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Direct Method of Finding First Integrals of Finite Dimensional Systems and Construction of Nondegenerate Poisson Structures

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Abstract

We present a novel method of finding first integrals and nondegenerate Poisson structures for a given system. We consider the given system as a system of differential 1-forms. After multiplying this system by a set of multiplicative functions, we demand the existence of first integrals. More interesingly these multipliers play a crucial role in constructing the required Poisson structures, if it exists. We illustrate this procedure with a class of physically interesting systems.

1 Introduction

Integrable systems have been perceived as an important modern development in many areas of mathematics and physics [1–11]. Recently, a number of methods were formulated to study such systems exactly. Yet, it is not completely clear, what is the unifying underlying phenomenon of integrability. The most important method that has been available is the inverse scattering which is applicable to both finite and infinite dimensional systems [3–5, 10]. It has been recognised that the systems belonging to this class can have more than one Hamiltonian formalism with respect to different Poisson structures. More significantly, Magri [12, 13] introduced the concept of a second Hamiltonian structure, popularly known as a Pair of Poisson structures, for integrable, popularly partial differential equations (PDE´s). One should emphasize that existence of a second Poisson structure is crucial for establishing integrability algebraically [6, 12–24]. It is well known that the Poisson pairs has close connections with Lax pairs, recursion operators, master symmetries , and R-matrices, etc. [25, 26]. The existence of n integrals on involution for a system with n degrees of freedom the integrability of the system in Liouville´s sense [1]. There are few interesting methods available in the literature to find the first integrals for a finite dimensional system. They are, for example, direct methods [27–34] and other methods proceed by symmetry analysis [35–45]. Apart from finding first integrals, many researchers have recently tried to find different Poisson structures for finite dimensional systems [46–51]. Wojciechowski [52] constructed Poisson structures for finite dimensional systems using Casimir functions. Mostly, these Poisson structures belong to the degenerate class. Very recently, nondegenerate Poisson structures have been constructed by Caboz et al [53, 54] for the Henon–Heiles system utilising separation of variables. However, it is well known that the construction of separation of variables for a given system is a formidable task.

The motivation for this paper is to construct the second Poisson structures without using either the separation of variables or symmetries. We present a direct method of finding other integrals apart from the Hamiltonian for the given potential. Here, we wish to remark that there is no need to compute symmetries or to use Noeather´s theorem to obtain other first integrals. We adopt a slightly different approach to integrate the set of differential 1-forms obtained from equations of motion. We multiply each 1-forms by an unknown function. Then, we require the existence of integrals such that the sum of the 1-forms multiplied by the unknown functions is exact. Under certain specific conditions on the multipliers, this approach leads systematically to obtaining the other required integrals. Secondly, we set as our goal the use of these multipliers to construct nondegenerate Poisson structures very naturally. It is significant that these Poisson structures are given explicitly in terms of Darboux variables (p, q) only.

The plan of the paper is a follows. In section 2, we present our method of finding first integrals for a given system, and in section 3, we illustrate it with a number of well known potentials. In section 4, we derive a systematic approach to the construction of nondegenerate Poisson structures, and we demonstrate this approach explicitly in section 5, by deriving a second Poisson structure for various systems. In section 6, we give our concluding remarks.

2 Theory and method

In this section, we present a method of finding first integrals for a given autonomous Hamiltonian system with two degrees of freedom

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2), \tag{2.1}$$

and the corresponding equations of motion are given by

$$\frac{dq_i}{dt} = p_i, \quad \frac{dp_i}{dt} = -V_{q_i}, \quad i = 1, 2 \tag{2.2}$$

where $V(q_1, q_2)$ is the given potential of the system. Equation (2.2) can be equivalently written in terms of the following six differential 1-forms:

$$p_2 dq_1 - p_1 dq_2 = 0, \tag{2.3a}$$

$$p_1 dp_1 - V_{q_1} dq_1 = 0, \tag{2.3b}$$

$$p_1 dp_2 - V_{q_2} dq_1 = 0, \tag{2.3c}$$

$$p_2 dp_1 - V_{q_1} dq_2 = 0, \tag{2.3d}$$

$$p_2 dp_2 - V_{q_2} dq_2 = 0, \tag{2.3e}$$

$$V_{q_1} dp_2 - V_{q_2} dp_1 = 0. \tag{2.3f}$$

We multiply equations (2.3a)–(2.3f) by various unknowns P, Q, R, S, T and U which are functions of (q_1, q_2, p_1, p_2), called multipliers and sum up the resulting equations. If the resulting equation is exact, say DI, then

$$\begin{aligned} DI &= p(p_2 dq_1 - p_1 dq_2) + Q(p_1 dp_1 + V_{q_1} dq_1) + \\ &\quad R((p_1 dp_2 + V_{q_2} dq_1) + S(p_2 dp_1 + V_{q_1} dq_2) + \\ &\quad T((p_2 dp_2 + V_{q_2} dq_2) + U(V_{q_1} dp_2 - V_{q_2} dp_1), \end{aligned} \tag{2.4}$$

where $I = I(q_1, q_2, p_1, p_2)$ and D is the total differential operator holds.

Expanding left hand side of (2.4) and equating the coefficients of dq_1, dq_2, dp_1, dp_2 in (2.4) on both sides, we get

$$I_{q_1} = Pp_2 + QV_{q_1} + RV_{q_2}, \tag{2.5a}$$

$$I_{q_2} = -Pp_1 + SV_{q_1} + TV_{q_2}, \tag{2.5b}$$

$$I_{p_1} = Qp_1 + Sp_2 - UV_{q_2}, \tag{2.5c}$$

$$I_{p_2} = Rp_1 + Tp_2 + UV_{q_1}. \tag{2.5d}$$

The compatibility conditions

$$I_{q_i q_j} = I_{q_j q_i} I_{p_i p_j} = I_{p_j p_i} I_{q_i p_j} = I_{p_j q_i}, \quad i, j = 1, 2,$$

lead to six coupled linear PDE´s :

$$\begin{aligned} p_2 P_{q_2} + p_1 P_{q_1} + (Q_{q_2} - S_{q_1}) V_{q_1} + (R_{q_2} - T_{q_1}) V_{q_2} + \\ (Q - T) V_{q_1 q_2} + RV_{q_2 q_2} - SV_{q_1 q_1} = 0, \end{aligned} \tag{2.6a}$$

$$p_2(P_{p_1} - S_{q_1}) - p_1 Q_{q_1} + Q_{p_1} V_{q_1} + (R_{p_1} + U_{q_1}) V_{q_2} + UV_{q_1 q_2} = 0, \tag{2.6b}$$

$$P + p_2(P_{p_2} - T_{q_1}) - p_1 R_{q_1} + (Q_{p_2} - U_{q_1}) V_{q_1} + R_{p_2} V_{q_2} - UV_{q_1 q_1} = 0, \tag{2.6c}$$

$$-P - p_2(P_{p_1} + Q_{q_2}) - p_2 S_{q_2} + S_{p_1} V_{q_1} + (T_{p_1} + U_{q_2}) V_{q_2} + UV_{q_2 q_2} = 0, \tag{2.6d}$$

$$p_1(P_{p_2} + R_{q_2}) + (S_{p_2} - U_{q_2}) V_{q_1} + T_{p_2} V_{q_2} - p_2 T_q - UV_{q_1 q_2} = 0, \tag{2.6e}$$

$$p_1(Q_{p_2} - R_{p_1}) + p_2(S_{p_2} - T_{p_1}) + S - R - U_{p_1} V_{q_1} - U_{p_2} V_{q_2} = 0. \tag{2.6f}$$

In order to solve the above equations, in general, we need to make some suitable assumptions on P, Q, R, S, T and U. In this direction as a first case, we take

$$
\begin{aligned}
&Q = Q(q_1, q_2, p_1), \quad T = T(q_1, q_2, p_2), \quad R = S(q_1, q_2),\\
&P = P(q{+}_1, q_2, p_1, p_2), \quad P_{p_1} = S_{q_1}, \quad P_{p_2} = R_{q_2}, \quad U = 0.
\end{aligned}
\tag{2.7}
$$

Substituting (2.7) into (2.6b) and (2.6e) and integrating, we get

$$Q = \alpha_1 \left[\frac{p_1^2}{2} + f_1(q_1, q_2)\right] + f_2(q_2) + \alpha_2, \tag{2.8a}$$

$$T = \alpha_3 \left[\frac{p_2^2}{2} + f_3(q_1, q_2)\right] + f_4(q_1) + \alpha_4, \tag{2.8b}$$

where $f_1 = \int V_{q_1} dq_1$ and $f_3 = \int V_{q_2} dq_2$ and $\alpha_1, \alpha_2, \alpha_3$ and α_4 are constants and f_1, f_2, f_3 and f_4 functions of the arguments. Adding equations (2.6c) and (2.6d) and further using (2.7), we obtain

$$2p_2 P_{p_2} - 2p_1 P_{p_1} - p_1 Q_{q_2} - p_2 T_{q_1} = 0.$$

Solving the linear PDE together with (2.8a) and (2.8b), we find

$$P = \frac{p_2}{2}(\alpha_3 f_{3q_1} + f_{4q_1}) - \frac{p_1}{2}(\alpha_1 f_{1q_2} + f_{2q_2}). \tag{2.9}$$

Using equations (2.7)–(2.9) in (2.6), we arrive at

$$p_1 \left\{\frac{1}{2}(\alpha_1 f_{1q_2} + f_{2q_2}) + S_{q_1}\right\} + p_2 \left\{\frac{1}{2}(\alpha_3 f_{3q_1} + f_{4q_1}) + S_{q_2}\right\} = 0. \tag{2.10}$$

Since the functions f_1, f_2, f_3, f_4 and S in (2.10) are independent of p_1, p_2, we equate the coefficients of p_1 and p_2 to zero. Then we have

$$S_{q_1} = -\frac{1}{2}(\alpha_1 f_{1q_2} + f_{2q_2}), \tag{2.11a}$$

$$S_{q_2} = -\frac{1}{2}(\alpha_3 f_{3q_1} + f_{4q_1}). \tag{2.11b}$$

By the compatibility condition $S_{q_2 q_1} = S_{q_1 q_2}$, we derive

$$\alpha_1 f_{1q_2q_2} + f_{2q_2q_2} - \alpha_3 f_{3q_1q_1} - f_{4q_1q_1} = 0. \tag{2.12}$$

By utilizing the equations (2.8), (2.9) and (2.12) in (2.6) after some simple calculation, we get

$$
\begin{aligned}
&3(\alpha_1 f_{1q_2} + f_{2q_2})V_{q_1} - 3(\alpha_3 f_{3q_1} + f_{4q_1})V_{q_2} + 2(\alpha 2 f_1 - \\
&\qquad \alpha_3 f_3 + f_2 - f_4 + \alpha_2 - \alpha_4)V_{q_1q_2} + 2S(V_{q_2q_2} - V_{q_1q_1}) = 0.
\end{aligned}
\tag{2.13}
$$

For a given V, f_1 and f_3 are immediately available and the remaining unknowns f_2 and f_4, can be determined from equation (2.12). Once we obtain f_1, f_2, f_3 and f_4, the values of P, Q, R, S and T follow from relations (2.8), (2.9) and (2.11).

As a second case, we shall assume that

$$R = S(q_1, q_2, p_1, p_2), \quad Q = Q(q_2)T, \quad T = T(q_1),$$
$$P = P(q_1, q_2, p_1, p_2), \quad U = 0. \tag{2.14}$$

Solving equation (2.6) together with (2.14), we have

$$R = \alpha_1[p_1p_2 + g_1(q_1, q_2)], \tag{2.15}$$

where g_1 is a function of the argument and α_1 is a constant. Again, using (2.12) and (2.15) in equations (2.6b) and (2.6c), we obtain

$$P_{p_1} = R_{q_1} - \alpha_1 V_{q_2}, \tag{2.16a}$$

$$P_{p_2} = -R_{q_2} - \alpha_1 V_{q_1}. \tag{2.16b}$$

From equations (2.15) and (2.16), we find that

$$P = \alpha_1\{p_2(V_{q_1} - g_{1q_2}) + p_1(g_{1q_1} - V_{q_2})\}. \tag{2.17}$$

Adding equations (2.6c) and (2.6d) together with (2.14), (2.15) and (2.16), we derive the equation

$$p_1(-2\alpha_1 g_{1q_1} + 2\alpha_1 V_{q_2} - Q_{q_2}) + p_2(-2\alpha_1 g_{1q_2} + 2\alpha_1 V_{q_1} - T_{q_1}) = 0. \tag{2.18}$$

Since functions q_1, Q and T in (2.18) are free of the variables p_1, p_2, we equate the coefficients of p_1 and p_2 to zero. Then, we get

$$Q_{q_2} = 2\alpha_1(V_{q_2} - g_{1q_1}), \tag{2.19a}$$

$$T_{q_1} = 2\alpha_1(V_{q_1} - g_{1q_2}). \tag{2.19b}$$

Differentiating equations (2.19a) and (2.19b) with respect to q_1 and q_2 respectively and using (2.14), we arrive at

$$g_{1q_2q_2} = V_{q_1q_2}, \quad g_{1q_1q_1} = V_{q_2q_1}. \tag{2.20}$$

By the compatibility condition of V, from (2.20), we determine

$$g_1 = f_1(q_1 + q_2) + f_2(q_1 - q_2), \tag{2.21}$$

where f_1 and f_2 are arbitrary functions of their arguments. From (2.20), we calculate the potential:

$$V = f_1(q_1 + q_2) - f_2(q_1 - q_2) + f_3(q_1) + f_4(q_2), \tag{2.22}$$

where f_3 and f_4 are functions of q_1 and q_2 respectively. Substituiting equations (2.21) and (2.22) in equations (2.19), we find

$$Q = 2\alpha_1 f_4(q_2) + \alpha_2, \tag{2.23a}$$

$$T = 2\alpha_2 f_3(q_1) + \alpha_3, \tag{2.23b}$$

where α_2 and α_3 are arbitrary constants. By introducing equations (2.14), (2.17) and (2.23) together with equations (2.21) and (2.22) into equation (2.6a), we obtain

$$-3f_1'(f_3'-f_4')-3f_2'(f_3'+f_4')+2(f_4-f_3)(f_1''+f_2'')+ \\ (f_1+f_2)(f_4''-f_3'')=0, \tag{2.24}$$

where f_i' and f_i'', $i=1,...,4$ denote first and second order derivatives of the functions with respect to their arguments. Equation (2.24) is an integrability condition for the potential (2.22). Trivially, if $f_3=f_4=0$ in (2.24), then the equation is satisfied. Therefore, the system with the potential

$$V=f_1(q_1+q_2)-f_2(q_1-q_2), \tag{2.25}$$

is integrable. Then, by using the values of P,R,S,T and U in equations (2.5a)–(2.5d), we determine the integral I:

$$I=\{p_1p_2+f_1(q_1+q_2)+f_2(q_1-q_2)\}^2. \tag{2.26}$$

We next consider the following choices:

$$\text{i)}\quad Q=Q(q_1),\quad T=T(q_2),\quad R=S(p_1,p_2,q_1,q_2),\quad U=0,$$

$$\text{ii)}\quad Q=Q(q_1),\quad T=T(q_1),\quad R=S(p_1,p_2,q_1,q_2),\quad U=0,$$

$$\text{iii)}\quad Q=Q(q_2),\quad T=T(q_2),\quad R=S(p_1,p_2,q_1,q_2),\quad U=0.$$

We have repeated the procedure as earlier for cases (i)–(iii) and obtained the same potential (2.25) and the integral (2.26). We verified that the above analysis for all other possible combinations in the arguments of P,Q,R,S,T and U in terms of (q_1,q_2,p_1,p_2) yield only trivial solution for multipliers namely all of them are constants, so that they do not yield any second integral. In the next section, we shall illustrate our method with a number of physically importanat systems and derive their first integrals.

3 Applications

In this section, we apply the procedure presented in section 2 to Sextic anharmonic oscillator [38]:

$$V(q_1,q_2)=Aq_1^2+Bq_2^2+Cq_1^6+Dq_2^6+Eq_1^4q_2^2+Fq_1^2q_2^4. \tag{3.1}$$

Since $f_1\int V_{q_1}dq_1$ and $\int V_{q_2}dq_2$, from (3.1), we can easily find the values of f_1 and f_3 as:

$$f_1=Aq_1^2+Cq_1^6+Eq_1^4q_2^2+Fq_1^2q_2^4, \tag{3.2a}$$

$$f_3=Bq_2^2+Dq_2^6+Eq_1^4q_2^2+Fq_1^2q_2^4. \tag{3.2b}$$

By making use of equations (3.2a) and (3.2b) into equation (2.12), we obtain

$$\alpha_1 = \alpha_2 = 0,$$
$$f_2 = \frac{\alpha_5}{2} q_2^2 + \alpha_6 q_2, \tag{3.3a}$$
$$f_4 = \frac{\alpha_5}{2} q_1^2 + \alpha_7 q_2, \tag{3.3b}$$

where α_5, α_6 and α_7 are constants. Hence, the equations (2.8a) and (2.8b) become

$$Q = \frac{\alpha_5}{2} q_2^2 + \alpha_6 q_2 + \alpha_2, \tag{3.4a}$$
$$T = \frac{\alpha_5}{2} q_1^2 + \alpha_7 q_1 + \alpha_4. \tag{3.4b}$$

Solving equations (2.11), we get

$$R = S = -\frac{\alpha_5}{2} q_1 q_2, -\frac{\alpha_6}{2} q_1 - \frac{\alpha_7}{2} q_2 + \alpha_8, \tag{3.5}$$

where α_8 is a constant. Also, from equation (2.9), we obtain

$$P = \frac{p_2}{2}\{\alpha_5 q_1 + \alpha_7\} - \frac{p_1}{2}\{\alpha_5 q_2 + \alpha_6\}. \tag{3.6}$$

Now, by introducing equations (3.4)–(3.6) together with (3.1) into equation (2.13), we recover the well known integrable cases:

$$\begin{aligned} &\text{a)}\ \ \alpha_7 \neq 0, \quad A = 4B, \quad C = 64D, \quad E = 80D, \quad F = 24D, \\ &\qquad\qquad \alpha_1 = \alpha_3\alpha_5 = \alpha_6\alpha_8 = 0, \quad \alpha_2 = \alpha_4; \\ &\text{b)}\ \ \alpha_8 \neq 0, \quad A = B, \quad C = D, \quad E = F = 15D, \\ &\qquad\qquad \alpha_1 = \alpha_3\alpha_5 = \alpha_6\alpha_8 = 0, \quad \alpha_2 = \alpha_4; \\ &\text{c)}\ \ \alpha_5 \neq 0, \quad A = B, \quad C = D, \quad E = F = 3D, \\ &\qquad \alpha_1 = \alpha_3\alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 0, \quad \alpha_2 = \alpha_4. \end{aligned} \tag{3.7}$$

In case (a) of (3.7), from equations (3.4)–(3.6), the values of the multipliers are given by

$$P = \frac{p_2}{2}, \quad Q = U = 0, \quad R = S = -\frac{q_2}{2}, \quad T = q_1. \tag{3.8}$$

Introducing these multipliers (3.8) into (2.5a)–(2.5d), we obtain the integral I:

$$I = p_2(q_2 p_1 - q_1 p_2) + 2q_1 q_2^2\{B + 3Dq_2^4 + 16Dq_1^2(q_1^2 + q_2^2)\}. \tag{3.9}$$

Similarly for case (b) of (3.7), we have

$$P = Q = T = U = 0, \quad R = S = 1$$

and

$$I = p_1 p_2 + q_1 q_2\{A + 4Dq_1^2 q_2^2 + 3D(q_1^2 + q_2^2)^2\}. \tag{3.10}$$

Again for case (c) we get

$$P = \frac{1}{2}(q_1 p_2 - g_2 p_1), \quad Q = \frac{q_2^2}{2}, \quad T = \frac{q_1^2}{2}, \quad R = S = -\frac{q_1 q_2}{2}, \quad U = 0$$

and

$$I = (q_2 p_1 - q_1 p_2)^2. \tag{3.11}$$

In a similar fashion, we apply our procedure to the following potentials and present the results in table I.

2) Perturbed Kepler System [56]:

$$V = -G(q_1^2 + q_2^2)^{-\frac{1}{2}} + Aq_1^M + Bq_2^N, \tag{3.12}$$

3) Quartic anharminic oscillator [57]:

$$V = Aq_1^2 + Bq_2^2 + Cq_1^4 + Dq_2^4 + Eq_1^2 q_2^2, \tag{3.13}$$

4) Inverse square potential [58]:

$$V = Aq_1^2 + Bq_2^2 + (q_1^2 + q_2^2)^2 + Cq_1^{-2} + Dq_2^{-2}, \tag{3.14}$$

5) Fifth order nonhomogeneous potential [31]:

$$\begin{aligned} V = Aq_1^5 + Bq_1^3 q_2^2 + Cq_1 g_2^4 + Dq_1^4 + Eq_1^2 q_2^2 + Fq_2^4 + \\ Gq_1^3 + Hq_1 q_2^2 + Iq_1^2 + Jq_2^2 + Kq_1, \end{aligned} \tag{3.15}$$

6) Henon–Heiles system [55]:

$$V = \frac{1}{2}(Aq_1^2 + Bq_2^2) + Dq_1^2 q_2 - \frac{C}{3} q_2^3, \tag{3.16}$$

where $a, b, g, A, B, C, D, E, F, G, H, I, J$ and K are all parameters. In system (3), for $A = B = 0$, we obtain the integrable potential of fourth order homogeneous type . Similarly for $D = E = F = I = J = 0$ in system (5), we get the known integrable potential of the fifth order homogeneous type.

In the next section, we make use of the multiplier functions P, Q, R, S and T to construct a nongenerate second Poisson structure for a given system.

4 Nondegenerate Poisson structures

In this section, we present a simple method of constructing a nondegenerate Poisson structures for a given system. We first review certain well known results on Poisson manifolds [8].

Definition: A Poisson bracket on a smooth manifold M is an operation that assigns a smooth real-valued function $[F, H]$ on M to each pair F, H of smooth, real valued functions, with the following properties:

(a) Bilinearity:

$$\{\alpha F + \beta P, H\} = \alpha\{F, H\} + \beta\{P, H\},$$
$$\{F, \alpha H + \beta P\} = \alpha\{F, H\} + \beta\{F, P\}, \quad \alpha, \beta \in R, \tag{4.1}$$

(b) Skew-symmetry:

$$\{F, H\} = -\{H, F\}, \tag{4.2 }$$

(c) Jacobi identity:

$$\{\{F, H\}, P\} + \{\{P, F\}, H\} + \{H, P\}, F\} = 0, \tag{4.3 }$$

(d) Leibniz rule:

$$\{F, H\, P\} = \{F, H\}\, P + H\, \{F, P\}, \tag{4.4 }$$

where F, H and P are arbitrary smooth real–valued functions on M.

The equation of motion can be written as

$$\frac{dx}{dt} = \{x, H\} = J(x) \cdot \bigtriangledown H(x), \tag{4.5}$$

where $J(x) = (J^{ij}(x))$ is the structure matrix of order m, and the coefficient $J^{ij}(x)\{x^i, x^j\}$, $i, j = 1, ..., m$, being the structure functions of the coordinates x^i and x^j , enable us to define a Poisson bracket of any pair (F, H) of functions on the smooth manifold M of dimension m with local coordinates $\{x^1, ..., x^m\}$ in the following form

$$\{F, H\} = \sum_{i=1}^{m}\sum_{j=1}^{m} J^{ij}(x)\, \frac{\partial F}{\partial x^i}\, \frac{\partial H}{\partial x^j}. \tag{4.6}$$

By a proposition from Olver´ s book [8], we say $J(x)$ is the structure matrix for a Poisson bracket defined over M if and only if it has the following properties:

(a) Skew symmetry:

$$J^{ij}(x) = -J^{ji}(x), \tag{4.7a}$$

(b) Jacobi identity :

$$\sum_{i=1}^{m} \{J^{ij}\partial_l J^{jk} + J^{kl}\partial_l J^{ij} + J^{jl}\partial_l J^{ki}\} = 0, \qquad i, j, k = 1, ..., m, \tag{4.7b}$$

for all $x \in M$, where $\partial_l = \dfrac{\partial}{\partial x^l}$.

The equation (4.7b), being a system of nonlinear PDE´s becomes linear whenever $J(x)$ has an inverse say $K(x) = [J(x)]^{-1}$. Then equations (4.7a) and (4.7b) can be rewritten as

(a) Skew symmetry:

$$K^{ij}(x) = K^{ji}(x). \qquad i, j = i, ..., m, \tag{4.8a}$$

(b) Jacobi identity:

$$\partial_i K^{jk} + \partial_j K^{ki} + \partial_k K^{ij} = 0, \qquad i, j, k = 1, ..., m. \tag{4.8b}$$

Hence, whenewer $J(x)$ has an inverse, we can very well use equation (4.8) instead of equation (4.7).

Definition: We say that a Hamiltonian system possess a pair of Poisson structures if there exists a function $f(q_1, q_2, p_1, p_2) \neq 0$ such that

$$\dot{x} = J_1 \bigtriangledown H = f(\rho J_2) \bigtriangledown I, \tag{4.9 }$$

where $x = (q_1, q_2, p_1, p_2)^T$, $\rho = f^{-1}$ and I is the first integral, independent of H, and J_1, ρJ_2 and $\alpha_1 J_1 + \alpha_2 \rho J_2$ satisfy equations (4.7a) and (4.7b).

In some sense our structure (4.9) is a weak bi–Hamiltonian structure, when compared with the Magri´s definition. Now, from equations (2.5a)–(2.5d) and $U = 0$, we get

$$fJ_2^{-1} = \begin{bmatrix} 0 & P & -Q & -R \\ -P & 0 & -S & -T \\ Q & S & 0 & 0 \\ R & T & 0 & 0 \end{bmatrix}. \tag{4.10}$$

In order to find 'f' we use the linear Jacobi identity condition equation (4.8b) for fJ_2^{-1} instead of equation (4.7b), since the later one is practically not useful for this purpose. Consequently, we obtain

$$f(Q_{p_2} - R_{p_1}) + Qf_{p_2} - R_{p_1} = 0, \tag{4.11a}$$

$$f(T_{p_1} - S_{p_2}) + Tf_{p_1} - S_{p_2} = 0, \tag{4.11b}$$

$$f(P_{p_1} - S_{q_1} + Q_{q_2}) + Pf_{p_1} - S_{q_1} - Qf_{q_2} = 0, \tag{4.11c}$$

$$f(P_{p_2} + R_{q_2} - T_{q_1}) + Pf_{p_2} + R_{q_2} - Tf_{q_2} = 0. \tag{4.11d}$$

Solving equations (4.11a)–(4.11d), we obtain the value of f. Hence,

$$\rho J_2 = \frac{f^{-1}}{(QT - RS)} \begin{bmatrix} 0 & 0 & T & -S \\ 0 & 0 & -R & Q \\ -T & R & 0 & P \\ S & -Q & -P & 0 \end{bmatrix}, \tag{4.12}$$

where $QT - RS \neq 0$. We perform this analysis in the following section for all the potentials discussed in section 3.

5 Applications

A Poisson structures for A given potential

In this section, we demonstrate the procedure presented in section 4, with certain physically important systems. As a first example, we consider Sextic anharmonic os cillator (3.1). Using our technigue, we have recovered known three integrable cases. For each integrable case, we have determined the corresponding multipliers P, Q, R, S and T explicitly. For the case (a), by making use of the multipliers (3.8) in (4.11a)–(4.11d), we find

$$f_{q_1} = 0, \qquad f_{q_2} = -\frac{2f}{q_2}, \qquad f_{p_1} = 0, \qquad f_{p_2} = 0. \tag{5.1}$$

Hence, by (5.1), we get

$$f = -\frac{4}{q_2^2} \text{ and } \rho = -\frac{q_2^2}{4}. \tag{5.2}$$

Therefore, from (4.12) together with (5.2), the Poisson structure ρJ_2 has the form

$$\rho J_2 = \begin{bmatrix} 0 & 0 & q_1 & \frac{q_2}{2} \\ 0 & 0 & \frac{q_2}{2} & 0 \\ -q_1 & -\frac{q_2}{2} & 0 & \frac{p_2}{2} \\ -\frac{q_2}{2} & 0 & -\frac{p_2}{2} & 0 \end{bmatrix}.$$

We have verified that this ρJ_2 satisfies the Jacobi identity (4.7b). Also, we have checkend that $\alpha J_1 + \alpha_2(\rho J_2)$ satisfies equations (4.7a) and (4.7b), where

$$J_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

is the usual Poisson structure satisfying equations (4.7a) and (4.7b) and equation (4.9) is satisfied identically for the above I, J_1 and ρJ_2.

Similaraly, for case (b), we determine from (4.11)

$$f = -1 \text{ and } \rho = -1.$$

Hence, the second Poisson structure ρJ_2 has the form

$$\rho J_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Since, for case (c), $QT - RS = 0$, the value of f does not exist. Hence, we are unable to construct a second Poisson structure for this case. Similarly, we find the values of f and the nondegenerate second Poisson structures for the potentials considered in the Table I and present the results in Table II.

B Poisson structures for *A* given integral

In section A, we have constructed the nondegenerate Poisson structures for a given potential using multipliers P, Q, R, S and T. Now, we construct the same structures for a given integral. First, we find the corresponding multipliers for the integral. Then by the same procedure as given in the section A, we derive the second Poisson structures. To illustrate this concept, we first consider an integral of motion,

$$1) \quad I = (q_1p_2 - q_2p_1)^2 + Up_1^2 + \frac{2}{A^2 - B^2}[B^2 g(A) - A^2 h(B)], \tag{5.3}$$

where

$$\begin{aligned} 2A^2 &= r^2 + c + \left[(r^2 + C)^2 - 4C_1^2\right]^{\frac{1}{2}}, \\ 2B &= r^2 + C + \left[(r^2 + C)^2 - 4C_1^2\right]^{\frac{1}{2}}, \\ r^2 &= q_1^2 + q_2^2. \end{aligned}$$

The potential corresponding to this integral is given by

$$V = \frac{g(A) - h(B)}{A^2 - B^2}, \quad A \neq B. \tag{5.4}$$

In this case the multipliers become

$$\begin{aligned} &P = 2(q_1p_2 - q_2p_1), \qquad Q = 2(q_2^2 + C), \\ &T = 2q_1^2, \qquad R = S = -2q_1q_2 \end{aligned} \tag{5.5}$$

and introducing equation (5.5) in equations (4.11), we get

$$f = \frac{1}{4Cq_1^2}.$$

Hence, the second Poisson structures ρJ_2 for the potential correspoinding to the integral (5.3) is given by

$$\rho J_2 = \begin{bmatrix} 0 & 0 & 2q_1^2 & 2q_1q_2 \\ 0 & 0 & 2q_1q_2 & 2(q_2^2 + c) \\ -2q_1 & -2q_1q_2 & 0 & 2(q_1p_2 - q_2p_1) \\ -2q_1q_2 & -2(q_2^2 + c) & -2(q_1p_2 - q_2p_1) & 0 \end{bmatrix}.$$

We establish this converse procedure explicitly by considering numerous first integrals for a class of physically interesting systems. We present our results in table III.

$$2)\quad I = (q_1p_2 - q_2p_1)^2 + Up_1(p_1 \pm up_2) + \frac{2}{A^2 - B^2}\left[B^2 g(A) - A^2 h(B)\right], \tag{5.6}$$

where

$$2A^2 = r^2 + [r^4 - 2C(q_1 \pm iq_2)^2]^{\frac{1}{2}},$$

$$2B^2 = r^2 + [r^4 - 2C(q_1 \pm iq_2)^2]^{\frac{1}{2}},$$

with the potential $V(q_1, q_2)$, (5.4) but different A and B.

$$3)\quad I = p_1(q_2p_1 - q_1p_2) + \frac{1}{r}[(r + q_2)h(r - q_2) - (r - q_2)g(r + q_2)], \tag{5.7}$$

with the potential

$$V = \frac{1}{r}[g(r + q_2) + h(r - q_2)].$$

$$\begin{aligned} 4)\quad I = (q_2p_1 - q_1p_2)(p_1 + p_2) + \frac{i}{8}(p_1 - up_2)^2 + \\ i\left(1 - \frac{z}{\sqrt{w}}\right)g(z + \sqrt{w}) + i\left(1 - \frac{z}{\sqrt{w}}\right)h(z - \sqrt{w}), \end{aligned} \tag{5.8}$$

where $z = q_1 + q_2, \quad w = \bar{z}$ with

$$V = w^{-\frac{1}{2}}[g(z + \sqrt{w}) + h(z - \sqrt{w})].$$

$$\begin{aligned} 5)\quad I = (q_2p_1 - q_1p_2)(p_1 \pm up_2) - i(q_1 \pm iq_2)[g(q_1 \pm iq_2) + \\ h'(q_1 \pm iq_2) + h(q_1 \pm iq_2)] \end{aligned} \tag{5.9}$$

with

$$V = \frac{1}{r}g(q_1 \pm iq_2) + h'(q_1 \pm iq_2).$$

$$\begin{aligned} 6)\quad I = p_1(p_1 \pm up_2) + r^2 g''(q_1 \pm iq_2) + h(q_1 \pm iq_2) + \\ 2(q_1 \pm iq_2) + g'(q_1 \pm iq_2) - 2g(q_1 \pm iq_2) \end{aligned} \tag{5.10}$$

with

$$V = r^2 g''(q_1 \pm iq_2) + h(q_1 \pm iq_2).$$

$$7)\quad I = (q_1p_2 - q_2p_1)^2 + 2g\left[\frac{q_1}{q_2}\right] \tag{5.11}$$

with

$$V = h(r) + g\left[\frac{q_1}{q_2}\right]r^{-2}.$$

Table I

Potentials	Integrable cases	Multipliers				Integrals of Motion
		P	Q	$R = S$	T	I
Perturbed Kepler problem	a) A, B: arbitrary $M = 1$, $N = -2$	p_2	0	$-q_2$	$2q_1$	$p_2(q_2p_1 - q_1p_2)$ $+Gq_1(q_1^2 + q_2^2)^{-\frac{1}{2}}$ $+\frac{A}{2}q_2^2 - 2Bq_1q_2^{-2}$
	b) A, B: arbitrary $M = N = 1$	$Bp_1 - Ap_2$	$-2Bq_2$	$Bq_1 + Aq_2$	$-2Aq_1$	$(-Bp_1 + Ap_2)(q_2p_1 - q_1p_2)$ $+G(Aq_1 + Bq_2)(q_1^2 + q_2^2)^{-1/2}$
	c) $A = 4B$ $M = N = 2$	$-p_2$	0	q_2	$-2q_1$	$p_2(q_2p_1 - q_1p_2)$ $+Gq_1(q_1^2 + q_2^2)^{-1/2} + 2Bq_1q_2^2$
	d) A, B: arbitrary $M = N = -2$	$2(q_1p_2 - q_2p_1)$	$2q_2^2$	$-2q_1q_2$	$2q_1^2$	$(q_2p_1 - q_1p_2)^2$ $+2Bq_1q_2^{-2} + 2Aq_2^2q_1^{-2}$ $+\frac{1}{2}(Aq_2 - Bq_1)^2$
	e) $A = B$: $M = N = 2$	1	0	0	0	$q_1p_2 - q_2p_1$

Quartic	a) A, B:	$\frac{1}{2}(-q_1p_2 + q_2p_1)$	$\frac{1}{2}(-q_2^2 + \frac{A-B}{C})$	$\frac{q_1q_2}{2}$	$-\frac{q_1^2}{4}$	$-\frac{p_1^2q_2^2}{4} + \frac{p_1p_2q_1q_2}{2}$
anharmonic	arbitrary					$-\frac{p_2^2q_1^2}{4} + \frac{A-B}{2C}(\frac{p_1^2}{2}$
oscilator	$E = 2C$,	$D = C$				$+Aq_1^2 + Cq_1^4 + Cq_1^2q_2^2)$
	b) $A = 4B$,	$\frac{p_2}{2}$	0	$-\frac{q_2}{2}$	q_1	$\frac{p_2}{2}(q_1p_2 - q_2p_1)$
	$C = 16D$,					$-q_1q_2^2(B + 2Dq_2^2 + 4Dq_1^2)$
	$E = 12D$					
	c) $A = B$,	0	0	1	0	$p_1p_2 + 2Aq_1q_2$
	$E = 6C$,					$+4Cq_1q_2(q_1^2 + q_2^2)$
	$D = C$					
	d) $A = 4B$,	$6Dq_1q_2^2p_2$	Dq_2^4	$-2Dq_1q_2^3$	$\frac{p_2^2}{2}$	$+\frac{p_2^4}{4} + \frac{p_2^2}{2}(Bq_2^2 + Dq_2^4$
		$-2Dq_2^3p_1$				$+6Dq_1^2q_2^2) + p_1p_2(-2Dq_1q_2^3)$
	$C = 8D$				Bq_2^2	$+\frac{p_1^2}{2}Dq_2^4 + 2BDq_1^2q_2^4$
	$E = 6D$				$+Dq_2^4$	$+2D^2q_1^4q_2^4 + 2D^2q_1^2q_2^6$
					$+6Dq_1^2q_2^2$	$+\frac{B^2}{2}q_2^4 + BDq_2^6 + \frac{B^2}{2}q_2^8$

Inverse square potential	$A, B, C,$ $D:$ arbitrary	$2(q_1p_2 - q_2p_1)$	$2(q_2^2 + B - A)$	$-2q_1q_2$	$2q_1^2$	$(q_2p_1 - q_1p_2)^2 + 2Cq_2^2q_1^{-2}$ $+2Dq_1^2q_2^{-2} + (B-A)(p_1^2$ $+2q_1^4 + 2q_1^2q_2^2 + 2Aq_1^2 + 2Cq_1^{-2})$
Fifth order nongomo-geneous potential	a) $B = 10A$ $C = 5A$ $E = 6F$	0	0	1	0	$p_1p_2 + q_2[A(5q_1^4 + 10q_1^2q_2^2$ $+q_2^4) + 4D(q_1^3 + q_1q_2^2)$ $+G(3q_1^2 + q_2^2) + 21q_1 + K]$
	b) $A = B$ $C = \frac{3}{16}A$ $E = \frac{3}{4}D$ $F = \frac{D}{16}$ $H = \frac{G}{2},$ $J = \frac{I}{4}$	$-p_2$	0	q_2	$-2q_1$	$p_2(q_1p_2 - q_2p_1)$ $+\frac{q_2^2}{2}[4(q_1^4 + \frac{3}{4}q_1^2q_2^2 + \frac{1}{16}q_2^4)$ $+D(q_1^3 + \frac{1}{2}q_1q_2^2)+$ $+G(q_1^2 + \frac{1}{4}q_2^2) + Iq_1K]$

Henon	a) $C=-6D$	$-\frac{p_2}{2}$	$q_2+\frac{B-4A}{4D}$	$-\frac{q_1}{2}$	0	$-\frac{p_1}{2}(-q_2p_1+q_1p_2)$
Heiles	A, B:					$+\frac{B-4A}{4D}(p_1^2+Aq_1^2)-$
System	arbitrary					$-\frac{q_1^2}{8}(4Aq_2+Dq_1^2+4Dq_2^2)$
	b) $16A=B$,	$-\frac{1}{2}Dq_1^2p_1$	$\frac{1}{2}p_1^2+\frac{1}{2}Aq_1^2$	$-\frac{1}{6}Dq_1^3$	0	$\frac{1}{8}p_1^4+(A+2Dq_2)q_1^2p_1^2$
	$C=-16D$		$+Dq_1^2q_2$			$-\frac{1}{6}q_1^3p_1p_2+\frac{1}{8}A^2q_1^4$
						$-\frac{1}{6}D(A+Dq_2)q_1^4q_2$
						$-\frac{D^2}{36}q_1^6$
	c) $A=B$,	0	0	1	0	$p_1p_2+Aq_1q_2$
	$C=-D$					$+\frac{D}{3}q_1(q_1^2+3q_2^2)$

Here $U=0$.

Table II

Potentials as in Tab. I	f	J_{12}	J_{13}	$J_{14}=J_{23}$	J_{24}	J_{34}
2 a)	$\frac{-1}{q_2^2}$	0	$2q_1$	q_2	0	p_2
b)	$\frac{-1}{(Bq_1-Aq_2)^2}$	0	$-2Aq_1$	$-(Bq_1+Aq_2)$	$-2Bq_2$	Bp_1-Ap_2
c)	$\frac{-1}{q_2^2}$	0	$-2q_1$	$-q_2$	0	$-p_2$
d)	f does not exist since $QT-RS=0$					
e)	- do -					
3 a)	$\frac{-4C}{(A-B)q_1^2}$	0	$-\frac{q_1^2}{2}$	$-\frac{q_1q_2}{2}$	$-\frac{q_2^2}{2}+\frac{A-B}{2C}$	$\frac{1}{2}(q_2p_1-q_1p_2)$
b)	$\frac{-4}{q_2^2}$	0	q_1	$\frac{q_2}{2}$	0	$\frac{p_2}{2}$
c)	-1	0	0	-1	0	0
d)	f is not found					
4	$\frac{1}{4(B-A)q_1^2}$	0	$2q_1^2$	$2q_1q_2$	$2(q_2^2+B-A)$	$2(q_1p_2-q_2p_1)$
5 a)	-1	0	0	-1	0	0
b)	$\frac{-1}{q_2^2}$	0	$-2q_1$	$-q_2$	0	$-p_2$
6 a)	$\frac{-4}{q_1^2}$	0	0	$\frac{q_1}{2}$	$q_2+\frac{B-4A}{4D}$	$\frac{-p_1}{2}$
b)	$\frac{-36}{D^2q_1^6}$	0	0	$\frac{D}{6}q_1^3$	$\frac{p_1^2}{2}+\frac{q_1^2}{2}(A+2Dq_2)$	$-\frac{D}{2}p_1q_1^2$
c)	-1	0	0	-1	0	0

$\rho=f^{-1}$, $J_2=\rho(J_{ij})$. Here $J_{ij}=0$, $J_{ij}=-J_{ji}$, $i,j=1,...,4$.

Table III

Integrals of motion	f	J_{12}	J_{13}	$J_{14}=J_{23}$	J_{24}	J_{34}
1	$\frac{1}{4Cq_1^2}$	0	$2q_1^2$	$2q_1q_2$	$2(q_2^2+C)$	$2(q_1p_2-q_2p_1)$
2	$\frac{1}{4Cq_1^2 \pm C^2 \pm 4iCq_1q_2}$	0	$2q_1^2$	$2q_1q_2 \pm Ci$	$2(q_2^2+C)$	$2(q_1p_2-q_2p_1)$
3	$-\frac{1}{q_1^2}$	0	0	q_1	$2q_2$	$-p_1$
4	$\frac{1}{q_1-q_1^2+q_2^2-iq_2(1+2q_1)}$	0	$-i(2q_1+\frac{1}{4})$	$(q_1-\frac{1}{4}-iq_2)$	$2q_2+\frac{i}{4}$	$-(p_1+ip_2)$
5	$\frac{-1}{\pm 4iq_1q_2+(-q_1 \pm iq_2)^2}$	0	$\mp 2iq_1$	$q_1 \mp iq_2$	$2q_2$	$-(p_1 \pm ip_2)$
6	1	0	0	$\mp i$	2	0

$\rho = f^{-1}$, $J_2 = \rho(J_{ij})$. Here $J_{ii}=0, J_{ij}=-J_{ji}$, $i,j=1,...,4$.

For case (7) of table III, $QT - RS = 0$, therefore the second Poisson structure-matrix does not exist.

6 Conclussion

We have presented a simple and systematic method for finding first integrals for dynamical systems, using multiplier functions. Here, we wish to emphasize that this method neither requires generalized symmetries nor Noether´s theorem to obtain the sufficient integrals. On utilising these multiplier functions we have proposed a method to constructing nondegenerate Poisson structures with respect to which a given Hamiltonian system with two degrees of freedom is rather a weak bi–Hamiltonian. We have applied this procedure to a class of physically interesting systems and determined their integrals and second Poisson structures.

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