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On the Weak Supersymmetry

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Abstract

Analyzing the spectrum of the Schrödinger-Pauli Hamiltonian for a particle of spin $s > 1/2$ we find that some energy levels are degenerated while the other are not. We investigate the symmetry (which is neither super- nor parasymmetry) causing this specific degeneration.

In quantum mechanics, supersymmetry [1] and parasupersymmetry [2, 3] appear as fine extensions of the usual Lie or dynamical symmetries. In particular they enable to explain such specific degenerations of energy spectra which are not caused by a dynamical symmetry [1].

In this paper we want to point out that there exist a symmetry (called the weak supersymmetry (WSS) in the following) which is in some sense more fine than symmetries of super- and parasupersymmetric quantum mechanics. Like supersymmetry, WSS is realized by charges commuting with the Hamiltonian and causing the specific degeneration of its spectrum. But in contrast with super- and parasupersymmetries, weak supercharges do not generate the Hamiltonian in a unique fashion, and moreover, the mentioned degeneration characterizes the limited set of energy levels.

Consider the Schrödinger-Pauli equation for an arbitrary spin particle

$$i\frac{\partial\Psi}{\partial t} = H\Psi \equiv \left(\frac{\pi^2}{2m} + \frac{eg}{2m}\mathbf{S}\cdot\mathbf{H}\right)\psi, \qquad g = \frac{1}{s}, \tag{1}$$

where

$$\pi^2 = \pi_1^2 + \pi_2^2 + \pi_3^2, \quad \pi_a = p_a - eA_a, \quad p_a = -i\partial/\partial x_a,\ a = 1, 2, 3, \quad \mathbf{H} = i\pi\times\pi.$$

Here S_a are the spin matrices realizing an irreducible representation $D(s)$ of the algebra $AO(3)$, so that

$$[S_a, S_b] = i\varepsilon_{abc}S_c, \qquad S_aS_a = s(s+1). \tag{2}$$

Equation (1) appears as a nonrelativistic approximation of Poincaré-invariant equations for arbitrary spin particles, interacting minimally with an external field, see, e.g., [4]. Moreover, the Galilei-invariant wave equations [5, 6] also reduce to the form (1). In both cases we have $g = 1/s$, but this value can be corrected by introducing the anomalous interaction [7].

In the case of a particle in a constant magnetic field directed along the third axis we have in (1)

$$A_0 = A_3 = 0, \qquad A_1 = -x_a/2,$$
$$A_2 = x_1 H/2, \quad H_1 = H_2 = 0, \quad H_3 = H = \text{const.} \tag{3}$$

Moreover, we set $p_3 = 0$ for simplicity.

If $s = 1/2$ then $S_a = \sigma_a/2, \sigma_a$ are the Pauli matrices. The eigenvalues of the corresponding Hamiltonian H of (1), (3) have the form [8]

$$E_{n\nu} = \frac{1}{2}\left(2n + 1 + \frac{\nu}{s}\right)\frac{eH}{2m}, \tag{4}$$

where $n = 0, 1, \ldots, \nu = \pm s = \pm 1/2$. These eigenvalues (Landau levels) are double degenerated since

$$E_{n \pm s} = E_{n\pm 1\ \mp s}, \tag{5}$$

where the only exception is the ground state level E_{0-s} which is a singlet.

A possible explanation of this degeneration is that the corresponding equation (1), (3) admits two specific symmetries (supercharges)

$$Q_1 = \frac{1}{\sqrt{2m}}(\sigma_1\pi_1 + \sigma_2\pi_2), \qquad Q_2 = \frac{1}{\sqrt{2m}}(\sigma_1\pi_2 - \sigma_2\pi_1) \tag{6}$$

which satisfy the following relations

$$[Q_1, H] = [Q_2, H] = 0, \tag{7}$$

$$[Q_1, Q_2] = 0, \qquad Q_1^2 = Q_2^2 = H. \tag{8}$$

The relations (7), (8) determine the superalgebra $sqm(2)$ and characterize a dinamical system whose energy values (i.e., the eigenvalues of the Hamiltonian H) have the same degeneracy as the Landau levels [1].

In the case of $s > 1/2$ the spectrum of the Hamiltonian (1), (3) is given again by formula (4), where, however, $\nu = s, s-1, \ldots, s$ [4]. Moreover, the energy levels corresponding to $\nu = \pm s$ have the typical supersymmetric degeneration (5) while the other levels are not degenerated.

Let us construct the weak supercharges causing this specific degeneration. For $s = 1$ we set in (6)

$$\sigma_1 = S_1^2 - S_2^2, \qquad \sigma_2 = S_1S_2 + S_2S_1. \tag{9}$$

Then, using (2) and the relations

$$S_aS_bS_c + S_cS_bS_a = \delta_{ab}S_c + \delta_{bc}S_a,$$

we find immediatelly that the operators (6), (9) are constants of motion, i.e., they commute with H of (1), (3). Furtheremore, these operators satisfy the anticommutation relations

(8) only on the subset of solutions $\Psi' = S\Psi$ corresponding to $\nu = \pm s$ and thus cause the supersymmetric degeneration of the related energy values.

Instead of (8) the weak supercharges satisfy the following double commutation relations

$$[Q_1, [Q_1, Q_2]] = Q_2 H, \qquad [Q_2, [Q_2, Q_1]] = Q_1 H \tag{10}$$

which are typical for parasupercharges, i.e., symmetries of parasupersymmetric quantum mechanics [3]. But in contrast with parasupercharges, the weak supercharges do not define the Hamiltonian in a unique fassion. Indeed, if Q_1, Q_2 are of the form (6), (9), the relations (10) are invariant with respect to a change

$$H \to H + (1 - S_3^2)\hat{H}(1 - S_3^2)$$

where $\hat{H}$ is an arbitrary operator defined on the three-component wave functions Ψ.

Thus we find a specific symmetry of the equation (1), (3) for $s = 1$. It is generated by the weak supercharges Q_A of (6), (9), which satisfy the relations (7), (10) typical for parasupersymmetric models, but cause the supersymmetric degeneration of the subset of the Hamiltonian eigenvalues. Moreover, the weak supercharges do not satisfy the relations (7), (10) characterizing usual supercharges.

The limited number of degenerated eigenvalues and absence of the strong condition (8) for charges (which is replaced by the weaker condition (10)) are the distinquishing features of the weak supersymmetry in comparison with the usual supersymmetry.

We notice that the weak supersymmetry is valid for the Schrödinger-Pauli equation (1) describing a particle of arbitrary spin in the uniform magnetic field, where

$$A_0 = A_3 = 0, \qquad A_1 = A_1(x_1, x_2), \qquad A_2 = A_2(x_1, x_2). \tag{11}$$

The corresponding weak supercharges for $s = 1/2$ and $s = 1$ are given by relations (6), (11) and (9). For $s = 3/2$ we can set in (6)

$$\sigma_1 = \frac{1}{12}(S_1^3 - S_1^2 S_2 - S_2 S_1^2 - S_1 S_2 S_1),$$

$$\sigma_2 = \frac{1}{12}(S_2^3 + S_2 S_1^2 + S_1^2 S_2 + S_2 S_1 S_2) \tag{12}$$

and for arbitrary s

$$\sigma_1 = \frac{1}{k_s}\left[(S_1 - iS_2)^{2s} + (S_1 + iS_2)^{2s}\right],$$

$$\sigma_2 = \frac{1}{k_s}\left[(S_1 - iS_2)^{2s} - (S_1 + iS_2)^{2s}\right], \tag{13}$$

$$k_s = \prod_{n=1}^{2s-1} [2sn - n(n-1)]^{\frac{1}{2}}\, C_{2s-1}^{2s-n}$$

where C_b^a is the number of combinations from b elements by a.

The considered equation admits weak supersymmetry also for the case of any value of $g = 1/\nu_0$ with a fixed integer (for integer s) or half integer (for integer s) $\nu_0, -s \le \nu_0 \le s$.

For example, if $s = 3/2, g = 2$, then the correcponding supercharges have the form (6), (11) with the following σ-matrices

$$\sigma_1 = \frac{1}{16\sqrt{3}}(9 - 4S_3^2)S_1(9 - 4S_3^2), \qquad \sigma_2 = \frac{1}{16\sqrt{3}}(9 - 4S_3^2)S_2(9 - 4S_3^2).$$

For another symmetries of a type "between super- and parasupersymmetries" see [9], [10]. We notice that the charges found in [9] are linear combinations of parasupercharges obtained in [11].

In conclusion we note that relations (7), (8), satisfied by weak supercharges, are valid for a lot of symmetries. It is the case for supercharges (since (7), (9) is a consequence of (7), (8)), for parasupercharges [3], and for the hidden symmetries of Rubakov-Spiridonov parasupersymmetric quantum mechanics [2] found in [12]. The two dimensional Schrödinger equation with the Coulomb potential

$$i\frac{\partial}{\partial t}\psi = \left[\frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{e}{x}\right]\Psi, \qquad x = \sqrt{x_1^2 + x_2^2}$$

also admits the two symmetries (Runge-Lentz vector components)

$$Q_1 = \frac{x_1}{x} + \frac{1}{em}(x_1 p_1^2 - x_2 p_1 p_2 + i p_1/2),$$
$$Q_2 = \frac{x_2}{x} + \frac{1}{em}(x_2 p_1^2 - x_1 p_1 p_2 + i p_2/2),$$

satisfying (7), (10).

Thus, WSS and the symmetries mentioned above are nothing but different realizations of the algebra (7), (9).

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