod_{j=1}^{m} B_{j}}^{\int} \frac{\int_{(p, z)}^{i} y d \mu_{i}(y)}{\mu_{i}\left(\bar{X}_{(p, z)}^{i}\right)} P_{1, \ldots, m}\left(p, d z_{1}, \ldots, d z_{m}\right),
$$

therefore

$$
\begin{equation*}
\frac{\left\langle p, \bar{x}_{i}(p)\right\rangle}{K_{i}(p)}=\sum_{k=1}^{n} \gamma_{i k}(p)=\frac{1}{K_{i}(p)} \int_{\prod_{j=1}^{m} B_{j}} \frac{\bar{X}_{(p, z)}^{i}}{\int_{i}\left(\bar{X}_{(p, z)}^{i}\right)}\langle p, y\rangle d \mu_{i}(y) \quad P_{1, \ldots, m}\left(p, d z_{1}, \ldots, d z_{m}\right) . \tag{3}
\end{equation*}
$$

From (3) and the definition of $\bar{X}_{(p, z)}^{i}$ there holds the inequality

$$
\frac{\left\langle p, \bar{x}_{i}(p)\right\rangle}{K_{i}(p)} \leq \frac{1}{K_{i}(p)} \int_{\prod_{j=1}^{m} B_{j}} K_{i}\left(p, z_{1}, \ldots, z_{m}\right) P_{1, \ldots, m}\left(p, d z_{1}, \ldots, d z_{m}\right)=1
$$

or

$$
\sum_{k=1}^{n} \gamma_{i k}(p) \leq 1, \quad \gamma_{i k}(p) \geq 0
$$

If a consumer is insatiable then

$$
\sum_{k=1}^{n} \gamma_{i k}(p)=1
$$

Economic sense of $\gamma_{i k}(p)$ is the following: the part of the average profit which the $i$-th consumer spends to buy the $k$-th goods.

Since the number of consumers is $l$, then we describe the society by demand matrix

$$
\left(\begin{array}{cc}
\gamma_{11}(p), \ldots, & \gamma_{1 n}(p)  \tag{4}\\
\ldots & \\
\gamma_{l 1}(p), \ldots, & \gamma_{l n}(p)
\end{array}\right)
$$

In this paper we assume that $\gamma_{i k}(p)$ are continuous functions of the price vector $p=$ $\left(p_{1}, \ldots, p_{n}\right)$. From economic reasons $\gamma_{i k}(p)$ must not depend on scale of prices, therefore we assume that $\gamma_{i k}(p)$ are homogeneous functions of zero degree, that is

$$
\begin{equation*}
\gamma_{i k}(t p)=\gamma_{i k}(p) \tag{5}
\end{equation*}
$$

Without loss of generality we assume that all consumers are insatiable, therefore

$$
\begin{equation*}
\sum_{k=1}^{n} \gamma_{i k}(p)=1, \quad i=\overline{1, l} \tag{6}
\end{equation*}
$$

This assumption seems to restrict the field of application of the proposed model but almost all assertions proved in the paper hold when a more general condition

$$
\sum_{k=1}^{n} \gamma_{i_{k}}(p) \leq 1, \quad \gamma_{i_{k}}(p) \geq 0
$$

is fulfilled [2]. Moreover we describe any $i$-th consumer by the vector of share in profits of firms $\alpha_{i}=\left(\alpha_{i 1}(p), \ldots, \alpha_{i m}(p)\right)$, where $\alpha_{i s}(p)$ is the part of the average profit of the $i$-th consumer which she obtains from share in profit of the $s$-th firm. It may be the part which makes up the gain in the average profit of the $i$-th consumer or dividends of stocks of the $s$-th firm and so on. We describe the society by a matrix of share in profits of $m$ firms $\left\|\alpha_{i j}(p)\right\|_{i=1}^{l} m_{j=1}^{m}$. The condition for the whole average profit of the $s$-th firm to be divided between consumers is the equality

$$
\begin{equation*}
\sum_{i=1}^{l} \alpha_{i s}(p)=1 \tag{7}
\end{equation*}
$$

From economic point of view it is natural to assume that $\alpha_{i s}(p)$ are homogeneous functions of zero degree, that is

$$
\alpha_{i s}(t p)=\alpha_{i s}(p), \quad t>0
$$

Further we also assume that $\alpha_{i s}(p)$ are continuous functions of $p$. On account of zero degree homogeneity of $\gamma_{i k}(p)$ and $\alpha_{i s}(p)$ it is sufficient to define them on the set $P$

$$
P=\left\{p=\left(p_{1}, \ldots, p_{n}\right), \quad p_{i} \geq 0, \quad \sum_{i=1}^{n} p_{i}=1\right\}
$$

We characterize the society by else two matrices: the income tax matrix $\left\|\pi_{i j}(p)\right\|_{i, j=1}^{l}$, and the profit redistribution one $\left\|r_{i j}(p)\right\|_{i, j=1}^{l}$. We also assume that both $\pi_{i j}(p)$ and $r_{i j}(p)$ are continuous on $R_{+}^{n}$ and homogeneous of zero degree. We also suppose that

$$
\begin{gather*}
\sum_{i=1}^{l} \pi_{i j}(p)=1, \quad \sum_{j=1}^{l} r_{j k}(p)=1,  \tag{8}\\
\pi_{i j}(t p)=\pi_{i j}(p), \quad r_{i j}(t p)=r_{i j}(p), \quad t>0
\end{gather*}
$$

The main assumption about the profit function of the $i$-th consumer is that it has the following structure

$$
\begin{gather*}
K_{i}\left(p, z_{1}, \ldots, z_{m}\right)=\sum_{j=1}^{l} \pi_{i j}(p) \sum_{k=1}^{l} r_{j k}(p)\left[\sum_{s=1}^{m} \alpha_{k s}(p)\left\langle y_{s}-x_{s}, p\right\rangle+\left\langle b_{k}, p\right\rangle\right] \\
z_{i}=\left(x_{i}, y_{i}\right), \quad i=\overline{1, m} \tag{9}
\end{gather*}
$$

$b_{k}$ is the initial stock of goods of the $k$-th consumer at the beginning of economy functioning. If $\left(\bar{x}_{j}(p), \bar{y}_{j}(p)\right)$ is the average strategy of the $j$-th firm behaviour, $j=\overline{1, m}$, then

$$
K_{i}(p)=M K_{i}\left(p, \eta_{1}(p), \ldots, \eta_{m}(p)\right)=
$$

$$
\sum_{j=1}^{l} \pi_{i j}(p) \sum_{k=1}^{l} r_{j k}(p)\left[\sum_{s=1}^{m} \alpha_{k s}(p)\left\langle\bar{y}_{s}(p)-\bar{x}_{s}(p), p\right\rangle+\left\langle b_{k}, p\right\rangle\right] .
$$

Let $t_{i}(p)=\left\{t_{i k}(p)\right\}_{k=1}^{n}$ be a realization of the random field $\xi_{i}(p)$ of the $i$-th consumer $i=\overline{1, l}$ and $\left(x_{s}(p), y_{s}(p)\right)=z_{s}(p)$ be a realization of the random field $\eta_{s}(p), s=\overline{1, m}$. Then the profit of the $i$-th consumer, corresponding to behaviour strategies of firms, is given by the formula

$$
D_{i}(p)=K_{i}\left(p, z_{1}(p), \ldots, z_{m}(p)\right) .
$$

The demand vector of the $i$-th consumer, corresponding to the realization of the random field $\xi_{i}(p)$, is the following

$$
\gamma_{i}(p)=\left\{\gamma_{i k}(p)=\frac{t_{i k}(p) p_{k}}{D_{i}(p)}\right\}_{k=1}^{n}, \quad i=\overline{1, l} .
$$

In this case the demand matrix is defined by

$$
\left\|\gamma_{i k}(p)\right\|_{i=1, k=1}^{l, n}=\left\|\frac{t_{i k}(p) p_{k}}{D_{i}(p)}\right\|_{i=1, k=1}^{l, n}
$$

If all consumers are insatiable then the last demand matrix satisfies conditions (5) and (6).

In the further investigation we assume that the state of economy is defined if we know realizations of all consumers and firms random fields or all average strategies of both firms and consumers behaviour. We prove theorems for those realizations of random fields which generate the continuous demand matrix, profit functions of all consumers and the supply vector of the whole society.

For the model of economy with regular interests of consumers the demand matrix, in general, is not continuous. The notion of a consumer demand vector is more convenient than the notion of a consumer behaviour strategy in spite of a simple relation between them. We shall appreciate the use of this notion in the model of economy with regular interests of consumers. Further we consider that the notion of the consumer demand vector is original due to simple economic sense of it. Thus for the whole society the original notion describing consumers is the demand matrix. For example, it is very important on describing opened economic system or credit and income tax regulation of firms behaviour strategies.

From here we consider that the state of economy is defined if we are given:

1) some demand matrix $\left\|\gamma_{i j}(p)\right\|_{i=1}^{l}{ }_{j=1}^{n}$ defined on $P$ and satisfying conditions (5) and (6);
2) the matrix of share in firms profits $\left\|\alpha_{i j}(p)\right\|_{i=1}^{l}{ }_{j=1}^{m}$ defined on $P$ and satisfying condition (7);
3) some income tax matrix $\left\|\pi_{i j}(p)\right\|$ and the profit redistribution matrix $\mid r_{i j}(p) \|_{i, j=1}^{l}$ defined on $P$ and satisfying conditions (8);
4) the profit function of the $i$-th consumer given by formula (9);
5) strategy of behaviour of every $j$-th firm $\left(x_{j}(p), y_{j}(p)\right), \quad j=\overline{1, m}$, defined on $P$, $\left(\left(x_{j}(t p)=x_{j}(p), y_{j}(t p)=y_{j}(p)\right.\right.$, for all $\left.t>0\right)$. Any strategy is defined by its technological mapping and credit policy determining the expenditure set of technological mapping.

For arbitrary strategy of firm behaviour $\left(x_{j}(p), y_{j}(p)\right)$ the net profit of the $i$-th consumer is given by the formula

$$
\begin{equation*}
D_{i}(p)=\sum_{j=1}^{l} \pi_{i j}(p) \sum_{k=1}^{l} r_{i k}(p)\left[\sum_{s=1}^{m} \alpha_{k s}(p)\left\langle y_{s}(p)-x_{s}(p), p\right\rangle+\left\langle b_{k}, p\right\rangle\right] . \tag{10}
\end{equation*}
$$

Let us introduce the vector

$$
\begin{equation*}
\Phi(p)=\left(\Phi_{1}(p), \ldots, \Phi_{n}(p)\right) \tag{11}
\end{equation*}
$$

where

$$
\Phi_{k}(p)=\frac{1}{p_{k}} \sum_{i=1}^{l} \gamma_{i k}(p) D_{i}(p) .
$$

The vector (11) is called the demand vector of society with strategies of firm behaviour $\left(x_{j}(p), y_{j}(p)\right), j=\overline{1, m}$, and the profit function $D_{i}(p)$ of the $i$-th consumer given by the formula (10).

The supply vector of society is denoted by $\psi(p)=\left(\psi_{1}(p), \ldots, \psi_{n}(p)\right)$, where

$$
\begin{aligned}
& \psi_{k}(p)=\sum_{i=1}^{l} b_{i k}+\sum_{j=1}^{m}\left[y_{j k}(p)-x_{j k}(p)\right], \quad b_{i}=\left(b_{i 1}, \ldots, b_{i n}\right), \\
& y_{j}(p)=\left(y_{j 1}(p), \ldots, y_{j n}(p)\right), \quad x_{j}(p)=\left(x_{j 1}(p), \ldots, x_{j n}(p)\right) .
\end{aligned}
$$

It is easy to see that if we choose as strategies of firms behaviour their average strategies $\left(\bar{x}_{j}(p), \bar{y}_{j}(p)\right), \quad j=1, m$, and the demand matrix given by formulae (1), (2), (4) then we obtain that equalities hold

$$
\begin{equation*}
\bar{x}(p)=\sum_{i=1}^{l} \bar{x}_{i}(p)=\bar{\Phi}(p), \quad \bar{y}(p)=\sum_{j=1}^{m} \bar{y}_{j}(p)=\bar{\psi}(p), \tag{12}
\end{equation*}
$$

where $\bar{x}(p)$ is the average demand vector of society, $\bar{y}(p)$ is the average supply vector of society introduced in [1]

$$
\begin{array}{ll}
\bar{\Phi}(p)=\left(\bar{\Phi}_{1}(p), \ldots, \bar{\Phi}_{n}(p)\right), & \bar{\Phi}_{k}(p)=\frac{1}{p_{k}} \sum_{i=1}^{l} \gamma_{i k}(p) K_{i}(p), \\
\bar{\psi}(p)=\left(\bar{\psi}_{1}(p), \ldots, \bar{\psi}_{n}(p)\right), & \bar{\psi}_{k}(p)=\sum_{i=1}^{l} b_{i k}+\sum_{j=1}^{m}\left[\bar{y}_{j k}(p)-\bar{x}_{j k}(p)\right],
\end{array}
$$

in other words

$$
\bar{\psi}(p)=\sum_{k=1}^{l} b_{k}+\sum_{j=1}^{m}\left[\bar{y}_{j}(p)-\bar{x}_{j}(p)\right] .
$$

Definition 1 The economic system is in the state of the Walras equlibrium in some period of its functioning if there exists the price vector $p^{*}, m$ productive processes

$$
\left(x_{i}^{*}\left(p^{*}\right), y_{i}^{*}\left(p^{*}\right)\right), \quad i=\overline{1, m},
$$

$$
x_{i}^{*}\left(p^{*}\right) \in X_{i}, \quad y_{i}^{*}\left(p^{*}\right) \in F_{i}\left(x_{i}^{*}\left(p^{*}\right)\right)
$$

such that

$$
\begin{align*}
\phi\left(p^{*}\right) & \leq \psi\left(p^{*}\right)  \tag{13}\\
\left\langle\phi\left(p^{*}\right), p^{*}\right\rangle & =\left\langle\psi\left(p^{*}\right), p^{*}\right\rangle \tag{14}
\end{align*}
$$

where $p^{*} \in P, \quad p^{*}=\left(p_{1}^{*}, \ldots, p_{n}^{*}\right)$ is the equilibrium price vector, $F_{i}(x)$ is technological mapping of the $i$-th firm.

Inequality (13) means that society demand for the equlibrium price vector $p^{*}$ does not exceed supply of society, and equality (14) means that the cost of goods which society wants to buy is equal to the cost of goods proposed to be consumed.

Definition 2 The Walras equilibrium state is called the optimum one if it is the Walras equilibrium, and moreover

$$
\left\langle y_{i}^{*}\left(p^{*}\right)-x_{i}^{*}\left(p^{*}\right), p^{*}\right\rangle=\sup _{x \in X_{i}} \sup _{y \in F_{i}(x)}\left\langle y-x, p^{*}\right\rangle, \quad i=\overline{1, m}
$$

where $X_{i}$ is the expenditure set of the $i$-th firm, $F_{i}(x)$ is its technological mapping.
In the following theorem we assume that $\gamma_{i k}(p)=p_{k} \gamma_{i k}^{\circ}(p)$. In further theorems we remove this restriction.

Theorem 1 Let $F_{i}(x)$ be the technological mapping of the $i$-th firm defined on $X_{i}$ which is a closed convex set in $R_{+}^{n}$. Every technological mapping $F_{i}(x)$ takes the values in $2^{S}$ which are a convex compact subset of $S$, and there exists the compact $Y \in S$ such that

$$
F_{i}(x) \subseteq Y, \quad \forall i, \quad \forall x \in X_{i}
$$

As before, $S$ is the set of all feasible goods. Moreover let $F_{i}(x)$ be convex into down technological mapping on $X_{i}$. If $\alpha_{i j}(p), \gamma_{i k}^{\circ}(p), \pi_{i l}(p), r_{i l}(p)$ are continuous functions of $p \in P$ for all $i, j, k$ and safisfy conditions (5-6), (7-8) then there exists the vector $p^{*} \in P$ for which the economic system is in the Walras equilibrium state. In addition if technological mappings $F_{i}(x), i=\overline{1, m}$ are Kakutani continuous into up then there exists the optimum Walras equilibrium state.

Proof For simplicity here and further we assume that $S$ is a cube of sufficiently large sizes in $R_{+}^{n}$ that is

$$
S=\left\{x=\left(x_{1}, \ldots, x_{n}\right), \quad 0 \leq x_{i} \leq c\right\}
$$

Owing to lemma 9 from [1] for sufficiently small $\varepsilon>0$ there exist $m$ continuous firms behaviour strategies $\left(x_{i}^{\varepsilon}(p), y_{i}^{\varepsilon}(p)\right), \quad i=\overline{1, m}$ where $x_{i}^{\varepsilon}(p)$ is the input vector, $y_{i}^{\varepsilon}(p)$ is the output vector of the $i$-th firm, $y_{i}^{\varepsilon}(p) \in F_{i}\left(x_{i}^{\varepsilon}(p)\right)$, for which

$$
\forall i=\overline{1, m} \quad \sup _{p \in P}\left|\varphi_{i}(p)-\left\langle y_{i}^{\varepsilon}(p)-x_{i}^{\varepsilon}(p), p\right\rangle\right|<\varepsilon
$$

where

$$
\varphi_{i}(p)=\sup _{x \in X_{i}} \sup _{y \in F_{i}(x)}\langle y-x, p\rangle
$$

Let us assume that firms choose strategies of behaviour $\left(x_{i}^{\varepsilon}(p), y_{i}^{\varepsilon}(p)\right), i=\overline{1, m}$. Then the net profit of the $i$-th consumer is

$$
D_{i}^{\varepsilon}(p)=\sum_{j=1}^{l} \pi_{i j}(p) \sum_{k=1}^{l} r_{j k}(p)\left[\sum_{s=1}^{m} \alpha_{k s}(p)\left\langle y_{j}^{\varepsilon}(p)-x_{j}^{\varepsilon}(p), p\right\rangle+\left\langle b_{k}, p\right\rangle\right] .
$$

Demand vector of the whole society is $\phi^{\varepsilon}(p)=\left(\phi_{1}^{\varepsilon}(p), \ldots, \phi_{n}^{\varepsilon}(p)\right)$, where

$$
\phi_{k}^{\varepsilon}(p)=\frac{1}{p_{k}} \sum_{i=1}^{l} \gamma_{i k}(p) D_{i}^{\varepsilon}(p)
$$

Supply vector for chosen strategies of firms behaviour is

$$
\psi^{\varepsilon}(p)=\left(\psi_{1}^{\varepsilon}(p), \ldots, \psi_{n}^{\varepsilon}(p)\right)
$$

where

$$
\begin{gathered}
\psi_{k}^{\varepsilon}(p)=\sum_{i=1}^{l} b_{i k}+\sum_{j=1}^{m}\left[y_{j k}^{\varepsilon}(p)-x_{j k}^{\varepsilon}(p)\right] \\
y_{j}^{\varepsilon}(p)=\left(y_{j 1}^{\varepsilon}(p), \ldots, y_{j n}^{\varepsilon}(p)\right), \quad x_{j}^{\varepsilon}(p)=\left(x_{j 1}^{\varepsilon}(p), \ldots, x_{j n}^{\varepsilon}(p)\right)
\end{gathered}
$$

Let us consider the vector function

$$
\tau^{\varepsilon}(p)=\phi^{\varepsilon}(p)-\psi^{\varepsilon}(p)
$$

Note that $\tau^{\varepsilon}(p)$ is a continuous vector function on $P$ satisfying the equality

$$
\left\langle\tau^{\varepsilon}(p), p\right\rangle=0
$$

for all $p \in P$. The last equality follows from the structure of vector functions $\psi^{\varepsilon}(p)$ and $\phi^{\varepsilon}(p)$. It is called the Walras law in the weak sense. Let us construct auxiliary mapping $\alpha^{\varepsilon}(p)=\left(\alpha_{i}^{\varepsilon}(p), \ldots, \alpha_{n}^{\varepsilon}(p)\right)$ on $P$

$$
\alpha_{k}^{\varepsilon}(p)=\frac{p_{k}+\max \left(0, \tau_{k}^{\varepsilon}(p)\right)}{\sum_{k=1}^{n}\left[p_{k}+\max \left(0, \tau_{k}^{\varepsilon}(p)\right)\right]}
$$

The mapping $\alpha^{\varepsilon}(p)$ is continuous on $P$ and maps $P$ into itself. Due to convexity of $P$ and the Brower theorem there exists the fixed point of this mapping $p^{*} \in P$ such that $\alpha^{\varepsilon}\left(p^{*}\right)=p^{*}$. Thus

$$
p_{k}^{*}=\frac{p_{k}^{*}+\max \left(0, \tau^{\varepsilon}\left(p^{*}\right)\right)}{\sum_{k=1}^{n}\left[p_{k}^{*}+\max \left(0, \tau^{\varepsilon}\left(p^{*}\right)\right)\right]}, \quad k=\overline{1, n}
$$

or

$$
\max \left(0, \tau_{k}^{\varepsilon}\left(p^{*}\right)\right)=p_{k}^{*} \sum_{k=1}^{n} \max \left(0, \tau_{k}^{\varepsilon}\left(p^{*}\right)\right)
$$

If we multiply the last equality by $\tau_{k}^{\varepsilon}\left(p^{*}\right)$ and sum from 1 to $n$ we obtain

$$
\sum_{k=1}^{n} \tau_{k}^{\varepsilon}\left(p^{*}\right) \max \left(0, \tau_{k}^{\varepsilon}\left(p^{*}\right)\right)=\sum_{k=1}^{n} p_{k}^{*} \tau_{k}^{\varepsilon}\left(p^{*}\right) \sum_{k=1}^{n} \max \left(0, \tau_{k}^{\varepsilon}\left(p^{*}\right)\right)
$$

From the Walras law it follows

$$
\sum_{k=1}^{n} p_{k}^{*} \tau_{k}^{*}\left(p^{*}\right)=0
$$

whence

$$
\sum_{k=1}^{n} \tau_{k}^{\varepsilon}\left(p^{*}\right) \max \left(0, \tau_{k}^{\varepsilon}\left(p^{*}\right)\right)=0
$$

or $\tau_{k}^{\varepsilon}\left(p^{*}\right) \leq 0$. So we have $\phi^{\varepsilon}\left(p^{*}\right) \leq \psi^{\varepsilon}\left(p^{*}\right)$.
Notice 1 If no component of $p^{*}$ equals to zero then

$$
\phi^{\varepsilon}\left(p^{*}\right)=\psi^{\varepsilon}\left(p^{*}\right) .
$$

Really, from the Walras law in weak sense there follows

$$
\sum_{k=1}^{n}\left[\phi_{k}^{\varepsilon}\left(p^{*}\right)-\psi_{k}^{\varepsilon}\left(p^{*}\right)\right] p_{k}^{*}=0
$$

Since $p_{k}^{*} \neq 0$ for all $k=\overline{1, n}$ then

$$
\phi_{k}^{\varepsilon}\left(p^{*}\right)=\psi_{k}^{\varepsilon}\left(p^{*}\right), \quad k=\overline{1, n} .
$$

Let us prove the second part of Theorem 1, using Kakutani continuity of technological mappings. Choose some sequence $\varepsilon_{n}$ converging to zero. For every $\varepsilon_{n}>0$ there exists the equilibrium price vector $p_{n}^{*}$ such that

$$
\phi^{\varepsilon_{n}}\left(p_{n}^{*}\right) \leq \psi^{\varepsilon_{n}}\left(p_{n}^{*}\right)
$$

moreover

$$
\begin{equation*}
\sup _{p \in P}\left|\varphi_{i}(p)-\left\langle y_{i}^{\varepsilon_{n}}(p)-x_{i}^{\varepsilon_{n}}(p), p\right\rangle\right|<\varepsilon_{n} \tag{15}
\end{equation*}
$$

for all $i=\overline{1, m}$. Without loss of generality we assume that the sequences of vectors $x_{i}^{\varepsilon_{n}}\left(p_{n}^{*}\right)$, $y_{i}^{\varepsilon_{n}}\left(p_{n}^{*}\right), p_{n}^{*}$ are convergent.

Really, if the last assumption is not valid then due to compactness of $P, X_{i}$ and existence of the common compact $Y$ such that $F_{i}(x) \subseteq Y$ we may choose such sequences for which it holds.

Let us denote

$$
\lim _{n \rightarrow \infty} p_{n}^{*}=p^{*}, \quad \lim _{n \rightarrow \infty} y_{i}^{\varepsilon_{n}}\left(p_{n}^{*}\right)=\bar{y}_{i}\left(p^{*}\right), \quad \lim _{n \rightarrow \infty} x_{i}^{\varepsilon_{n}}\left(p_{n}^{*}\right)=\bar{x}_{i}\left(p^{*}\right) .
$$

From Kakutani continuity into up and (15) there follows

$$
\bar{x}_{i}\left(p^{*}\right) \subseteq X_{i}, \quad \bar{y}_{i}\left(p^{*}\right) \subseteq F\left(\bar{x}_{i}\left(p^{*}\right)\right)
$$

and

$$
\varphi_{i}\left(p^{*}\right)=\left\langle\bar{y}_{i}\left(p^{*}\right)-\bar{x}_{i}\left(p^{*}\right), p^{*}\right\rangle, \quad i=\overline{1, m} .
$$

Taking the limit we obtain

$$
\begin{equation*}
\phi\left(p^{*}\right) \leq \phi\left(p^{*}\right) . \tag{16}
\end{equation*}
$$

Theorem 1 is proved.
Notice 2 If $p^{*}>0$ that is all components of $p^{*}$ are positive then in (16) only equality takes place.
Notice 3 Proving Theorem 1 we use continuous strategies of firms behaviour closed to the optimal one. This is necessary for the proof of the second part of the Theorem 1. In the proof of the first part of Theorem 1 we could choose any continuous strategies of firms behaviour. From the proof of the Theorem 1 it follows that for any continuous demand matrix, defining consumers behaviour strategies, and for any continuous strategies of firms behaviour there exists its Walras equilibrium state. Therefore if we suppose that random fields $\xi_{i}(p), \quad i=\overline{1, l}, \quad \eta_{j}(p), \quad j=\overline{1, m}$ are continuous with probability equal to one then for almost all realizations of random fields, describing economy, there exists its Walras equilibrium state. Moreover it is necessary to note that, in general, optimal Walras equilibrium states can be absent among them. In general, random fields describing economy, can be such that for not all realizations the Walras equilibrium state exists. The investigation of this question is a separate problem.

Lemma 1 Let

$$
\gamma_{i k}(p)=\frac{p_{k}}{p_{k}+\varepsilon} \gamma_{i k}^{\circ}(p)
$$

where $\varepsilon>0, \gamma_{i k}^{\circ}(p)$ are continuous functions on $P$. There exists $\delta>0$ independent of $p \in P$ such that

$$
\sum_{i=1}^{l} \gamma_{i k}^{\circ}(p) \geq \delta, \quad \forall p \in P, \quad k=\overline{1, n}
$$

Moreover if there exists $d>0$ independent of $p$ such that

$$
D_{i}(p) \geq d>0
$$

and

$$
0<\triangle<\infty, \quad \triangle=\max _{k} \sup _{p \in P} \psi_{k}(p)
$$

then any solution of the inequality

$$
\begin{equation*}
\phi(p) \leq \psi(p) \tag{17}
\end{equation*}
$$

satisfies inequalities

$$
p_{k}>\frac{\delta}{2} \frac{d}{\triangle}, \quad p=\left(p_{1} \ldots p_{n}\right)
$$

for $0 \leq \varepsilon \leq \frac{\delta}{2} \frac{d}{\triangle}$.
Proof From conditions of Lemma 1 it follows that if for any $p$ inequality (17) holds then the $k$-th component of the vector $p$ satisfies the inequality

$$
\begin{equation*}
\frac{\delta d}{p_{k}+\varepsilon}<\triangle \tag{18}
\end{equation*}
$$

Solving (18) we obtain

$$
p_{k}>\frac{\delta d}{\triangle}-\varepsilon
$$

Since

$$
0 \leq \varepsilon \leq \frac{\delta}{2} \frac{d}{\triangle}
$$

then we prove the assertion of Lemma 1.
Consequence If

$$
\gamma_{i k}(p)=\gamma_{i k}^{\circ}(p),
$$

where $\gamma_{i k}^{\circ}(p)$ are continuous functions on $P$, there exists $\delta>0$ independent of $p \in P$ such that

$$
\sum_{i=1}^{l} \gamma_{i k}^{\circ}(p) \geq \delta, \quad \forall p \in P, \quad \forall k=\overline{1, n}
$$

moreover, there exists $d>0$ independent of $p$ and $0<\Delta<\infty$ such that

$$
\max _{k} \sup _{p \in P} \psi_{k}(p)=\triangle
$$

then any solution $p$ of inequality (17) with $\gamma_{i k}=\gamma_{i k}^{\circ}(p)$ satisfies inequalities

$$
p_{k}>\frac{\delta}{2} \frac{d}{\triangle}, \quad k=\overline{1, n}, \quad p=\left(p_{1}, \ldots, p_{n}\right) .
$$

At first sight the assumption about a structure of the demand matrix

$$
\gamma_{i k}(p)=p_{k} \gamma_{i k}^{\circ}(p)
$$

in the Theorem 1 seems to be restrictive. It restricts behaviour of $\gamma_{i k}(p)$ in the neighbourhood of zero only. In the next theorem we remove this assumption, moreover we obtain the set of equations for the equilibrium price vector $p$.

Theorem 2 Let $\gamma_{i k}(p), \alpha_{i j}(p), r_{i j}(p), \pi_{i j}(p)$ be continuous functions of $p \in P$ defined before. Technological mappings $F_{i}(x) \quad i=\overline{1, m}$ satisfy conditions of Theorem 1. Moreover let $\left(x_{i}(p), y_{i}(p)\right), \quad i=\overline{1, m}$ be continuous strategies of firms behaviour. If they are such that

$$
D_{i}(p)>0, \quad \sum_{i=1}^{m} y_{i}(p)-x_{i}(p)+\sum_{i=1}^{l} b_{i}>0, \quad p \in P
$$

then there exists the equilibrium price vector $\bar{p}$ for which the economic system is in the Walras equilibrium state and satisfies the set of equations

$$
\sum_{i=1}^{l} \gamma_{i k}(\bar{p}) D_{i}(\bar{p})=\bar{p}_{k}\left[\sum_{i=1}^{m}\left[y_{i}(\bar{p})-x_{i}(\bar{p})\right]_{k}+\sum_{i=1}^{l} b_{i k}\right] .
$$

$\mathbf{P r o o f}$ Let us consider an auxiliary sequence of the demand matrix with matrix elements

$$
\bar{\gamma}_{i k}^{\varepsilon_{n}}(p)=\frac{p_{k}}{p_{k}+\varepsilon_{n}}\left(\gamma_{i k}(p)+\delta_{1}\right)\left(\sum_{l=1}^{n} \frac{p_{l}}{p_{l}+\varepsilon_{n}}\left(\gamma_{i l}(p)+\delta_{1}\right)\right)^{-1}=p_{k} \bar{\gamma}_{i k}^{\circ, \varepsilon_{n}}(p)
$$

where

$$
\begin{gathered}
\bar{\gamma}_{i k}^{\mathrm{o}, \varepsilon_{n}}(p)=\frac{1}{p_{k}+\varepsilon_{n}}\left(\gamma_{i k}(p)+\delta_{1}\right)\left(\sum_{l=1}^{n} \frac{p_{l}}{p_{l}+\varepsilon_{n}}\left(\gamma_{i l}(p)+\delta_{1}\right)\right)^{-1}, \\
\delta_{1}>0, \quad \varepsilon_{n}>0, \quad \varepsilon_{n} \rightarrow 0 .
\end{gathered}
$$

For every fixed $\delta_{1}>0, \quad \varepsilon_{n}>0, \quad \bar{\gamma}_{i k}^{0, \varepsilon_{n}}(p)$ are continuous functions on $P$, satisfy condition (6) and the inequality

$$
\begin{equation*}
\sum_{i=1}^{l} \bar{\gamma}_{i k}^{0, \varepsilon_{n}}(p) \geq \frac{l \delta_{1}}{1+\delta_{1} n} \frac{1}{p_{k}+\varepsilon_{n}} \tag{19}
\end{equation*}
$$

At the beginning we assume that the $i$-th consumer has the initial net profit

$$
\tilde{D}_{i}(p)=D_{i}(p)+d\langle e, p\rangle
$$

where $d>0, e=(1, \ldots, 1)$. Let us denote

$$
\begin{aligned}
\phi_{k}^{\varepsilon_{n}}(p) & =\frac{1}{p_{k}} \sum_{i=1}^{l} \bar{\gamma}_{i k}^{\varepsilon_{n}}(p) \tilde{D}_{i}(p), & \phi^{\varepsilon_{n}}(p) & =\left(\phi_{k}^{\varepsilon_{n}}(p)\right)_{k=1}^{n} \\
\psi_{k}^{d}(p) & =\sum_{j=1}^{m}\left[y_{j}(p)-x_{j}(p)\right]_{k}+\sum_{j=1}^{l}\left[b_{j k}+d\right], & \psi^{d}(p) & =\left(\psi_{k}^{d}(p)\right)_{k=1}^{n}
\end{aligned}
$$

According to Theorem 1 there exists the equilibrium price vector $p_{\varepsilon_{n}}^{*}$ such that

$$
\phi^{\varepsilon_{n}}\left(p_{\varepsilon_{n}}^{*}\right) \leq \psi^{d}\left(p_{\varepsilon_{n}}^{*}\right)
$$

From inequality (19) and from

$$
\tilde{D}_{i}(p) \geq d\langle p, e\rangle=d>0
$$

on P ,

$$
\max _{p \in P, k}\left[\sum_{i=1}^{m}\left[y_{i}(p)-x_{i}(p)\right]_{k}+\sum_{j=1}^{l}\left(b_{j k}+d\right)\right]=\triangle<\infty
$$

it follows that the conditions of Lemma 1 hold if $\varepsilon_{n}$ satisfies the inequality

$$
0 \leq \varepsilon_{n}<\frac{1}{2} \frac{\delta d}{\triangle}=\frac{1}{2} \frac{l d \delta_{1}}{\left(1+\delta_{1} n\right) \triangle}
$$

From Lemma 1 components $p_{\varepsilon_{n}, k}^{*}$ of $p_{\varepsilon_{n}}^{*}$ satisfy inequalities

$$
p_{\varepsilon_{n}, k}^{*}>\frac{1}{2} \frac{l \delta_{1} d}{\left(1+\delta_{1} n\right) \triangle}
$$

Since $p_{\varepsilon_{n}}^{*}$ is the equilibrium price vector then according to the notice 1 to Theorem 1 we have

$$
\begin{equation*}
\phi^{\varepsilon_{n}}\left(p_{\varepsilon_{n}}^{*}\right)=\psi^{d}\left(p_{\varepsilon_{n}}^{*}\right) \tag{20}
\end{equation*}
$$

Without loss of generality we assume that $p_{\varepsilon_{n}}^{*}$ is a convergent sequence on account of $p$ is compact. Let $p^{*}\left(d, \delta_{1}\right)$ be the limit of a sequence $p_{\varepsilon_{n}}^{*}$, that is

$$
p^{*}\left(d, \delta_{1}\right)=\lim _{\varepsilon_{n} \rightarrow 0} p_{\varepsilon_{n}}^{*}
$$

The 1.h.s. of (20) has the limit

$$
\phi\left(p^{*}\right)=\left(\frac{1}{p_{k}^{*}} \sum_{i=1}^{l} \bar{\gamma}_{i k}\left(p^{*}\right) \tilde{D}_{i}\left(p^{*}\right)\right)_{k=1}^{n} .
$$

The r.h.s. of (20) has the limit too

$$
\psi^{d}\left(p^{*}\right)=\left(\sum_{i=1}^{m}\left[y_{i}\left(p^{*}\right)-x_{i}\left(p^{*}\right)\right]_{k}+\sum_{j=1}^{l}\left(b_{j_{k}}+d\right)\right)_{k=1}^{n}
$$

where

$$
\bar{\gamma}_{i k}(p)=\frac{\gamma_{i k}(p)+\delta_{1}}{1+n \delta_{1}} .
$$

Thus $\phi\left(p^{*}\right)=\psi^{d}\left(p^{*}\right)$. Multiplying the equality for the $k$-th component by $p_{k}^{*}$ we obtain

$$
\begin{equation*}
\sum_{i=1}^{l} \bar{\gamma}_{i k}\left(p^{*}\right) \tilde{D}_{i}\left(p^{*}\right)=p_{k}^{*}\left[\sum_{i=1}^{m}\left[y_{i}\left(p^{*}\right)-x_{i}\left(p^{*}\right)\right]_{k}+\sum_{j=1}^{l}\left(b_{j_{k}}+d\right)\right], \quad k=\overline{1, n} . \tag{21}
\end{equation*}
$$

The vector $p^{*}$ depends on $d$ and $\delta_{1}$. Let us choose sequences $d_{m}>0, \delta_{1, m}>0$ which converge to zero. Let $p_{m}^{*}$ be the sequence of equlibrium price vectors which corresponds to $d_{m}$ and $\delta_{1, m}$. This sequence satisfies the set of equations (21) if instead of $d$ and $\delta_{1}$ to choose $d_{m}$ and $\delta_{1, m}$ respectively. We may consider that $p_{m}^{*} \in P$ converges to $\bar{p} \in P$. Taking the limit in the left and right sides of (21) we show the existence of the equilibrium price vector $\bar{p}$ which satisfies the set of equations

$$
\begin{gather*}
\sum_{i=1}^{l} \bar{\gamma}_{i k}(p) D_{i}(p)=p_{k}\left[\sum_{i=1}^{m}\left[y_{i}(\bar{p})-x_{i}(\bar{p})\right]_{k}+\sum_{j=1}^{l} b_{j k}\right],  \tag{22}\\
k=\overline{1, n}, \quad \bar{p} \in P, \quad \bar{p} \neq 0 .
\end{gather*}
$$

Till now we nowhere used any properties of the demand vector. Now suppose that strategies of firms behaviour satisfy the condition

$$
\begin{equation*}
\sum_{i=1}^{m}\left[y_{i}(p)-x_{i}(p)\right]+\sum_{j=1}^{l} b_{j}>0, \tag{23}
\end{equation*}
$$

that is every commodity consumed is produced by the certain firm or was produced earlier.
From the assumption of Theorem $D_{i}(p)>0, p \in P$, therefore an additional assumption on the supply vector (23) permits to assert that $\bar{p}_{s}=0$ for some $1 \leq s \leq n$ if and only if

$$
\forall i=\overline{1, l} \quad \gamma_{i s}(\bar{p})=0 .
$$

Really, from the fact that $\bar{p}_{s}=0$ and $\min _{i} D_{i}(\bar{p})>0$ it follows

$$
\min _{i} D_{i}(\bar{p}) \sum_{i=1}^{l} \gamma_{i s}(\bar{p}) \leq 0,
$$

or

$$
\sum_{i=1}^{l} \gamma_{i s}(\bar{p})=0 \quad \Rightarrow \quad \gamma_{i s}(\bar{p})=0, \quad i=\overline{1, l}
$$

On the contrary, from the fact that $\gamma_{i_{s}}(\bar{p})=0, i=\overline{1, l}$ it follows $\bar{p}_{s}=0$. Indeed, then

$$
\bar{p}_{s}\left[\sum_{i=1}^{m}\left[y_{i}(\bar{p})-x_{i}(\bar{p})\right]_{s}+\sum_{j=1}^{l} b_{j s}\right]=0
$$

from where $\bar{p}_{s}=0$. The last assertion has a very important economic sense. Price of produced goods is equal to zero if and only if the demand of it is equal to zero .

Thus, if the matrix $\left\|\gamma_{i k}(\bar{p})\right\|_{i=1}^{l}{ }_{k=1}^{m}$ is such that no column of this matrix equals to zero then no component $\bar{p}_{s}=0, s=\overline{1, m}$. For the price vector $\bar{p}>0$ demand and suply of the $s$-th commodity are

$$
\begin{gathered}
\phi_{s}(\bar{p})=\frac{1}{\bar{p}_{s}} \sum_{i=1}^{l} \gamma_{i s}(\bar{p}) D_{i}(\bar{p}) \\
\psi_{s}(\bar{p})=\sum_{i=1}^{m}\left[y_{i}(\bar{p})-x_{i}(\bar{p})\right]_{s}+\sum_{j=1}^{l} b_{j s}
\end{gathered}
$$

respectively. From (22) it follows that

$$
\phi_{s}(\bar{p})=\psi_{s}(\bar{p}), \quad s=\overline{1, n}
$$

In this case Theorem is proved. If for clarity the first $t$ columns of the matrix $\left\|\gamma_{i k}(\bar{p})\right\|$ are equal to zero and others are nonzero then

$$
\bar{p}_{1}=\bar{p}_{2}=, \ldots,=\bar{p}_{t}=0, \quad \bar{p}_{t+1}>0, \ldots, \bar{p}_{n}>0
$$

Equilibrium demand and supply vectors have the form

$$
\phi=\left(\phi_{1}(\bar{p}), \ldots, \phi_{n}(\bar{p})\right)
$$

where

$$
\begin{gathered}
\phi_{s}(\bar{p})=0, \\
\phi_{s}(\bar{p})=\frac{1}{p_{s}} \sum_{i=1}^{l} \gamma_{i s}(\bar{p}) D_{i}(\bar{p}), \quad \overline{1, t}, \\
\psi=\left(\psi_{1}(\bar{p}), \ldots \psi_{n}(\bar{p})\right), \\
\psi_{s}(\bar{p})=\sum_{i=1}^{m}\left[y_{j}(\bar{p})-x_{j}(\bar{p})\right]_{s}+\sum_{j=1}^{l} b_{j s}, \\
\phi(\bar{p}) \leq \psi(\bar{p}), \quad\langle\phi(\bar{p}), \bar{p}\rangle=\langle\psi(\bar{p}), \bar{p}\rangle .
\end{gathered}
$$

Theorem is proved.
Theorem $3 \operatorname{Let} \gamma_{i k}(p), \alpha_{i j}(p), \pi_{i j}(p), r_{i j}(p)$ be continuous functions on P. Technological mappings satisfy conditions of Theorem 1. If $\left(x_{i}, y_{i}\right), i=\overline{1, m}$ s some set of productive processes and

$$
\left(x_{i}(p)=x_{i}, y_{i}(p)=y_{i}\right)
$$

is the strategy of behaviour of the $i$-th firm satisfying conditions

$$
D_{i}(p)>0, \quad p \in P, \quad \sum_{i=1}^{m}\left(y_{i}-x_{i}\right)+\sum_{i=1}^{l} b_{i}>0
$$

then there exists an equlibrium price vector $\bar{p}$ which satisfies the set of equations

$$
\sum_{i=1}^{l} \gamma_{i k}(\bar{p}) D_{i}(\bar{p})=p_{k}\left[\sum_{i=1}^{m}\left[y_{i}-x_{i}\right]_{k}+\sum_{i=1}^{l} b_{i k}\right]
$$

This theorem is the consequence of theorem 2 .
Theorem $4 \operatorname{Let} \gamma_{i k}(p), \alpha_{i j}(p), \pi_{i j}(p), r_{i j}(p)$ be continuous functions on P. Technological mappings $F_{i}(x), i=\overline{1, m}$ safisfy conditions of Theorem 1. If any set of optimal strategies of firms behaviour $\left\{x_{i}(p), y_{i}(p)\right\}, i=\overline{1, m}$ is such that

$$
D_{i}(p)>0, \quad \sum_{i=1}^{m}\left[y_{i}(p)-x_{i}(p)\right]+\sum_{i=1}^{l} b_{i}>0, \quad p \in P
$$

then there exists the optimal Walras equilibrium state.
Proof As in the proof of Theorem 2 let us consider an auxiliary sequence of demand matrices $\left\|\bar{\gamma}_{i k}^{0, \varepsilon_{n}}(p)\right\|$,

$$
\bar{\gamma}_{i k}^{\varepsilon_{n}}(p)=\frac{p_{k}}{p_{k}+\varepsilon_{n}}\left(\gamma_{i k}(p)+\delta_{1}\right)\left(\sum_{l=1}^{n} \frac{p_{l}}{p_{l}+\varepsilon_{n}}\left(\gamma_{i l}(p)+\delta_{1}\right)\right)^{-1}=p_{k} \bar{\gamma}_{i k}^{0, \varepsilon_{n}}(p)
$$

Suppose that the net profit of the $i$-th consumer is

$$
D_{i}(p)+d\langle p, e\rangle=\tilde{D}_{i}(p)
$$

Let $\phi^{\varepsilon}(p)$ and $\psi^{d}(p)$ be the same as in the Theorem 2 . Since we assume $F_{i}(x)$ to be Kakutani continuous into up then by Theorem 1 there exists the optimal Walras equilibrium state. As in the proof of Theorem 2 we have

$$
\begin{equation*}
\phi^{\varepsilon_{n}}\left(p_{\varepsilon_{n}}^{*}\right)=\psi^{d}\left(p_{\varepsilon_{n}}^{*}\right) \tag{24}
\end{equation*}
$$

where $p_{\varepsilon_{n}}^{*}$ is the equilibrium price vector

$$
\begin{gathered}
\phi^{\varepsilon_{n}}\left(p_{\varepsilon_{n}}^{*}\right)=\left(\phi_{k}^{\varepsilon_{n}}\left(p_{\varepsilon_{n}}^{*}\right)\right)_{k=1}^{n}, \quad \phi_{k}^{\varepsilon_{n}}\left(p_{\varepsilon_{n}}^{*}\right)=\frac{1}{p_{k, \varepsilon_{n}}^{*}} \sum_{i=1}^{l} \bar{\gamma}_{i k}^{\varepsilon_{n}}\left(p_{\varepsilon_{n}}^{*}\right) \tilde{D}\left(p_{\varepsilon_{n}}^{*}\right) \\
\psi^{d}\left(p_{\varepsilon_{n}}^{*}\right)=\left(\psi_{k}^{d}\left(p_{\varepsilon_{n}}^{*}\right)\right)_{k=1}^{n}, \quad \psi_{k}^{d}\left(p_{\varepsilon_{n}}^{*}\right)=\sum_{j=1}^{m}\left[\bar{y}_{j}\left(p_{\varepsilon_{n}}^{*}\right)-x_{j}\left(p_{\varepsilon_{n}}^{*}\right)\right]+\sum_{j=1}^{l}\left(b_{j_{k}}+d\right) \\
\bar{x}_{j}\left(p_{\varepsilon_{n}}^{*}\right) \subseteq X_{i}, \quad \bar{y}_{j}\left(p_{\varepsilon_{n}}^{*}\right) \in F\left(\bar{x}_{j}\left(p_{\varepsilon_{n}^{*}}\right)\right. \\
\left.\varphi_{j}\left(p_{\varepsilon_{n}}^{*}\right)=\left\langle\bar{y}_{j}\left(p_{\varepsilon_{n}}^{*}\right)-\bar{x}_{j}\left(p_{\varepsilon_{n}}^{*}\right), p_{\varepsilon_{n}}^{*}\right)\right\rangle \\
\varphi_{j}(p)=\sup _{x \in X_{j}} \sup _{y \in F_{j}(x)}\langle y-x, p\rangle, \quad j=\overline{1, m}
\end{gathered}
$$

Without loss of generality we assume that

$$
p_{\varepsilon_{n}}^{*} \rightarrow p^{*}, \quad \bar{x}_{j}\left(p_{\varepsilon_{n}}^{*}\right) \rightarrow \bar{x}_{j}\left(p^{*}\right), \quad \bar{y}_{j}\left(p_{\varepsilon_{n}}^{*}\right) \rightarrow \bar{y}_{j}\left(p^{*}\right)
$$

as $\varepsilon \rightarrow 0$. Left and right sides of equality (24) have limits and this equality is preserved under taking limit, moreover all the conditions of Lemma 1 hold. Multiplying the $k$-th component of equality (24) by $p_{k}$ it may be written in the form

$$
\begin{gather*}
\sum_{i=1}^{l} \bar{\gamma}_{i k}\left(p^{*}\right)\left\{\sum_{j=1}^{l} \pi_{i j}\left(p^{*}\right) \sum_{k=1}^{l} r_{j k}\left(p^{*}\right) \times\right. \\
\left.\left[\sum_{s=1}^{m} \alpha_{k s}\left(p^{*}\right)\left\langle\bar{y}_{s}\left(p^{*}\right)-\bar{x}_{s}\left(p^{*}\right), p^{*}\right\rangle+\left\langle b_{k}, p^{*}\right\rangle\right]+\langle p, e\rangle d\right\}= \\
p_{k}^{*}\left[\sum_{j=1}^{m}\left[\bar{y}_{j}\left(p^{*}-\bar{x}_{j}\left(p^{*}\right)\right]_{k}+\sum_{j=1}^{l}\left(b_{j k}+d\right)\right], \quad k=\overline{1, n}\right. \\
\bar{x}_{j}\left(p^{*}\right) \subseteq X_{j}, \quad \bar{y}_{i}\left(p^{*}\right) \subseteq F\left(\bar{x}_{j}\left(p^{*}\right)\right) \\
\varphi_{j}\left(p^{*}\right)=\left\langle\bar{y}_{j}\left(p^{*}\right)-\bar{x}_{j}\left(p^{*}\right), p^{*}\right\rangle \\
\bar{\gamma}_{i k}(p)=\frac{\gamma_{i k}(p)+\delta_{1}}{1+n \delta_{1}} . \tag{25}
\end{gather*}
$$

The price vector $p^{*}$ depends on $d, \delta_{1}$. The proof of the Theorem is completed by the scheme of the proof of Theorem 2 with the only difference that in Theorem $2\left(x_{j}(p), y_{j}(p)\right)$ is a continuous strategy of behaviour, and it is sufficient to follow convergence of the sequence $p_{m}^{*}$, which corresponds to $d_{m}$ and $\delta_{1 m}$ tending to zero. Here owing to compactness of $X_{i}$ and $Y, F_{i}(x) \subseteq Y, \quad x \in X_{i}, \quad i=\overline{1, m}$, Kakutani continuity of $F_{i}(x)$, let us choose a convergent sequence $p_{m}^{*}$ such that $\bar{x}_{j}\left(p_{m}^{*}\right)$ and $\bar{y}_{j}\left(p_{m}^{*}\right)$ are convergent too. Let us denote

$$
\lim _{m \rightarrow \infty} p_{m}^{*}=\bar{p}, \quad \lim _{m \rightarrow \infty} \bar{y}_{j}\left(p_{m}^{*}\right)=\tilde{y}_{j}(\bar{p}), \quad \lim _{m \rightarrow \infty} \bar{x}_{j}\left(p_{m}^{*}\right)=\tilde{x}_{j}(\bar{p})
$$

From Kakutani continuity of $F_{i}(x)$ and (25) we obtain

$$
\begin{gather*}
\tilde{y}_{s}(\bar{p}) \subseteq F\left(\tilde{x}_{j}(\bar{p})\right), \quad \bar{p} \in P \\
\sum_{i=1}^{l} \gamma_{i k}(\bar{p}) \sum_{j=1}^{l} \pi_{i j}(\bar{p}) \sum_{k=1}^{l} r_{j k}(\bar{p})\left[\sum_{s=1}^{m} \alpha_{k s}(\bar{p}) \varphi_{s}(\bar{p})+\left\langle b_{k}, \bar{p}\right\rangle\right]= \\
\bar{p}_{k}\left[\sum_{j=1}^{m}\left[\tilde{y}_{j}(\bar{p})-\tilde{x}_{j}(\bar{p})\right]_{k}+\sum_{j+1}^{l} b_{j k}\right]  \tag{26}\\
\varphi_{s}(\bar{p})=\left\langle\tilde{y}_{s}(\bar{p})-\tilde{x}_{s}(\bar{p}), \bar{p}\right\rangle
\end{gather*}
$$

It is impossible to consider equalities (26) as the set of equations for finding $\bar{p}$ because $\tilde{x}_{j}(\bar{p})$ and $\tilde{y}_{j}(\bar{p})$ are unknown. The only known thing is that they exist. It is the main lack of both (26) and the Arrow-Debreu theorem at all [5]. Further arguments to prove Theorem are the same as for Theorem 2. For definiteness let the equilibrium price vector
be $\bar{p}=\left(0, \ldots, 0, \bar{p}_{t+1}, \ldots \bar{p}_{n}\right), \bar{p}_{i} \neq 0, t+1 \leq i \leq n$. Then equlibrium demand and supply vectors, for example, are

$$
\begin{gathered}
\phi=\left(\phi_{1}(\bar{p}), \ldots, \phi_{n}(\bar{p})\right) \\
\phi_{s}(\bar{p})=0, \quad 0 \leq s \leq t \\
\phi_{s}(\bar{p})=\frac{1}{\bar{p}_{s}} \sum_{i=1}^{l} \gamma_{i k}(\bar{p}) \sum_{j=1}^{l} \pi_{i j}(\bar{p}) \sum_{k=1}^{l} r_{j k}(\bar{p})\left[\sum_{s=1}^{m} \alpha_{k s}(\bar{p}) \varphi_{s}(\bar{p})+\left\langle b_{k}, \bar{p}\right\rangle\right] \\
t+1 \leq s \leq n \\
\psi=\left(\psi_{1}(\bar{p}), \ldots, \psi_{n}(\bar{p})\right) \\
\psi_{s}(\bar{p})=\sum_{j=1}^{m}\left[\tilde{y}_{j}(\bar{p})-\tilde{x}_{j}(\bar{p})\right]_{k}+\sum_{j=1}^{l} b_{j k}
\end{gathered}
$$

respectively. Moreover

$$
\phi(\bar{p}) \leq \psi(\bar{p}), \quad\langle\phi(\bar{p}), \bar{p}\rangle=\langle\psi(\bar{p}), \bar{p}\rangle \quad \varphi_{j}(\bar{p})=\left\langle\tilde{y}_{j}(\bar{p})-\tilde{x}_{j}(\bar{p}), \bar{p}\right\rangle
$$

Theorem 4 is proved.
Notice If in the Theorem 4 we the assumed that technological mappings $F_{i}(x) i=\overline{1, m}$ are strict convex into down then the optimal strategy of the $i$-th firm behaviour $\left(x_{i}(p), y_{i}(p)\right)$ is unique and continuous (see [1]). In that case in (26) instead of $\left(\tilde{x}_{i}(\bar{p}), \tilde{y}_{i}(\bar{p})\right)$ we should substitute $\left(x_{j}(\bar{p}), y_{j}(\bar{p})\right)$. Then we may consider (26) as the set of equations for finding the equilibrium price vector.

## 2 Theory of interindustry equilibrium

We use here the results of the section 1 to construct a Walras equilibrium model of economy which consists of $n$ interdependent branches or firms. We assume that every branch produces one type of goods which may be used as a raw material to produce other goods or to be consumer goods. Technological mapping is given by the Leontieff "input-output" matrix $A$. With respect to $A$ we assume that there are vanishing neither columns nor rows. Thus, technological mapping $F(x)$ is given by the expenditure set $X$ and for every $x \in X$ by the set of plans

$$
\begin{equation*}
F(x)=\left\{y \in R_{+}^{n}, \quad A y \leq x\right\} \tag{27}
\end{equation*}
$$

Let us describe $X$. Every $i$-th branch in the previous period of functioning has a stock of money $\beta_{i}$. If $p^{\circ}=\left(p_{1}^{\circ} \ldots p_{n}^{\circ}\right)$ is the price vector in this previous period then this branch may purchase any bundle of goods $x \in X_{i}$ where

$$
X_{i}=\left\{x=\left(x_{1}, \ldots, x_{n}\right) \in S, \quad \sum_{j=1}^{n} p_{j}^{\circ} x_{j} \leq \beta_{i}\right\}
$$

Thus, the expenditure set of $n$ branches are

$$
\begin{equation*}
X=X_{1}+X_{2}+\ldots+X_{n} \tag{28}
\end{equation*}
$$

$X$ is a convex polyhedron. Let us consider the set $Q \subseteq S$ which is defined by

$$
Q=\{x \in S, \quad A x<x\}
$$

If $Q$ is a nonempty set then $Q$ is a convex cone in $R_{+}^{n}$. The sufficient condition for $Q$ to be nonempty is the following:

$$
\max _{i} \sum_{j=1}^{n} a_{i j}<1
$$

Really, due to the Perron-Frobenius theorem the maximum eigenvalue of the problem $A x=\lambda x$ is positive and $\lambda<1$. Moreover $x$ is strictly positive, therefore $A x<x$. Further we assume that $Q \cap X \neq \emptyset, \emptyset$ is an empty set. If $x \in Q \cap X$ then in $F(x)$ there exists the output vector $y$ such that $y-x>0$. Indeed, if

$$
x \in Q=\{x \in S, \quad A x<x\}
$$

then to every component of $x$ we can add a sufficiently small positive number that is to add the positive vector $x^{\circ}$ to $x$ such that the inequality

$$
A\left(x+x^{\circ}\right) \leq x
$$

is preserved. Then for $y$ we can choose $x+x^{\circ}$. Let us assume that the economic system chooses strategy of behaviour $(x(p)=x, y(p)=y), p \in P, y \in F(x) . y-x>0$. Then this strategy of behaviour is profitable for all $p \in P$, that is, $\langle y-x, p\rangle>0$. If $x=x_{1}+\ldots+x_{n}$, $x_{i} \in X_{i}$ then the profit of the $i$-th branch is $\varphi_{i}(p)=p_{i} y_{i}-\left\langle p, x_{i}\right\rangle$. If $b_{i}$ is the initial stock of goods of the $i$-th consumer then the net profit of the $i$-th consumer is

$$
\begin{equation*}
D_{i}(p)=\sum_{j=1}^{l} \pi_{i j}(p) \sum_{k=1}^{l} r_{j k}(p)\left[\sum_{s=1}^{n} \alpha_{k_{s}}(p) \varphi_{s}(p)+\left\langle b_{k}, p\right\rangle\right] \tag{29}
\end{equation*}
$$

where $\left\|\pi_{i j}(p)\right\|,\left\|r_{i j}(p)\right\|,\left\|\alpha_{i j}(p)\right\|$ are matrices introduced earlier.
Theorem 5 Let A be the Leontieff "input-output" matrix which describes $n$ branches of economy. Technological mapping is given by (27) on the expenditure set (28). Let $Q \cap X$ be nonempty set. If

$$
(x(p)=x, y(p)=y)
$$

is the strategy of behaviour of $n$ branches described above then there exists the equlibrium price vector $\bar{p} \in P$ safisfying the set of equations

$$
\sum_{i=1}^{l} \gamma_{i k}(\bar{p}) D_{i}(\bar{p})=\bar{p}_{k}\left[(y-x)_{k}+\sum_{i=1}^{l} b_{i k}\right]
$$

provided that $D_{i}(p)>0$ on $P$ for all $i=\overline{1, l}$, where $D_{i}(p)$ is given by formula (29).
$\mathbf{P r o o f}$ This theorem is the consequence of Theorem 2.

## 3 The model of economy with regular interests of consumers

This model is the approximate model of economy which may serve as a zero approximation to further more complicated models. The preferences of statistical description are already revealed in this model. The proposed model of economy is conditionally maked up of $n$ "branches" each of which produces one good. We say conditionally because by a branch we understand, for example, one worker who produces one goods such as labour power. Such an economy we describe by structure matrix $A$ which is completely analogous to the Leontieff matrix. This matrix $A$ is named the matrix of regular interests of consumers which are simultaneously firms. Economic sense of this matrix is the following: the component $a_{k i}$ of the $i$-th column $a_{i}=\left\{a_{k i}\right\}_{k=1}^{n}$, which describes the $i$-th consumer, is the quantity of units of the $k$-th goods to produce one unit of the $i$-th goods. Regularity of interests means that the $i$-th consumer spends all her profit to purchase proportionally all components of the vector $a_{i}$ or, in other words, if $p=\left(p_{1}, \ldots, p_{n}\right)$ is the price vector then the average demand vector of the $i$-th consumer is written in the form

$$
\gamma_{i}(p)=\left\{\frac{a_{k i} p_{k}}{\sum_{l=1}^{n} a_{l i} p_{l}}\right\}_{k=1}^{n}
$$

Let us suppose that in a certain period of the functioning of economy the $i$-th consumer produces $b_{i}$ units of the $i$-th goods. For the price vector $p=\left(p_{1}, \ldots, p_{n}\right)$ the profit, which the $i$-th consumer obtains, will be

$$
D_{i}(p)=b_{i}\left[p_{i}-\sum_{l=1}^{n} a_{l i} p_{l}\right] .
$$

The demand of the whole society for the $k$-th goods can be expressed in the form

$$
\phi_{k}(p)=\frac{1}{p_{k}} \sum_{i=1}^{n} \gamma_{i k}(p) D_{i}(p)=\sum_{i=1}^{n} \frac{a_{k i} b_{i}\left[p_{i}-\sum_{l=1}^{n} a_{l i} p_{l}\right]}{\sum_{l=1}^{n} a_{l i} p_{l}} .
$$

The vector demand of the whole society has the form

$$
\phi=\left\{\phi_{k}(p)\right\}_{k=1}^{n}
$$

The supply vector of the $i$-th consumer is

$$
y_{i}-x_{i}=\left\{b_{i}\left(\delta_{k i}-a_{k i}\right)\right\}_{k=1}^{n}
$$

Thus, the supply vector of the whole society is the following

$$
\psi=\sum_{i=1}^{n}\left(y_{i}-x_{i}\right)=\left\{b_{k}-\sum_{i=1}^{n} b_{i} a_{k i}\right\}_{k=1}^{n}
$$

Then the equilibrium price vector must satisfy the following set of inequalities

$$
\sum_{i=1}^{n} \frac{a_{k i} b_{i}\left[p_{i}-\sum_{l=1}^{n} a_{l i} p_{l}\right]}{\sum_{l=1}^{n} a_{l i} p_{l}} \leq b_{k}-\sum_{i=1}^{n} b_{i} a_{k i}
$$

or

$$
\sum_{i=1}^{n} \frac{a_{k i} b_{i} p_{i}}{\sum_{l=1}^{n} a_{l i} p_{l}} \leq b_{k}
$$

Thus, if $p^{\circ}$ is the Walras equilibrium price vector then it must satisfy such a set of inequalities

$$
\begin{align*}
& \sum_{i=1}^{n} \frac{a_{k i} b_{i} p_{i}}{\sum_{l=1}^{n} a_{l i} p_{l}}-b_{k}=0, \quad k \in I \\
& \sum_{i=1}^{n} \frac{a_{k i} b_{i} p_{i}}{\sum_{l=1}^{n} a_{l i} p_{l}}-b_{k}<0, \quad k \in J \tag{30}
\end{align*}
$$

where $I$ and $J$ are subsets of the set $N=\{1,2, \ldots n\}, \quad I \bigcap J=\emptyset$ is an empty set, $J \bigcup I=N$.

The purpose of the paper is to describe all solutions of the set of inequalities (30). (30) means that demand does not exceed supply for the equlibrium price vector.

Lemma 2 Let the vector $p^{\circ}=\left(p_{1}^{\circ}, \ldots p_{n}^{\circ}\right) \in R_{+}^{n}$ be any solution of the set of inequalities (30) then the components of the vector $p^{\circ}$ belonging to $J$ are equal to zero.
$\mathbf{P r o o f}$ If $p^{\circ}$ is a solution of (30) then multiplying every inequality of (30) by $p_{k}^{\circ}$ and summing over all $k$ we obtain

$$
0 \equiv \sum_{k=1}^{n}\left(\sum_{i=1}^{n} \frac{a_{k i} b_{i} p_{i}^{\circ}}{\sum_{l=1}^{n} a_{l i} p_{l}^{\circ}}-b_{k}\right) p_{k}^{\circ}=\sum_{k \in J}\left(\sum_{i=1}^{n} \frac{a_{k i} b_{i} p_{i}^{\circ}}{\sum_{l=1}^{n} a_{l i} p_{l}^{\circ}}-b_{k}\right) p_{k}^{\circ}
$$

If at least one component $p_{k}>0, k \in J$ then the r.h.s. of this equality is strictly negative and the l.h.s. is identically equal to zero. This contradiction proves Lemma 2. The fact that $p^{\circ}$ is the solution of (30) means $\sum_{l=1}^{n} a_{l i} p_{l}^{\circ}>0, \forall i=\overline{1, m}$. Let us denote

$$
y_{i}^{\circ}=\frac{p_{i}^{\circ}}{\sum_{l=1}^{n} a_{l i} p_{l}^{\circ}}
$$

Theorem 6 Let the vector $y^{\circ}=\left(y_{i}^{\circ}, \ldots, y_{n}^{\circ}\right)$ be a solution of the set of inequalities

$$
\begin{equation*}
\sum_{i=1}^{n} \frac{a_{k i} b_{i}}{b_{k}} y_{i}=1, \quad k \in I, \quad \sum_{i=1}^{n} \frac{a_{k i} b_{i}}{b_{k}} y_{i}<1, \quad k \in J \tag{31}
\end{equation*}
$$

moreover $y_{i}^{\circ} \geq 0, \quad i=\overline{1, n}, I \bigcup J=N, y_{i}=0, \quad i \in J$, then there exists a positive solution $p^{\circ}=\left(p_{1}^{\circ}, \ldots, p_{n}^{\circ}\right)$ to the problem

$$
p_{i}^{\circ}=y_{i}^{\circ} \sum_{l=1}^{n} a_{l i} p_{l}^{\circ}
$$

If the solution is such that

$$
\sum_{l=1}^{n} a_{l i} p_{l}^{\circ} \neq 0, \quad \forall i \in N
$$

then the vector $p^{\circ}$ is the solution of the problem (30).
Proof Let $I$ be a nonempty set, then those components of $y^{\circ}$, which are solutions of inequalities (31) and belong to the set $I$, are determined by the set of equations

$$
\begin{equation*}
\sum_{i \in I} \frac{a_{k i} b_{i}}{b_{k}} y_{i}^{\circ}=1, \quad k \in I \tag{32}
\end{equation*}
$$

Let us denote by $C_{y^{\circ}}^{I}=\left\|C_{k i} y_{i}^{\circ}\right\|$ the matrix of size $|I|$ by $|I|$ the elements of which are

$$
c_{k i} y_{i}^{\circ}=\frac{a_{k i} b_{i}}{b_{k}} y_{i}^{\circ}, \quad k, i \in I
$$

$|I|$ is the number of elements of the set $I$. The operator $C_{y^{\circ}}^{I}$ constructed on the basis of the matrix $\left\|C_{k i} y_{i}^{\circ}\right\|$ in the space of a dimension $|I|$ has the norm which does not exceed 1 in the norm $\|x\|=\max _{i \in I}\left|x_{i}\right|$.

From (32) there follows that 1 is an eigenvalue of the problem

$$
C_{y^{\circ}}^{I} \tilde{x}=\tilde{x}
$$

with the eigenvector $\tilde{x}=(1, \ldots, 1)$. The conjugate matrix to $C_{y^{\circ}}^{I}$ is a transposed one. Since eigenvalues of the transposed matrix are the same as those of $C_{y^{\circ}}^{I}$ then 1 is the maximal eigenvalue of the transposed matrix $\bar{C}_{y^{\circ}}^{I}$.

By the Perron-Frobenius theorem there exists a nonnegative vector $\tilde{p}^{\circ}$ of the conjugate problem

$$
\sum_{k \in I} c_{k i} y_{i}^{\circ} \tilde{p}_{k}^{\circ}=\tilde{p}_{i}^{\circ}, \quad i \in I
$$

or

$$
\begin{equation*}
b_{i} y_{i}^{\circ} \sum_{k \in I} a_{k i} \frac{1}{b_{k}} \tilde{p}_{k}^{\circ}=\tilde{p}_{i}^{\circ}, \quad i \in I \tag{33}
\end{equation*}
$$

Let us define the vector $p^{\circ}=\left(p_{1}^{\circ}, \ldots, p_{n}^{\circ}\right)$ by the following rule: if components of $p^{\circ}$ belong to the set $J$ then they equal to zero; those components of $p^{\circ}$ which belong to the set $I$ are given by the formula

$$
p_{i}^{\circ}=\frac{\tilde{p}_{i}^{\circ}}{b_{i}}, \quad i \in I
$$

So the vector constructed is the solution to the problem

$$
\begin{equation*}
y_{i}^{\circ} \sum_{k=1}^{n} a_{k i} p_{k}^{\circ}=p_{i}^{\circ} \tag{34}
\end{equation*}
$$

If the solution is such that $\sum_{k=1}^{n} a_{k i} p_{k}^{\circ} \neq 0, \forall i=\overline{1, n}$, then

$$
y_{i}^{\circ}=\frac{p_{i}^{\circ}}{\sum_{k=1}^{n} a_{k i} p_{k}^{\circ}} .
$$

Substituting the expression for $y_{i}^{\circ}$ into (31) we complete the proof of the theorem.
Theorem 7 If $a_{k i} \neq 0, \forall i, k=\overline{1, n}$, then the set of inequalities (30) is solvable.
This theorem is the consequence of the Theorem 2 .
Theorem 8 If for same $k a_{k k} \neq 0$ and for $\forall l=\overline{1, n}, \quad l \neq k$ the inequality

$$
\frac{a_{l k} b_{k}}{a_{k k}}<b_{l}
$$

holds then the vector $p^{\circ}=(0, \ldots 0,1,0, \ldots 0)$ with unit on the $k$-th position only and the rest of components is vanishing is the solution to the problem (30).

Theorem 9 If the matrix $A$ is indecomposable and the vector $y^{\circ}$ is such that $y_{i}^{\circ}>0$, $\forall i \in N$, then the vector $p^{\circ}=\left(p_{1}^{\circ}, \ldots, p_{n}^{\circ}\right)$ solving the problem (30) is such that $p_{i}^{\circ} \neq 0$, $\forall i \in N$.

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