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Symmetry reduction and exact solutions of the Navier-Stokes equations. II*

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6 Symmetry properties and exact solutions of system (3.12)

As was mentioned in Sec. 3, ansatzes (3.4)–(3.7) reduce the NSEs (1.1) to the systems of PDEs of a similar structure that have the general form (see (3.12)):

$$\begin{aligned} &w^i w^1_i - w^1_{ii} + s_1 + \alpha_2 w^2 = 0,\\ &w^i w^2_i - w^2_{ii} + s_2 - \alpha_2 w^1 + \alpha_1 w^3 = 0,\\ &w^i w^3_i - w^3_{ii} + \alpha_4 w^3 + \alpha_5 = 0,\\ &w^i_i = \alpha_3, \end{aligned} \tag{6.1}$$

where α_n $(n = \overline{1,5})$ are real parameters.

Setting $\alpha_k = 0$ $(k = \overline{2,5})$ in (6.1), we obtain equations describing a plane convective flow that is brought about by nonhomogeneous heating of boudaries [25]. In this case w^i are the coordinates of the flow velocity vector, w^3 is the flow temperature, s is the pressure, the Grasshoff number λ is equal to $-\alpha_1$, and the Prandtl number σ is equal to 1. Some similarity solutions of these equations were constructed in [22]. The particular case of system (6.1) for $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = 0$ and $\alpha_3 = 1$ was considered in [31].

In this section we study symmetry properties of system (6.1) and construct large sets of its exact solutions.

Theorem 6.1 *The MIA of (6.1) is the algebra*

1. $E_1 = <\partial_1,\, \partial_2,\, \partial_s>$ *if* $\alpha_1 \neq 0,\, \alpha_4 \neq 0$.
2. $E_2 = <\partial_1,\, \partial_2,\, \partial_s,\, \partial_{w^3} - \alpha_1 z_2 \partial_s>$ *if* $\alpha_1 \neq 0,\, \alpha_4 = 0,\, (\alpha_1, \alpha_2, \alpha_5) \neq (0, 0, 0)$.

3. $E_3 = <\partial_1,\ \partial_2,\ \partial_s,\ \partial_{w^3} - \alpha_1 z_2 \partial_s,\ \tilde{D} - 3w^3\partial_{w^3}>$ *if* $\alpha_1 \neq 0,\ \alpha_k = 0,\ k = \overline{2,5}.$

4. $E_4 = <\partial_1,\ \partial_2,\ \partial_s,\ J,\ (w^3 + \alpha_5/\alpha_4)\partial_{w^3}>$ *if* $\alpha_1 = 0,\ \alpha_4 \neq 0.$

5. $E_5 = <\partial_1,\ \partial_2,\ \partial_s,\ J,\ \partial_{w^3}>$ *if* $\alpha_1 = \alpha_4 = 0,\ (\alpha_2, \alpha_3) \neq (0,0),\ \alpha_5 \neq 0.$

6. $E_6 = <\partial_1,\ \partial_2,\ \partial_s,\ J,\ \partial_{w^3},\ w^3\partial_{w^3}>$ *if* $\alpha_1 = \alpha_4 = \alpha_5 = 0,\ (\alpha_2, \alpha_3) \neq (0,0).$

7. $E_7 = <\partial_1,\ \partial_2,\ \partial_s,\ J,\ \partial_{w^3},\ \tilde{D} + 2w^3\partial_{w^3}>$ *if* $\alpha_5 \neq 0,\ \alpha_l = 0,\ l = \overline{1,4}.$

8. $E_8 = <\partial_1,\ \partial_2,\ \partial_s,\ J,\ \partial_{w^3},\ \tilde{D}, w^3\partial_{w^3}>$ *if* $\alpha_n = 0,\ n = \overline{1,5}.$

Here $\tilde{D} = z_i\partial_i - w^i\partial_{w^i} - 2s\partial_s,\quad J = z_1\partial_2 - z_2\partial_1 + w^1\partial_{w^2} - w^2\partial_{w^1},\quad \partial_i = \partial_{z_i}.$

Note 6.1 *The bases of the algebras* E_6 *and* E_8 *contain the operator* $w^3\partial_{w^3}$ *that is not induced by elements of* $A(NS)$*.*

Note 6.2 *If* $\alpha_4 \neq 0$*, the constant* α_5 *can be made to vanish by means of local transformation*

$$\tilde{w}^3 = w^3 + \alpha_5/\alpha_4, \quad \tilde{s} = s - \alpha_1\alpha_5\alpha_4^{-1}z_2, \tag{6.2}$$

where the independent variables and the functions w^i *are not transformed. Therefore, we consider below that* $\alpha_5 = 0$ *if* $\alpha_4 \neq 0$ *.*

Note 6.3 *Making the non-local transformation*

$$\tilde{s} = s + \alpha_2\Psi, \tag{6.3}$$

where $\Psi_1 = w^2$, $\Psi_2 = -w^1$ *(such a function* Ψ *exists in view of the last equation of (6.1)), in system (6.1) with* $\alpha_3 = 0$*, we obtain a system of form (6.1) with* $\tilde{\alpha}_3 = \tilde{\alpha}_2 = 0$*. In some cases (*$\alpha_1 \neq 0,\ \alpha_3 = \alpha_4 = \alpha_5 = 0,\ \alpha_2 \neq 0;\ \alpha_1 = \alpha_3 = \alpha_4 = 0,\ \alpha_2 \neq 0$*) transformation (6.3) allows the symmetry of (6.1) to be extended and non-Lie solutions to be constructed. Moreover, it means that in the cases listed above system (6.1) is invariant under the non-local transformation*

$$\hat{z}_i = e^{\varepsilon}z_i, \quad \hat{w}^i = e^{-\varepsilon}w^i, \quad \hat{w}^3 = e^{\delta\varepsilon}w^3, \quad \hat{s} = e^{-2\varepsilon}s + \alpha_2(e^{-2\varepsilon} - 1)\Psi,$$

where $\delta = -3$ *if* $\alpha_3 = \alpha_4 = \alpha_5 = 0, \quad \alpha_1, \alpha_2 \neq 0;$
$\delta = 2$ *if* $\alpha_1 = \alpha_3 = \alpha_4 = 0, \quad \alpha_2, \alpha_5 \neq 0;$
$\delta = 0$ *if* $\alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = 0, \quad \alpha_2 \neq 0.$

Let us consider an ansatz of the form:

$$\begin{aligned} w^1 &= a_1\varphi^1 - a_2\varphi^3 + b_1\omega_2, \\ w^2 &= a_2\varphi^1 + a_1\varphi^3 + b_2\omega_2, \\ w^3 &= \varphi^2 + b_3\omega_2, \\ s &= h + d_1\omega_2 + d_2\omega_1\omega_2 + \tfrac{1}{2}d_3\omega_2^2, \end{aligned} \tag{6.4}$$

where $a_1^2 + a_2^2 = 1,\ \omega = \omega_1 = a_1z_2 - a_2z_1,\ \omega_2 = a_1z_1 + a_2z_2,\ B, b_a, d_a = \text{const},$

$$\begin{aligned} &b_i = Ba_i, \quad b_3(B + \alpha_4) = 0, \\ &d_2 = \alpha_2B - \alpha_1b_3a_1, \quad d_3 = -B^2 - \alpha_1b_3a_2, \end{aligned} \tag{6.5}$$

Table 1: Complete sets of inequivalent one-dimensional subalgebras of the algebras $E_1 - E_8$ (a and a_l ($l = \overline{1,4}$) are real constants)

Algebra	Subalgebras	Values of parameters
E_1	$< a_1\partial_1 + a_2\partial_2 + a_3\partial_s >,\ < \partial_s >$	$a_1^2 + a_2^2 = 1$
E_2	$< a_1\partial_1 + a_2\partial_2 + a_3(\partial_{w^3} - \alpha_1 z_2\partial_s) >$, $< \partial_1 + a_4\partial_s >, < \partial_{w^3} - \alpha_1 z_2\partial_s >, < \partial_s >$	$a_1^2 + a_2^2 = 1$, $a_4 \neq 0$
E_3	$< a_1\partial_1 + a_2\partial_2 + a_3(\partial_{w^3} - \alpha_1 z_2\partial_s) >, < \partial_1 + a_4\partial_s >$, $< \tilde{D} - 3w^3\partial_{w^3} >, < \partial_{w^3} - \alpha_1 z_2\partial_s >, < \partial_s >$	$a_1^2 + a_2^2 = 1$, $a_3 \in \{-1; 0; 1\}$, $a_4 \in \{-1; 1\}$
E_4	$< J + a_1\partial_s + a_2 w^3\partial_{w^3} >, < \partial_2 + a_1\partial_s + a_2 w^3\partial_{w^3} >$, $< w^3\partial_{w^3} + a_1\partial_s >, < \partial_s >$	
E_5	$< J + a_1\partial_s + a_2\partial_{w^3} >, < \partial_2 + a_1\partial_s + a_2\partial_{w^3} >$, $< \partial_{w^3} + a_1\partial_s >, < \partial_s >$	
E_6	$< J + a_1\partial_s + a_2 w^3\partial_{w^3} >, < \partial_2 + a_1\partial_s + a_2 w^3\partial_{w^3} >$, $< J + a_1\partial_s + a_3\partial_{w^3} >, < \partial_2 + a_1\partial_s + a_3\partial_{w^3} >$, $< w^3\partial_{w^3} + a_1\partial_s >, < \partial_{w^3} + a_1\partial_s >, < \partial_s >$	$a_2 \neq 0$, $a_3 \in \{-1; 0; 1\}$
E_7	$< \tilde{D} + aJ + 2w^3\partial_{w^3} >, < J + a_1\partial_s + a_2\partial_{w^3} >$, $< \partial_2 + a_1\partial_s + a_2\partial_{w^3} >, < \partial_{w^3} + a_2\partial_s >, < \partial_s >$	$a_2 \in \{-1; 0; 1\}$, $a_1 \in \{-1; 0; 1\}$ if $a_2 = 0$
E_8	$< \tilde{D} + aJ + a_3 w^3\partial_{w^3} >, < \tilde{D} + aJ + a_3\partial_{w^3} >$, $< J + a_1\partial_s + a_4 w^3\partial_{w^3} >, < \partial_2 + a_1\partial_s + a_4 w^3\partial_{w^3} >$, $< J + a_1\partial_s + a_2\partial_{w^3} >, < \partial_2 + a_1\partial_s + a_2\partial_{w^3} >$, $< w^3\partial_{w^3} + a_1\partial_s >, < \partial_{w^3} + a_1\partial_s >, < \partial_s >$	$a_i \in \{-1; 0; 1\}$, $a_4 \neq 0$

Here and below $\varphi^a = \varphi^a(\omega)$ and $h = h(\omega)$. Indeed, formulas (6.4) and (6.5) determine a whole set of ansatzes for system (6.1). This set contains both Lie ansatzes, constructed by means of subalgebras of the form

$$< a_1\partial_1 + a_2\partial_2 + a_3(\partial_{w^3} - \alpha_1 z_2\partial_s) + a_4\partial_s >, \tag{6.6}$$

and non-Lie ansatzes. Equation (6.5) is the necessary and sufficient condition to reduce (6.1) by means of an ansatz of form (6.3). As a result of reduction we obtain the following system of ODEs:

$$\begin{aligned}
&\varphi^3\varphi^1_\omega - \varphi^1_{\omega\omega} + \mu_{1j}\varphi^j + d_1 + d_2\omega + \alpha_2\varphi^3 = 0,\\
&\varphi^3\varphi^2_\omega - \varphi^2_{\omega\omega} + \mu_{2j}\varphi^j + \alpha_5 = 0,\\
&\varphi^3\varphi^3_\omega - \varphi^3_{\omega\omega} + h_\omega - \alpha_2\varphi^1 + \alpha_1 a_1\varphi^2 = 0,\\
&\varphi^3_\omega = \sigma,
\end{aligned} \tag{6.7}$$

where $\mu_{11} = -B$, $\mu_{12} = -\alpha_1 a_2$, $\mu_{21} = -b_3$, $\mu_{22} = -\alpha_4$, $\sigma = \alpha_3 - B$. If $\sigma = 0$, system (6.7) implies that

$$\begin{aligned}
&\varphi^3 = C_0 = \text{const},\\
&h = \alpha_2 \textstyle\int \varphi^1(\omega)d\omega - \alpha_1 a_1 \int \varphi^2(\omega)d\omega,
\end{aligned}$$

and the functions φ^i satisfy system (4.23), where $\nu_{11} = d_1 + \alpha_2 C_0$, $\nu_{21} = d_2$, $\nu_{12} = \alpha_5$, $\nu_{22} = 0$. If $\sigma \neq 0$, then $\varphi^3 = \sigma\omega$ (translating ω, the integration constant can be made to vanish),

$$h = -\tfrac{1}{2}\sigma^2\omega^2 + \alpha_2 \textstyle\int \varphi^1(\omega)d\omega - \alpha_1 a_1 \int \varphi^2(\omega)d\omega,$$

and the functions satisfy system (4.29), where $\nu_{11} = d_1$, $\nu_{21} = d_2 + \alpha_2\sigma$, $\nu_{12} = \alpha_5$, $\nu_{22} = 0$.

Note 6.4 *Step-by-step reduction of the NSEs (1.1) by means of ansatzes (3.4)–(3.7) and (6.4) is equivalent to a particular case of immediate reduction of the NSEs (1.1) to ODEs by means of ansatzes 5 and 6 from Subsec. 4.1.*

Now let us choose such algebras, among the algebras from Table 1, that can be used to reduce system (6.1) and do not belong to the set of algebras (6.6). By means of the chosen algebras we construct ansatzes that are tabulated in the form of Table 2.

Substituting the ansatzes from Table 2 into system (6.1), we obtain the reduced systems of ODEs in the functions φ^a and h:

1. $$\begin{aligned}
&\varphi^2\varphi^1_\omega - \varphi^1_{\omega\omega} - \varphi^1\varphi^1 - \varphi^2\varphi^2 - 2h + \alpha_1\varphi^3\sin\omega + 2\varphi^2_\omega = 0,\\
&\varphi^2\varphi^2_\omega - \varphi^2_{\omega\omega} + h_\omega - 2\varphi^1_\omega + \alpha_1\varphi^3\cos\omega = 0,\\
&\varphi^2\varphi^3_\omega - \varphi^3_{\omega\omega} - 3\varphi^1\varphi^3 - 9\varphi^3 = 0,\\
&\varphi^2_\omega = 0.
\end{aligned} \tag{6.8}$$

Table 2: Ansatzes reducing system (6.1) $(r = (z_1^2 + z_2^2)^{1/2})$

N	Values of α_n	Algebra	Invariant variable	Ansatz
1	$\alpha_1 \neq 0$, $\alpha_k = 0$, $k = \overline{2,5}$	$< \tilde{D} - 3w^3\partial_{w^3} >$	$\omega = \arctan\frac{z_2}{z_1}$	$w^1 = r^{-2}(z_1\varphi^1 - z_2\varphi^2)$, $w^2 = r^{-2}(z_2\varphi^1 + z_1\varphi^2)$, $w^3 = r^{-3}\varphi^3$, $s = r^{-2}h$
2	$\alpha_1 = 0$, $\alpha_5 = 0$	$< \partial_2 + a_1\partial_s + a_2w^3\partial_{w^3} >$, $a_2 \neq 0$	$\omega = z_1$	$w^1 = \varphi^1$, $w^2 = \varphi^2$, $w^3 = \varphi^3 e^{a_2 z_2}$, $s = h + a_1 z_2$
3	$\alpha_1 = 0$, $\alpha_4 = 0$	$< J + a_1\partial_s + a_2\partial_{w^3} >$	$\omega = r$	$w^1 = z_1\varphi^1 - z_2 r^{-2}\varphi^2$, $w^2 = z_2\varphi^1 + z_1 r^{-2}\varphi^2$, $w^3 = \varphi^3 + a_2\arctan\frac{z_2}{z_1}$, $s = h + a_1\arctan\frac{z_2}{z_1}$
4	$\alpha_1 = 0$, $\alpha_5 = 0$	$< J + a_1\partial_s + a_2 w_3\partial_{w^3} >$ $a_2 \neq 0$ if $\alpha_4 = 0$	$\omega = r$	$w^1 = z_1\varphi^1 - z_2 r^{-2}\varphi^2$, $w^2 = z_2\varphi^1 + z_1 r^{-2}\varphi^2$, $w^3 = \varphi^3 e^{a_2\arctan\frac{z_2}{z_1}}$, $s = h + a_1\arctan\frac{z_2}{z_1}$
5	$\alpha_5 \neq 0$, $\alpha_l = 0$, $l = \overline{1,4}$	$< \tilde{D} + aJ + 2w^3\partial_{w^3} >$	$\omega = \arctan\frac{z_2}{z_1} - a\ln r$	$w^1 = r^{-2}(z_1\varphi^1 - z_2\varphi^2)$, $w^2 = r^{-2}(z_2\varphi^1 + z_1\varphi^2)$, $w^3 = r^2\varphi^3$, $s = r^{-2}h$
6	$\alpha_n = 0$, $n = \overline{1,5}$	$< \tilde{D} + aJ + a_1\partial_{w^3} >$	$\omega = \arctan\frac{z_2}{z_1} - a\ln r$	$w^1 = r^{-2}(z_1\varphi^1 - z_2\varphi^2)$, $w^2 = r^{-2}(z_2\varphi^1 + z_1\varphi^2)$, $w^3 = \varphi^3 + a_1\ln r$, $s = r^{-2}h$
7	$\alpha_n = 0$, $n = \overline{1,5}$	$< \tilde{D} + aJ + a_1w^3\partial_{w^3} >$, $a_1 \neq 0$	$\omega = \arctan\frac{z_2}{z_1} - a\ln r$	$w^1 = r^{-2}(z_1\varphi^1 - z_2\varphi^2)$, $w^2 = r^{-2}(z_2\varphi^1 + z_1\varphi^2)$, $w^3 = r^{a_1}\varphi^3$, $s = r^{-2}h$

2. $$\begin{aligned}
&\varphi^1\varphi^1_\omega - \varphi^1_{\omega\omega} + \alpha_2\varphi^2 + h_\omega = 0,\\
&\varphi^1\varphi^2_\omega - \varphi^2_{\omega\omega} - \alpha_2\varphi^1 + a_1 = 0,\\
&\varphi^1\varphi^3_\omega - \varphi^3_{\omega\omega} + (a_2\varphi^2 + \alpha_4 - a_2^2)\varphi^3 = 0,\\
&\varphi^1_\omega = \alpha_3.
\end{aligned} \tag{6.9}$$

3. $$\begin{aligned}
&\omega\varphi^1\varphi^1_\omega - \varphi^1_{\omega\omega} + \varphi^1\varphi^1 - \omega^{-4}\varphi^2\varphi^2 - 3\omega^{-1}\varphi^1_\omega +\\
&\quad \alpha_2\omega^{-2}\varphi^2 + \omega^{-1}h_\omega = 0,\\
&\omega\varphi^1\varphi^2_\omega - \varphi^2_{\omega\omega} + \omega^{-1}\varphi^2_\omega - \alpha_2\omega^2\varphi^1 + a_1 = 0,\\
&\omega\varphi^1\varphi^3_\omega - \varphi^3_{\omega\omega} + a_2\omega^{-2}\varphi^2 - \omega^{-1}\varphi^3_\omega + \alpha_5 = 0,\\
&2\varphi^1 + \omega\varphi^1_\omega = \alpha_3.
\end{aligned} \tag{6.10}$$

4. $$\begin{aligned}
&\omega\varphi^1\varphi^1_\omega - \varphi^1_{\omega\omega} + \varphi^1\varphi^1 - \omega^{-4}\varphi^2\varphi^2 - 3\omega^{-1}\varphi^1_\omega +\\
&\quad \alpha_2\omega^{-2}\varphi^2 + \omega^{-1}h_\omega = 0,\\
&\omega\varphi^1\varphi^2_\omega - \varphi^2_{\omega\omega} + \omega^{-1}\varphi^2_\omega - \alpha_2\omega^2\varphi^1 + a_1 = 0,\\
&\omega\varphi^1\varphi^3_\omega - \varphi^3_{\omega\omega} + a_2\omega^{-2}\varphi^2\varphi^3 - \omega^{-1}\varphi^3_\omega + (\alpha_4 - a_2^2\omega^{-2})\varphi^3 = 0,\\
&2\varphi^1 + \omega\varphi^1_\omega = \alpha_3.
\end{aligned} \tag{6.11}$$

5. $$\begin{aligned}
&(\varphi^2 - a\varphi^1)\varphi^1_\omega - (1+a^2)\varphi^1_{\omega\omega} - \varphi^1\varphi^1 - \varphi^2\varphi^2 - ah_\omega - 2h = 0,\\
&(\varphi^2 - a\varphi^1)\varphi^2_\omega - (1+a^2)\varphi^2_{\omega\omega} - 2(a\varphi^2_\omega + \varphi^1_\omega) + h_\omega = 0,\\
&(\varphi^2 - a\varphi^1)\varphi^3_\omega - (1+a^2)\varphi^3_{\omega\omega} + 2\varphi^1\varphi^3 - 4\varphi^3 + 4a\varphi^3_\omega + \alpha_5 = 0,\\
&\varphi^2_\omega - a\varphi^1_\omega = 0.
\end{aligned} \tag{6.12}$$

6. $$\begin{aligned}
&(\varphi^2 - a\varphi^1)\varphi^1_\omega - (1+a^2)\varphi^1_{\omega\omega} - \varphi^1\varphi^1 - \varphi^2\varphi^2 - ah_\omega - 2h = 0,\\
&(\varphi^2 - a\varphi^1)\varphi^2_\omega - (1+a^2)\varphi^2_{\omega\omega} - 2(a\varphi^2_\omega + \varphi^1_\omega) + h_\omega = 0,\\
&(\varphi^2 - a\varphi^1)\varphi^3_\omega - (1+a^2)\varphi^3_{\omega\omega} + a_1\varphi^1 = 0,\\
&\varphi^2_\omega - a\varphi^1_\omega = 0.
\end{aligned} \tag{6.13}$$

$$\begin{aligned}
7.\quad & (\varphi^2 - a\varphi^1)\varphi^1_\omega - (1+a^2)\varphi^1_{\omega\omega} - \varphi^1\varphi^1 - \varphi^2\varphi^2 - ah_\omega - 2h = 0,\\
& (\varphi^2 - a\varphi^1)\varphi^2_\omega - (1+a^2)\varphi^2_{\omega\omega} - 2(a\varphi^2_\omega + \varphi^1_\omega) + h_\omega = 0,\\
& (\varphi^2 - a\varphi^1)\varphi^3_\omega - (1+a^2)\varphi^3_{\omega\omega} + a_1\varphi^1\varphi^3 - a_1^2\varphi^3 + 2aa_1\varphi^3_\omega = 0,\\
& \varphi^2_\omega - a\varphi^1_\omega = 0.
\end{aligned} \tag{6.14}$$

Numeration of reduced systems (6.8)–(6.14) corresponds to that of the ansatzes in Table 2. Let us integrate systems (6.8)–(6.14) in such cases when it is possible. Below, in this section, $C_k = \mathrm{const}\ \ (k=\overline{1,6})$.

1. We failed to integrate system (6.8) in the general case, but we managed to find the following particular solutions:

a) $\varphi^1 = -6\wp(\omega + C_3, \frac{1}{3}(4-2C_1), C_2) - 2,$

$\varphi^2 = \varphi^3 = 0, \quad h = 2\varphi^1 + C_1;$

b) $\varphi^1 = -6C_1^2 e^{2C_1\omega}\wp(e^{C_1\omega} + C_3, 0, C_2) + 3C_1^2 - 2,$

$\varphi^2 = 5C_1, \quad \varphi^3 = 0,$

$h = -12C_1^2 e^{2C_1\omega}\wp(e^{C_1\omega} + C_3, 0, C_2) - 2 - \frac{13}{2}C_1^2 - \frac{9}{4}C_1^4;$

c) $\varphi^1 = C_1, \quad \varphi^2 = C_2, \quad \varphi^3 = 0, \quad h = -\frac{1}{2}(C_1^2 + C_2^2).$

Here $\wp(\tau, \varkappa_1, \varkappa_2)$ is the Weierstrass function that satisfies the equation (see [19]):

$$(\wp_\tau)^2 = 4\wp^3 - \varkappa_1\wp - \varkappa_2. \tag{6.15}$$

2. If $\alpha_3 = 0$, the last equation of (6.9) implies that $\varphi^1 = C_1$. It follows from the other equations of (6.9) that

$$\varphi^2 = C_3 + C_2 e^{C_1\omega} - (a_1 C_1^{-1} - \alpha_2)\omega,$$

$$h = C_6 - \alpha_2 C_3\omega - \alpha_2 C_2 C_1^{-1} e^{C_1\omega} + \tfrac{1}{2}\alpha_2(a_1 C_1^{-1} - \alpha_2)\omega^2$$

if $C_1 \neq 0$, and

$$\varphi^2 = C_3 + C_2\omega + \tfrac{1}{2}a_1\omega^2,$$

$$h = C_6 - \alpha_2 C_3\omega - \tfrac{1}{2}\alpha_2 C_2\omega^2 - \tfrac{1}{6}\alpha_2 a_1\omega^3$$

if $C_1 = 0$. The function φ^3 satisfies the equation

$$\varphi^3_{\omega\omega} - C_1\varphi^3_\omega + (a_2^2 - \alpha_4 - a_2\varphi^2)\varphi^3 = 0. \tag{6.16}$$

We solve equation (6.16) for the following cases:

A. $C_2 = a_1 - \alpha_2 C_1 = 0$:

$$\varphi^3 = \begin{cases} e^{\frac{1}{2}C_1\omega}(C_4 e^{\mu^{1/2}\omega} + C_5 e^{-\mu^{1/2}\omega}), & \mu > 0, \\ e^{\frac{1}{2}C_1\omega}(C_4 + C_5\omega), & \mu = 0, \\ e^{\frac{1}{2}C_1\omega}(C_4 \cos((-\mu)^{1/2}\omega) + C_5 \sin((-\mu)^{1/2}\omega)), & \mu < 0, \end{cases}$$

where $\mu = \frac{1}{4}C_1^2 - a_2^2 + \alpha_4 + a_2 C_3$.

B. $C_1 = a_1 = 0$, $C_2 \neq 0$ ([19]):

$$\varphi^3 = \xi^{1/2} Z_{1/3}(\tfrac{2}{3}(-a_2 C_2)^{1/2}\xi^{3/2}),$$

where $\xi = \omega + (C_3 a_2 - a_2^2 - \alpha_4)/(a_2 C_2)$. Here $Z_\nu(\tau)$ is the general solution of the Bessel equation (4.22).

C. $C_1 = 0$, $a_1 \neq 0$ ([19]):

$$\varphi^3 = (\omega + C_2 a_1^{-1})^{-1/2} W(\nu, \tfrac{1}{4}, (\tfrac{1}{2}a_1 a_2)^{-1/2}(\omega + C_2 a_1^{-1})^2),$$

where $\nu = \frac{1}{4}(\frac{1}{2}a_1 a_2)^{-1/2}(a_2^2 - \alpha_4 - a_2 C_3 + \frac{1}{2}a_2 C_3^2 a_1^{-1})$. Here $W(\varkappa, \mu, \tau)$ is the general solution of the Whittaker equation (4.21).

D. $C_1 \neq 0$, $C_2 \neq 0$, $a_1 - \alpha_2 C_1 = 0$ ([19]):

$$\varphi^3 = e^{\frac{1}{2}C_1\omega} Z_\nu(2C_1^{-1}(-a_2 C_2)^{1/2} e^{\frac{1}{2}C_1\omega}),$$

where $\nu = C_1^{-1}(C_1^2 + 4(\alpha_4 + a_2 C_3 - a_2^2))^{1/2}$. Here $Z_\nu(\tau)$ is the general solution of the Bessel equation (4.22).

E. $C_1 \neq 0$, $a_1 - \alpha_2 C_1 \neq 0$, $C_2 = 0$ ([19]):

$$\varphi^3 = e^{\frac{1}{2}C_1\omega}\xi^{1/2} Z_{1/3}\Big(\tfrac{2}{3}(a_2(a_1 C_1^{-1} - \alpha_2))^{1/2}\xi^{3/2}\Big),$$

where $\xi = \omega + (a_2^2 - \frac{1}{4}C_1^2 - C_3 a_2 - \alpha_4)/(a_2(a_1 C_1^{-1} - \alpha_2))$. Here $Z_\nu(\tau)$ is the general solution of the Bessel equation (4.22).

If $\alpha_3 \neq 0$, then $\varphi^1 = \alpha_3\omega$ (translating ω, the integration constant can be made to vanish),

$$\varphi^2 = C_1 + C_2 \textstyle\int e^{\frac{1}{2}\alpha_3\omega^2} d\omega + a_1 \int e^{\frac{1}{2}\alpha_3\omega^2}\Big(\int e^{-\frac{1}{2}\alpha_3\omega^2} d\omega\Big) d\omega + \alpha_2\omega,$$

$$h = C_3 - \tfrac{1}{2}(\alpha_2^2 + \alpha_3^2)\omega^2 - \alpha_2 C_1\omega - \alpha_2 C_2\Big(\omega \textstyle\int e^{\frac{1}{2}\alpha_3\omega^2} d\omega - \alpha_3^{-1} e^{\frac{1}{2}\alpha_3\omega^2}\Big) -$$

$$\alpha_2 a_1\Big(\omega \textstyle\int e^{\frac{1}{2}\alpha_3\omega^2}(\int e^{-\frac{1}{2}\alpha_3\omega^2} d\omega) d\omega - \alpha_3^{-1} e^{\frac{1}{2}\alpha_3\omega^2} \int e^{-\frac{1}{2}\alpha_3\omega^2} d\omega + \alpha_3^{-1}\omega\Big),$$

and the function φ^3 satisfies the equation

$$\varphi^3_{\omega\omega} - \alpha_3\omega\varphi^3_\omega + (a_2^2 - \alpha_4 - a_2\varphi^2)\varphi^3 = 0. \tag{6.17}$$

We managed to find a solution of (6.17) only for the case $a_1 = C_2 = 0$, i.e.,

$$\varphi^3 = e^{\frac{1}{4}\alpha_3\omega^2} V(\alpha_3^{1/2}(\omega + 2a_2\alpha_2\alpha_3^{-2}), \nu),$$

where $\nu = 4\alpha_3^{-1}(\alpha_4 + a_2C_1 - a_2^2(\alpha_2^2\alpha_3^{-2} + 1))$. Here $V(\tau, \nu)$ is the general solution of the Weber equation

$$4V_{\tau\tau} = (\tau^2 + \nu)V. \qquad (6.18)$$

3. The general solution of system (6.10) has the form:

$$\varphi^1 = C_1\omega^{-2} + \tfrac{1}{2}\alpha_3, \qquad (6.19)$$

$$\begin{aligned}\varphi^2 = {} & C_2 + C_3 \int \omega^{C_1+1} e^{\frac{1}{4}\alpha_3\omega^2} d\omega - \tfrac{1}{2}\alpha_2\omega^2 + \\ & a_1 \int \omega^{C_1+1} e^{\frac{1}{4}\alpha_3\omega^2} \Big(\int \omega^{-C_1-1} e^{-\frac{1}{4}\alpha_3\omega^2} d\omega\Big) d\omega,\end{aligned} \qquad (6.20)$$

$$\begin{aligned}\varphi^3 = {} & C_4 + C_5 \int \omega^{C_1-1} e^{\frac{1}{4}\alpha_3\omega^2} d\omega + \\ & \int \omega^{C_1-1} e^{\frac{1}{4}\alpha_3\omega^2} \Big(\int \omega^{1-C_1} e^{-\frac{1}{4}\alpha_3\omega^2} (\alpha_5 + a_2\omega^{-2}\varphi^2) d\omega\Big) d\omega,\end{aligned}$$

$$h = C_6 - \tfrac{1}{8}\alpha_3^2\omega^2 - \tfrac{1}{2}C_1^2\omega^{-2} + \int (\varphi^2(\omega))^2\omega^{-3} d\omega - \alpha_2 \int \omega^{-1}\varphi^2(\omega) d\omega. \qquad (6.21)$$

4. System (6.11) implies that the functions φ^i and h are determined by (6.19)–(6.21), and the function φ^3 satisfies the equation

$$\varphi^3_{\omega\omega} - ((C_1 - 1)\omega^{-1} + \tfrac{1}{2}\alpha_3\omega)\varphi^3_\omega + (a_2\omega^{-2}(a_2 - \varphi^2) - \alpha_4)\varphi^3 = 0. \qquad (6.22)$$

We managed to solve equation (6.22) in following cases:

A. $C_3 = a_1 = 0$, $\alpha_3 \neq 0$:

$$\varphi^3 = \omega^{\frac{1}{2}C_1 - 1} e^{\frac{1}{8}\alpha_3\omega^2} W(\varkappa, \mu, \tfrac{1}{4}\alpha_3\omega^2),$$

where $\varkappa = \frac{1}{4}(2 - C_1 - (4\alpha_4 + 2\alpha_2 a_2)\alpha_3^{-1})$, $\mu = \frac{1}{4}(C_1^2 - 4a_2^2 + 4a_2C_2)^{1/2}$. Here $W(\varkappa, \mu, \tau)$ is the general solution of the Whittaker equation (4.21).

Let $\alpha_3 = 0$, then

$$\varphi^2 = \begin{cases} C_2 + C_3 \ln\omega + \frac{1}{4}(a_1 + 2\alpha_2)\omega^2, & C_1 = -2, \\ C_2 + \frac{1}{2}C_3\omega^2 + \frac{1}{2}a_1\omega^2(\ln\omega - \frac{1}{2}), & C_1 = 0, \\ C_2 + C_3(C_1 + 2)^{-1}\omega^{C_1+2} - \frac{1}{2}C_1^{-1}(a_1 - \alpha_2C_1)\omega^2, & C_1 \neq 0, -2. \end{cases}$$

B. $C_3 = a_1 - \alpha_2C_1 = 0$:

$$\varphi^3 = \begin{cases} \omega^{\frac{1}{2}C_1} Z_\nu(\mu^{1/2}\omega), & \mu \neq 0, \\ \omega^{\frac{1}{2}C_1}(C_5\omega^\nu + C_6\omega^{-\nu}), & \mu = 0, \nu \neq 0, \\ \omega^{\frac{1}{2}C_1}(C_5 + C_6 \ln\omega), & \mu = 0, \nu = 0, \end{cases} \qquad (6.23)$$

where $\mu = -\alpha_4$, $\nu = \frac{1}{2}(C_1^2 - 4a_2^2 + 4a_2C_2)^{1/2}$. Here and below $Z_\nu(\tau)$ is the general solution of the Bessel equation (4.22).

C. $C_3 = 0$, $C_1 \neq 0$: φ^3 is determined by (6.23), where

$$\mu = \tfrac{1}{2}a_2C_1^{-1}(a_1 - \alpha_2C_1) - \alpha_4, \quad \nu = \tfrac{1}{2}(C_1^2 - 4a_2^2 + 4a_2C_2)^{1/2}.$$

D. $C_1 = a_1 = 0$: φ^3 is determined by (6.23), where

$$\mu = -\tfrac{1}{2}a_2C_3 - \alpha_4, \quad \nu = (-a_2^2 + a_2C_2)^{1/2}.$$

E. $C_3 \neq 0$, $C_1 \notin \{0; -2\}$, $a_2(a_1 - \alpha_2C_1) - 2\alpha_4C_1 = 0$:

$$\varphi^3 = \omega^{\frac{1}{2}C_1} Z_\nu(\mu\omega^{1+\frac{1}{2}C_1}),$$

where $\mu = 2C_3^{1/2}(C_1 + 2)^{-3/2}$, $\nu = (C_1 + 2)^{-1}(C_1^2 - 4a_2^2 + 4a_2C_2)^{1/2}$.

F. $C_1 = -2$, $C_3 \neq 0$, $a_2(a_1 + 2\alpha_2) + 4\alpha_4 = 0$ ([19]):

$$\varphi^3 = \omega^{-1}\xi^{1/2}Z_{1/3}(\tfrac{2}{3}C_3^{1/2}\xi^{3/2}),$$

where $\xi = \ln\omega + C_3^{-1}(a_2^2 - a_2C_2 - 1)$.

G. $C_1 = 2$, $C_3 < 0$, $1 - a_2^2 + a_2C_2 \geq 0$:

$$\varphi^3 = W(\varkappa, \mu, \tfrac{1}{2}(-C_3)^{1/2}\omega^2),$$

where $\varkappa = \frac{1}{8}(-C_3)^{-1/2}(-4\alpha_4 + a_2^2 - 2\alpha_2a_2)$, $\mu = \frac{1}{2}(1 - a_2^2 + a_2C_2)^{1/2}$. Here $W(\varkappa, \mu, \tau)$ is the general solution of the Whittaker equation (4.21).

5–7. Identical corollaries of system (6.12), (6.13), and (6.14) are the equations

$$\varphi^2 = a\varphi^1 + C_1, \tag{6.24}$$

$$h = a(1 + a^2)\varphi^1_\omega + (2 + 2a^2 - aC_1)\varphi^1 + C_2, \tag{6.25}$$

$$(1 + a^2)\varphi^1_{\omega\omega} + (4a - C_1)\varphi^1_\omega + \varphi^1\varphi^1 + 4\varphi^1 + (1 + a^2)^{-1}(C_1^2 + 2C_2) = 0. \tag{6.26}$$

We found the following solutions of (6.26):

A. If $(1 + a^2)^{-1}(C_1^2 + 2C_2) < 4$:

$$\varphi^1 = (4 - (1 + a^2)^{-1}(C_1^2 + 2C_2))^{1/2} - 2. \tag{6.27}$$

B. If $C_1 = 4a$:

$$\varphi^1 = -6\wp\left(\frac{\omega}{(1 + a^2)^{1/2}} + C_4,\ \frac{4}{3} - \frac{(C_1^2 + 2C_2)}{3(1 + a^2)},\ C_3\right) - 2. \tag{6.28}$$

Here and below $\wp(\tau, \varkappa_1, \varkappa_2)$ is the Weierstrass function satisfying equation (6.15). If $C_2 = 2 - 6a^2$ and $C_3 = 0$, a particular case of (6.28) is the function

$$\varphi^1 = -6(1 + a^2)\omega^2 - 2 \tag{6.29}$$

(the constant C_4 is considered to vanish).

C. If $C_1 \neq 4a$, $(1+a^2)^{-1}(C_1^2+2C_2)-4=-9\mu^4$:

$$\varphi^1 = -6\mu^2 e^{-2\xi}\wp(e^{-\xi}+C_4,0,C_3)+3\mu^2-2, \tag{6.30}$$

where $\xi=(1+a^2)^{-1/2}\mu\omega$, $\mu=\frac{1}{5}(4a-C_1)(1+a^2)^{-1/2}$. If $C_3=0$, a paticular case of (6.30) is the function

$$\varphi^1 = -6\mu^2 e^{-2\xi}(e^{-\xi}+C_4)^{-2}+3\mu^2-2, \tag{6.31}$$

where the constant C_4 is considered not to vanish.

The function φ^3 has to be found for systems (6.12), (6.13), and (6.14) individually.

5. The function φ^3 satisfy the equation

$$(1+a^2)\varphi^3_{\omega\omega}-(C_1+4a)\varphi^3_\omega-(2\varphi^1-4)\varphi^3-\alpha_5=0.$$

If φ^1 is determined by (6.27), we obtain

$$\varphi^3=\exp(\tfrac{1}{2}(1+a^2)^{-1}(C_1+4a)\omega)\cdot$$

$$\left\{\begin{array}{ll} C_5\exp(\nu^{1/2}\omega)+C_6\exp(-\nu^{1/2}\omega), & \nu>0 \\ C_5\cos((-\nu)^{1/2}\omega)+C_6\sin((-\nu)^{1/2}\omega), & \nu<0 \\ C_5+C_6\omega, & \nu=0 \end{array}\right\}+$$

$$\left\{\begin{array}{lll} -\alpha_5(2\varphi^1-4)^{-1}, & 2\varphi^1-4\neq 0 & \\ -\alpha_5(4a+C_1)^{-1}\omega, & 2\varphi^1-4=0, & C_1+4a\neq 0 \\ \frac{1}{2}\alpha_5(1+a^2)^{-1}\omega^2, & 2\varphi^1-4=0, & C_1+4a=0 \end{array}\right\},$$

where $\nu=\frac{1}{4}(1+a^2)^{-2}(C_1+4a)^2-(1+a^2)^{-1}(4-2\varphi^1)$.

6. In this case φ^3 satisfy the equation

$$(1+a^2)\varphi^3_{\omega\omega}-C_1\varphi^3_\omega=a_1\varphi^1.$$

Therefore,

$$\varphi^3=C_5+C_6\exp((1+a^2)^{-1}C_1\omega)+a_1C_1^{-1}\Big(\int\varphi^1(\omega)d\omega+$$

$$\exp((1+a^2)^{-1}C_1\omega)\int\exp(-(1+a^2)^{-1}C_1\omega)\varphi^1(\omega)d\omega\Big)$$

for $C_1\neq 0$, and

$$\varphi^3=C_5+C_6\omega+a_1(1+a^2)^{-1}(\omega\int\varphi^1(\omega)d\omega-\int\omega\varphi^1(\omega)d\omega)$$

for $C_1=0$.

7. The function φ^3 satisfy the equation

$$(1+a^2)\varphi^3_{\omega\omega}-(C_1+2a_1a)\varphi^3_\omega+(a_1^2-a_1\varphi^1)\varphi^3=0. \tag{6.32}$$

A. If φ^1 is determined by (6.27), it follows that

$$\varphi^3 = \exp(\tfrac{1}{2}(1+a^2)^{-1}(C_1+2a_1a)\omega)\cdot$$

$$\left\{\begin{array}{ll} C_5\exp(\nu^{1/2}\omega)+C_6\exp(-\nu^{1/2}\omega), & \nu>0 \\ C_5\cos((-\nu)^{1/2}\omega)+C_6\sin((-\nu)^{1/2}\omega), & \nu<0 \\ C_5+C_6\omega, & \nu=0 \end{array}\right\},$$

where $\nu = \frac{1}{4}(1+a^2)^{-2}(C_1+2a_1a)^2-(1+a^2)^{-1}(a_1^2-a_1\varphi^1)$.

B. If $C_1=4a$, that is, φ^1 is determined by (6.27), we obtain

$$\varphi^3 = \exp(a(a_1+2)(1+a^2)^{-1}\omega)\theta(\tau),$$

where $\tau=(1+a^2)^{-1/2}\omega+C_4$. Here the function $\theta=\theta(\tau)$ is the general solution of of the following Lame equation ([19]):

$$\theta_{\tau\tau}+(6a_1\wp(\tau)+a_1^2+2a_1-a^2(2+a_1)^2(1+a^2)^{-1})\theta=0$$

with the Weierstrass function

$$\wp(\tau)=\wp\Big(\tau,\tfrac{1}{3}(4-(1+a^2)^{-1}(C_1^2+2C_2)),C_3\Big).$$

Consider the particular case when $C_2=2-6a^2$ and $C_3=0$ additionally, i.e., φ^1 can be given in form (6.29). Depending on the values of a and a_1, we obtain the following expression for φ^3:

Case 1. $a_1\neq-2$, $a_1\neq 2a^2$:

$$\varphi^3=|\omega|^{1/2}exp\left(\frac{a(2+a_1)}{1+a^2}\omega\right)Z_\nu\left(\frac{((2+a_1)(a_1-2a^2))^{1/2}}{1+a^2}\omega\right),$$

where $\nu=(\frac{1}{4}-6a_1)^{1/2}$.

Case 2. $a_1=-2$: $\quad\varphi^3=C_5\omega^4+C_6\omega^{-3}$.

Case 3. $a_1=2a^2$:

Case 3.1. $48a^2<1$: $\quad\varphi^3=|\omega|^{1/2}e^{2a\omega}(C_5\omega^\sigma+C_6\omega^{-\sigma})$,
where $\sigma=\frac{1}{2}\sqrt{1-48a^2}$.

Case 3.2. $48a^2=1$, that is, $a=\pm\frac{1}{12}\sqrt{3}$: $\quad\varphi^3=|\omega|^{1/2}(C_5+C_6\ln\omega)$.

Case 3.3. $48a^2>1$: $\quad\varphi^3=|\omega|^{1/2}e^{2a\omega}(C_5\cos(\gamma\ln\omega)+C_6\sin(\gamma\ln\omega))$,
where $\gamma=\frac{1}{2}\sqrt{48a^2-1}$.

C. Let the conditions

$$C_1\neq 4a,\quad (1+a^2)^{-1}(C_1^2+2C_2)-4=-9\mu^4$$

be satisfied, that is, let φ^1 be determined by (6.30). Transforming the variables in equation (6.32) by the formulas:

$$\varphi^3=\tau^{-1/2}\exp\Big(\tfrac{1}{2}(C_1+2aa_1)(1+a^2)^{-1}\omega\Big)\theta(\tau),$$

$$\tau = \exp(-\mu(1+a^2)^{-1/2}\omega),$$

we obtain the following equation in the function $\theta = \theta(\tau)$:

$$\tau^2\theta_{\tau\tau} + (6a_1\tau^2\wp(\tau + C_4, 0, C_3) + \sigma)\theta = 0, \tag{6.33}$$

where $\sigma = \mu^{-2}(a_1^2 + 2a_1 - \frac{1}{4}(1+a^2)^{-1}(C_1^2 + 2aa_1)^2) - 3a_1 + \frac{1}{4}$. If $\sigma = 0$, equation (6.33) is a Lame equation.

In the particular case when φ^1 is determined by (6.31), equation (6.33) has the form:

$$\tau^2(\tau + C_4)^2\theta_{\tau\tau} + (6a_1\tau^2 + \sigma(\tau + C_4)^2)\theta = 0. \tag{6.34}$$

By means of the following transformation of variables:

$$\theta = |\xi|^{\nu_1}|\xi - 1|^{\nu_2}\psi(\xi), \quad \xi = -C_4^{-1}\tau,$$

where $\nu_1(\nu_1 - 1) + \sigma = 0$ and $\nu_2(\nu_2 - 1) + 6a_1 = 0$, equation (6.34) is reduced to a hypergeometric equation of the form (see [19]):

$$\xi(\xi - 1)\psi_{\xi\xi} + (2(\nu_1 + \nu_2)\xi - 2\nu_1)\psi_\xi + 2\nu_1\nu_2\psi = 0.$$

If $\sigma = 0$, equation (6.34) implies that

$$(\tau + C_4)^2\theta_{\tau\tau} + 6a_1\theta = 0.$$

Therefore,

$$\theta = C_5|\tau + C_4|^{1/2-\nu} + C_6|\tau + C_4|^{1/2+\nu}$$

if $a_1 < \frac{1}{24}$, where $\nu = (\frac{1}{4} - 6a_1)^{1/2}$,

$$\theta = |\tau + C_4|^{1/2}(C_5 + C_6\ln|\tau + C_4|)$$

if $a_1 = \frac{1}{24}$, and

$$\theta = |\tau + C_4|^{1/2}(C_5\cos(\nu\ln|\tau + C_4|) + C_6\sin(\nu\ln|\tau + C_4|))$$

if $a_1 > \frac{1}{24}$, where $\nu = (6a_1 - \frac{1}{4})^{1/2}$.

7 Exact solutions of system (2.9)

Among the reduced systems from Sec. 2, only particular cases of system (2.9) have Lie symmetry operators that are not induced by elements from $A(NS)$. Therefore, Lie reductions of the other systems from Sec. 2 give only solutions that can be obtained by means of reducing the NSEs with two- and three-dimensional subalgebras of $A(NS)$.

Here we consider system (2.9) with ρ^i vanishing. As mentioned in Note 2.5, in this case the vector-function $\vec{m}$ has the form $\vec{m} = \eta(t)\vec{e}$, where $\vec{e} = \text{const}$, $|\vec{e}| = 1$, and $\eta = \eta(t) = |\vec{m}(t)| \neq 0$. Without loss of generality we can assume that $\vec{e} = (0,0,1)$, i.e.,

$$\vec{m} = (0, 0, \eta(t)).$$

For such vector $\vec{m}$, conditions (2.5) are satisfied by the following vector $\vec{n}^i$:

$$\vec{n}^1 = (1,0,0), \quad \vec{n}^2 = (0,1,0).$$

Therefore, ansatz (2.4) and system (2.9) can be written, respectively, in the forms:

$$
\begin{aligned}
&u^1 = v^1, \quad u^2 = v^2, \quad u^3 = (\eta(t))^{-1}(v^3 + \eta_t(t)x_3), \\
&p = q - \tfrac{1}{2}\eta_{tt}(t)(\eta(t))^{-1}x_3^2,
\end{aligned} \tag{7.1}
$$

where $v = v(y_1, y_2, y_3)$, $q = q(y_1, y_2, y_3)$, $y_i = x_i$, $y_3 = t$, and

$$
\begin{aligned}
&v^i_t + v^j v^i_j - v^i_{jj} + q_i = 0, \\
&v^3_t + v^j v^3_j - v^3_{jj} = 0, \\
&v^i_i + \rho^3 = 0,
\end{aligned} \tag{7.2}
$$

where $\rho^3 = \rho^3(t) = \eta_t/\eta$.

It was shown in Note 2.8 that there exists a local transformation which make ρ^3 vanish. Therefore, we can consider system (7.2) only with ρ^3 vanishing and extend the obtained results in the case $\rho^3 \neq 0$ by means of transformation (2.12). However it will be sometimes convenient to investigate, at once, system (7.2) with an arbitrary function ρ^3.

The MIA of (7.2) with $\rho^3 = 0$ is given by the algebra

$$B =< R_3(\bar\psi),\ Z^1(\lambda),\ D^1_3,\ \partial_t,\ J^1_{12},\ \partial_{v^3},\ v^3\partial_{v^3} >$$

(see notations in Subsec. 2.1). We construct complete sets of inequivalent one-dimensional subalgebras of B and choose such algebras, among these subalgebras, that can be used to reduce system (7.2) and do not lie in the linear span of the operators $R_3(\bar\psi)$, $Z^1(\lambda)$, J^1_{12}, i.e., the operators which are induced by operators from $A(NS)$ for arbitrary ρ^3. As a result we obtain the following algebras (more exactly, the following classes of algebras):

The one-dimentional subalgebras:

1. $B^1_1 =< D^1_3 + 2\varkappa J^1_{12} + 2\gamma v^3\partial_{v^3} + 2\beta\partial_{v^3} >$, where $\gamma\beta = 0$.
2. $B^1_2 =< \partial_t + \varkappa J^1_{12} + \gamma v^3\partial_{v^3} + \beta\partial_{v^3} >$, where $\gamma\beta = 0$, $\varkappa \in \{0; 1\}$.
3. $B^1_3 =< J^1_{12} + \gamma v^3\partial_{v^3} + Z^1(\lambda(t)) >$, where $\gamma \neq 0$, $\lambda \in C^\infty((t_0, t_1), \mathbb{R})$.
4. $B^1_4 =< R_3(\bar\psi(t)) + \gamma v^3\partial_{v^3} >$, where $\gamma \neq 0$, $\bar\psi(t) = (\psi^1(t), \psi^2(t)) \neq (0, 0)\ \forall t \in (t_0, t_1)$, $\psi^i \in C^\infty((t_0, t_1), \mathbb{R})$.

The two-dimentional subalgebras:

1. $B^2_1 =< \partial_t + \beta_2\partial_{v^3},\ D^1_3 + \varkappa J^1_{12} + \gamma v^3\partial_{v^3} + \beta_1\partial_{v^3} >$, where $\gamma\beta_1 = 0$, $(\gamma - 2)\beta_2 = 0$.
2. $B^2_2 =< D^1_3 + 2\gamma_1 v^3\partial_{v^3} + 2\beta_1\partial_{v^3},\quad J^1_{12} + \gamma_2 v^3\partial_{v^3} + \beta_2\partial_{v^3} + Z^1(\varepsilon|t|^{-1}) >$, where $\gamma_1\beta_1 = 0$, $\gamma_2\beta_2 = 0$, $\gamma_1\beta_2 - \gamma_2\beta_1 = 0$.
3. $B^2_3 =< D^1_3 + 2\varkappa J^1_{12} + 2\gamma_1 v^3\partial_{v^3} + 2\beta_1\partial_{v^3},\quad R_3(|t|^{\sigma+1/2}\cos\tau, |t|^{\sigma+1/2}\sin\tau) + \gamma_2 v^3\partial_{v^3} + \beta_2\partial_{v^3} + Z^1(\varepsilon|t|^{\sigma-1}) >$, where $\tau = \varkappa\ln|t|$, $(\gamma_1 + \sigma)\beta_1 - \gamma_2\beta_1 = 0$, $\sigma\gamma_2 = 0$, $\varepsilon\sigma = 0$.
4. $B^2_4 =< \partial_t + \gamma_1 v^3\partial_{v^3} + \beta_1\partial_{v^3},\ J^1_{12} + \gamma_2 v^3\partial_{v^3} + \beta_2\partial_{v^3} + Z^1(\varepsilon) >$, where $\gamma_1\beta_1 = 0$, $\gamma_2\beta_2 = 0$, $\gamma_1\beta_2 - \gamma_2\beta_1 = 0$.
5. $B^2_5 =< \partial_t + \varkappa J^1_{12} + \gamma_1 v^3\partial_{v^3} + \beta_1\partial_{v^3},\ R_3(e^{\sigma t}\cos\varkappa t, e^{\sigma t}\sin\varkappa t) + Z^1(\varepsilon e^{\sigma t}) + \gamma_2 v^3\partial_{v^3} +$

$\beta_2\partial_{v^3} >$, where $(\gamma_1+\sigma)\beta_1-\gamma_2\beta_1=0$, $\sigma\gamma_2=0$, $\varepsilon\sigma=0$.

6. $B_6^2 =< R_3(\bar\psi^1)+\gamma v^3\partial_{v^3},\ R_3(\bar\psi^2) >$, where $\bar\psi^i=(\psi^{i1}(t),\psi^{i2}(t))\neq(0,0)\ \forall t\in(t_0,t_1)$, $\psi^{ij}\in C^\infty((t_0,t_1),\mathbb{R})$, $\bar\psi^1_{tt}\cdot\bar\psi^2-\bar\psi^1\cdot\bar\psi^2_{tt}=0$, $\gamma\neq 0$. Hereafter $\bar\psi^1\cdot\bar\psi^2:=\psi^{1i}\psi^{2i}$.

Let us reduce system (7.2) to systems of PDEs in two independent variables. With the algebras B_1^1–B_4^1 we can construct the following complete set of Lie ansatzes of codimension 1 for system (7.2) with $\rho^3=0$:

$$
\begin{aligned}
1.\quad v^1 &= |t|^{-1/2}(w^1\cos\tau - w^2\sin\tau)+\tfrac12 y_1t^{-1}-\varkappa y_2t^{-1},\\
v^2 &= |t|^{-1/2}(w^1\sin\tau + w^2\cos\tau)+\tfrac12 y_2t^{-1}+\varkappa y_1t^{-1},\\
v^3 &= |t|^{\gamma}w^3+\beta\ln|t|,\\
q &= |t|^{-1}s+\tfrac12(\varkappa^2+\tfrac14)t^{-2}r^2,
\end{aligned}
\tag{7.3}
$$

where $\tau=\varkappa\ln|t|$, $\gamma\beta=0$,

$$z_1=|t|^{-1/2}(y_1\cos\tau+y_2\sin\tau),\qquad z_2=|t|^{-1/2}(-y_1\sin\tau+y_2\cos\tau).$$

Here and below $w^a=w^a(z_1,z_2)$, $s=s(z_1,z_2)$, $r=(y_1^2+y_2^2)^{1/2}$.

$$
\begin{aligned}
2.\quad v^1 &= w^1\cos\varkappa t-w^2\sin\varkappa t-\varkappa y_2,\\
v^2 &= w^1\sin\varkappa t+w^2\cos\varkappa t+\varkappa y_1,\\
v^3 &= w^3e^{\gamma t}+\beta t,\\
q &= s+\tfrac12\varkappa^2r^2,
\end{aligned}
\tag{7.4}
$$

where $\varkappa\in\{0;1\}$, $\gamma\beta=0$,

$$z_1=y_1\cos\varkappa t+y_2\sin\varkappa t,\qquad z_2=-y_1\sin\varkappa t+y_2\cos\varkappa t.$$

$$
\begin{aligned}
3.\quad v^1 &= y_1r^{-1}w^3-y_2r^{-2}w^1-\gamma y_2r^{-2},\\
v^2 &= y_2r^{-1}w^3+y_1r^{-2}w^1+\gamma y_1r^{-2},\\
v^3 &= w^2e^{\gamma\arctan y_2/y_1},\\
q &= s+\lambda(t)\arctan y_2/y_1,
\end{aligned}
\tag{7.5}
$$

where $z_1=t$, $z_2=r$, $\gamma\neq0$, $\lambda\in C^\infty((t_0,t_1),\mathbb{R})$.

$$
\begin{aligned}
4.\quad \bar v &= (\bar\psi\cdot\bar\psi)^{-1}\Big((w^1+\gamma)\bar\psi+w^3\bar\theta+(\bar\psi\cdot\bar y)\bar\psi_t-z_2\bar\theta_t\Big)\\
v^3 &= w^2\exp(\gamma(\bar\psi\cdot\bar\psi)^{-1}(\bar\psi\cdot\bar y))\\
q &= s-(\bar\psi\cdot\bar\psi)^{-1}(\bar\psi_{tt}\cdot\bar y)(\bar\psi\cdot\bar y)+\tfrac12(\bar\psi\cdot\bar\psi)^{-2}(\bar\psi_{tt}\cdot\bar\psi)(\bar\psi\cdot\bar y)^2,
\end{aligned}
\tag{7.6}
$$

where $z_1=t$, $z_2=(\bar\theta\cdot\bar y)$, $\gamma\neq0$, $\bar v=(v^1,v^2)$, $\bar y=(y_1,y_2)$, $\psi^i\in C^\infty((t_0,t_1),\mathbb{R})$, $\bar\theta=(-\psi^2,\psi^1)$.

Substituting ansatzes (7.3) and (7.4) into system (7.2) with $\rho^3 = 0$, we obtain a reduced system of the form (6.1), where

$$\alpha_1 = 0, \quad \alpha_2 = -1, \quad \alpha_3 = -2\varkappa, \quad \alpha_4 = \gamma, \quad \alpha_5 = \beta \quad \text{if} \quad t > 0 \quad \text{and}$$

$$\alpha_1 = 0, \quad \alpha_2 = 1, \quad \alpha_3 = 2\varkappa, \quad \alpha_4 = -\gamma, \quad \alpha_5 = -\beta \quad \text{if} \quad t < 0$$

for ansatz (7.3) and

$$\alpha_1 = 0, \quad \alpha_2 = 0, \quad \alpha_3 = -2\varkappa, \quad \alpha_4 = \gamma, \quad \alpha_5 = \beta$$

for ansatz (7.4). System (6.1) is investigated in Sec. 6 in detail.

Because the form of ansatzes (7.3) is not changed after transformation (2.12), it is convinient to substitute their into a system of form (7.2) with an arbitrary function ρ^3. As a result of substituting, we obtain the following reduced systems:

3. $$\begin{aligned} & w^3_1 + w^3 w^3_2 - z_2^{-3}(w^1+\gamma)^2 - (w^3_{22} + z_2^{-1} w^3_2 - z_2^{-2} w^3) + s_2 = 0, \\ & w^1_1 + w^3 w^1_2 - w^1_{22} + z_2^{-1} w^1_2 + \lambda = 0, \\ & w^2_1 + w^3 w^2_2 - w^2_{22} - z_2^{-1} w^2_2 + \gamma z_2^{-2} w^1 w^2 = 0, \\ & w^3_2 + z_2^{-1} w^3 = -\eta_1/\eta. \end{aligned} \tag{7.7}$$

4. $$\begin{aligned} & w^1_1 + w^3 w^1_2 - (\bar\psi\cdot\bar\psi) w^1_{22} = 0, \\ & w^3_1 + w^3 w^3_2 - (\bar\psi\cdot\bar\psi) w^3_{22} + (\bar\psi\cdot\bar\psi) s_2 + 2(w^1+\gamma)(\bar\psi\cdot\bar\theta)(\bar\psi\cdot\bar\psi)^{-1} - \\ & \quad 2(\bar\psi_t\cdot\bar\psi)(\bar\psi\cdot\bar\psi)^{-1} w^3 + (2\bar\psi_t\cdot\bar\psi_t - \bar\psi_{tt}\cdot\bar\psi)(\bar\psi\cdot\bar\psi)^{-1} z_2 = 0, \\ & w^2_1 + w^3 w^2_2 - (\bar\psi\cdot\bar\psi) w^2_{22} + \gamma(\bar\psi\cdot\bar\psi)^{-1}(w^1 + (\bar\psi_t\cdot\bar\theta)(\bar\psi\cdot\bar\psi)^{-1} z_2) w^2 = 0, \\ & w^3_2 + \eta_t/\eta = 0. \end{aligned} \tag{7.8}$$

Unlike systems 8 and 9 from Subsec. 3.2, systems (7.7) and (7.8) are not reduced to linear systems of PDEs.

Let us investigate system (7.7). The last equation of (7.7) immediately gives

$$(w^3_2 + z_2^{-1} w^3)_2 = w^3_{22} + z_2^{-1} w^3_2 - z_2^{-2} w^3 = 0, \quad w^3 = (\chi - 1) z_2^{-1} - \tfrac{1}{2}\eta_t \eta^{-1} z_2, \tag{7.9}$$

where $\chi = \chi(t)$ is an arbitrary differentiable function of $t = z_2$. Then it follows from the first equation of (7.7) that

$$s = \textstyle\int z_2^{-3}(w^1+\gamma)^2 dz_2 - \tfrac{1}{2}(\chi-1)^2 z_2^{-2} + \tfrac{1}{4} z_2^2\Big((\eta_t/\eta)_t - \tfrac{1}{2}(\eta_t/\eta)^2\Big) - \chi_t \ln|z_2|.$$

Substituting (7.9) into the remaining equations of (7.7), we get

$$\begin{aligned} & w^1_1 - w^1_{22} + (\chi z_2^{-1} - \tfrac{1}{2}\eta_t\eta^{-1} z_2) w^1_2 + \lambda = 0, \\ & w^2_1 - w^2_{22} + ((\chi-2) z_2^{-1} - \tfrac{1}{2}\eta_t\eta^{-1} z_2) w^2_2 + \gamma z_2^{-2} w^1 w^2 = 0. \end{aligned} \tag{7.10}$$

By means of changing the independent variables

$$\tau = \textstyle\int |\eta(t)| dt, \quad z = |\eta(t)|^{1/2} z_2, \tag{7.11}$$

system (7.10) can be transformed to a system of a simpler form:

$$w^1_\tau - w^1_{zz} + \hat\chi z^{-1} w^2_z + \hat\lambda|\hat\eta|^{-1} = 0,$$
$$w^2_\tau - w^2_{zz} + (\hat\chi - 2) z^{-1} w^2_z + \gamma z^{-2} w^1 w^2 = 0, \tag{7.12}$$

where $\hat\eta(\tau) = \eta(t)$, $\hat\chi(\tau) = \chi(t)$, and $\hat\lambda(\tau) = \lambda(t)$.

If $\lambda(t) = -2C\eta(t)(\chi(t) - 1)$ for some fixed constant C, particular solutions of (7.10) are functions

$$w^1 = C\eta(t) z_2^2, \quad w^2 = f(z_1, z_2) \exp(\gamma C \textstyle\int \eta(t) dt),$$

where f is an arbitrary solution of the following equation

$$f_1 - f_{22} + ((\chi - 2) z_2^{-1} - \tfrac{1}{2}\eta_t \eta^{-1} z_2) f_2 = 0. \tag{7.13}$$

In the variables from (7.11), equation (7.13) has form (5.22) with $\tilde\eta(\tau) = \chi(t) - 2$.

In the case $\lambda(t) = 8C(\chi(t)-1)\eta(t) \int \eta(t)(\chi(t)-3)dt$ ($C = \text{const}$), particular solutions of (7.10) are functions

$$w^1 = C\Big((\eta(t))^2 z_2^4 - 4 z_2^2 \eta(t) \textstyle\int \eta(t)(\chi(t) - 3) dt\Big),$$
$$w^2 = f(z_1, z_2) \exp(\tfrac{1}{2}(\gamma C)^{1/2} \eta(t) z_2^2 + \xi(t)),$$

where $\xi(t) = -(\gamma C)^{1/2} \int \eta(t)(\chi(t) - 3) dt + 4\gamma C \int \eta(t) (\int \eta(t)(\chi(t) - 3) dt) dt$ and f is an arbitrary solution of the following equation

$$f_1 - f_{22} + ((\chi - 2) z_2^{-1} - (\tfrac{1}{2}\eta_t \eta^{-1} + 2(\gamma C)^{1/2}) z_2) f_2 = 0. \tag{7.14}$$

After the change of the independent variables

$$\tau = \textstyle\int |\eta(t)| \exp(4(\gamma C)^{1/2} \int \eta(t) dt) dt, \quad z = |\eta(t)|^{1/2} \exp(2(\gamma C)^{1/2} \int \eta(t) dt) z_2$$

in (7.14), we obtain equation (5.22) with $\tilde\eta(\tau) = \chi(t) - 2$ again.

Let us continue to system (7.8). The last equation of (7.8) integrates with respect to z_2 to the following expression: $w^3 = -\eta_t \eta^{-1} z_2 + \chi$. Here $\chi = \chi(t)$ is an differentiable function of $z_1 = y_3 = t$. Let us make the transformation from the symmetry group of (7.2):

$$\tilde{\bar v}(t, \bar y) = \bar v(t, \bar y - \bar\xi(t)) + \bar\xi_t(t), \quad \tilde v^3 = v^3, \quad \tilde q(t, \bar y) = q(t, \bar y - \bar\xi(t)) - \bar\xi_{tt}(t) \cdot \bar y,$$

where $\bar\xi_{tt} \cdot \bar\psi - \bar\xi \cdot \bar\psi_{tt} = 0$ and

$$\bar\xi_t \cdot \bar\theta + \chi + \eta_t \eta^{-1} (\bar\xi \cdot \bar\theta) - |\bar\psi|^{-2} (\bar\xi \cdot \bar\psi)(\bar\psi_t \cdot \bar\theta) + |\bar\psi|^{-2} (\bar\xi \cdot \bar\theta)(\bar\theta_t \cdot \bar\theta) = 0.$$

Hereafter $|\bar\psi|^2 = \bar\psi \cdot \bar\psi$. This transformation does not modify ansatz (7.6), but it makes the function χ vanish, i.e., $\tilde w^3 = -\eta_t \eta^{-1} z_2$. Therefore, without loss of generality we may assume, at once, that $w^3 = -\eta_t \eta^{-1} z_2$.

Substituting the expression for w^3 in the other equations of (7.8), we obtain that

$$s = z_2^2 |\bar\psi|^{-2} \Big((\tfrac{1}{2}\bar\psi_{tt} \cdot \bar\psi - \bar\psi_t \cdot \bar\psi_t - (\bar\psi_t \cdot \bar\psi)\eta_t \eta^{-1}) |\bar\psi|^{-2} + \tfrac{1}{2}\eta_{tt} \eta^{-1} - (\eta_t)^2 \eta^{-2} \Big) -$$
$$2(\bar\psi_t \cdot \bar\theta) |\bar\psi|^{-2} \textstyle\int w^1(z_1, z_2) dz_2,$$

$$
\begin{aligned}
&w^1_1 - \eta_1\eta^{-1}z_2w^1_2 - |\bar\psi|^2w^1_{22} = 0,\\
&w^2_1 - \eta_1\eta^{-1}z_2w^2_2 - |\bar\psi|^2w^2_{22} + \gamma|\bar\psi|^{-2}(2(\bar\psi_t\cdot\bar\theta)|\bar\psi|^{-2}z_2 + w^1)w^2 = 0.
\end{aligned}
\tag{7.15}
$$

The change of the independent variables

$$
\tau = \textstyle\int(\eta(t)|\bar\psi|)^2dt, \quad z = \eta(t)z_2
$$

reduces system (7.15) to the following form:

$$
\begin{aligned}
&w^1_\tau - w^1_{zz} = 0,\\
&w^2_\tau - w^2_{zz} + \gamma|\hat{\bar\psi}|^{-4}\hat\eta^{-2}(2(\hat{\bar\psi}_t\cdot\hat{\bar\theta})\hat\eta z + w^1)w^2 = 0,
\end{aligned}
\tag{7.16}
$$

where $\hat{\bar\psi}(\tau) = \bar\psi(t)$, $\hat{\bar\theta}(\tau) = \bar\theta(t)$, $\hat\eta(\tau) = \eta(t)$.

Particular solutions of (7.15) are the functions

$$
\begin{aligned}
&w^1 = C_1 + C_2\eta(t)z_2 + C_3(\tfrac12(\eta(t)z_2)^2 + \textstyle\int(\eta(t)|\bar\psi|)^2dt),\\
&w^2 = f(t,z_2)\exp(\xi^2(t)z_2^2 + \xi^1(t)z_2 + \xi^0(t)),
\end{aligned}
$$

where $(\xi^2(t),\xi^1(t),\xi^0(t))$ is a particular solution of the system of ODEs:

$$
\begin{aligned}
&\xi^2_t - 2\eta_t\eta^{-1}\xi^2 - 4|\bar\psi|^2(\xi^2)^2 + \tfrac12C_3\gamma\eta^2|\bar\psi|^{-2} = 0,\\
&\xi^1_t - \eta_t\eta^{-1}\xi^1 - 4|\bar\psi|^2\xi^2\xi^1 + 2\gamma(\bar\psi_t\cdot\bar\theta)|\bar\psi|^{-4} + C_2\gamma\eta|\bar\psi|^{-2} = 0,\\
&\xi^0_t - 2|\bar\psi|^2\xi^2 - |\bar\psi|^2(\xi^1)^2 + \gamma(C_1 + C_3\textstyle\int(\eta(t)|\bar\psi|)^2dt)|\bar\psi|^{-2} = 0,
\end{aligned}
$$

and f is an arbitrary solution of the following equation

$$
f_1 - |\bar\psi|^2f_{22} + ((\eta_t\eta^{-1} + 4|\bar\psi|^2\xi^2)z_2 + 2|\bar\psi|^2\xi^1)f_2 = 0.
\tag{7.17}
$$

Equation (7.17) is reduced by means of a local transformation of the independent variables to the heat equation.

Consider the Lie reductions of system (7.2) to systems of ODEs. The second basis operator of the each algebra B^2_k, $k=\overline{1,5}$ induces, for the reduced system obtained from system (7.2) by means of the first basis operator, either a Lie symmetry operator from Table 2 or a operator giving a ansatz of form (6.4). Therefore, the Lie reduction of system (7.2) with the algebras $B^2_1 - B^2_5$ gives only solutions that can be constructed for system (7.2) by means of reducing with the algebras B^1_1 and B^1_2 to system (6.1).

With the algebra B^2_6 we obtain an ansatz and a reduced system of the following forms:

$$
\begin{aligned}
&\bar v = \bar\phi + \lambda^{-1}(\bar\theta^i\cdot\bar y)\bar\psi^i_t, \quad v^3 = \phi^3\exp(\gamma\lambda(\bar\theta^1\cdot\bar y)),\\
&s = h - \tfrac12\lambda^{-1}(\bar\psi^i_{tt}\cdot\bar y)(\bar\theta^i\cdot\bar y),
\end{aligned}
\tag{7.18}
$$

where $\phi^a = \phi^a(\omega)$, $h = h(\omega)$, $\omega = t$, $\lambda = \psi^{11}\psi^{22} - \psi^{12}\psi^{21} = \bar\psi^1\cdot\bar\theta^1 = \bar\psi^2\cdot\bar\theta^2$, $\bar\theta^1 = (\psi^{22}, -\psi^{21})$, $\bar\theta^2 = (-\psi^{12},\psi^{11})$, and

$$
\begin{aligned}
&\bar\phi_t + \lambda^{-1}(\bar\theta^i\cdot\bar\phi)\bar\psi^i_t = 0, \quad \phi^3_t + (\gamma\lambda^{-1}(\bar\theta^1\cdot\bar\phi) - \gamma^2\lambda^{-2}(\bar\theta^1\cdot\bar\theta^1))\phi^3 = 0,\\
&\lambda^{-1}(\bar\theta^i\cdot\bar\psi^i_t) + \eta_t\eta^{-1} = 0.
\end{aligned}
\tag{7.19}
$$

Let us make the transformation from the symmetry group of system (7.2):

$$\tilde{\bar{v}}(t,\bar{y}) = \bar{v}(t,\bar{y}-\bar{\xi}) + \bar{\xi}_t, \quad \tilde{v}^3(t,\bar{y}) = v^3(t,\bar{y}-\bar{\xi}), \quad \tilde{s}(t,\bar{y}) = s(t,\bar{y}-\bar{\xi}) - \bar{\xi}_{tt}\cdot\bar{y},$$

where

$$\bar{\xi}_t + \lambda^{-1}(\bar{\theta}^i\cdot\bar{\xi})\bar{\psi}^i_t + \bar{\phi} = 0. \tag{7.20}$$

It follows from (7.20) that $\bar{\xi}_{tt} = \lambda^{-1}(\bar{\theta}^i\cdot\bar{\xi})\bar{\psi}^i_{tt}$, i.e., $\bar{\theta}^i_{tt}\cdot\bar{\xi} - \bar{\theta}^i\cdot\bar{\xi}_{tt} = 0$. Therefore, this trasformation does not modify ansatz (7.18), but it makes the functions ϕ^i vanish. And without loss of generality we may assume, at once, that $\phi^i \equiv 0$. Then

$$\phi^3 = C\exp\Bigl(\textstyle\int(\gamma\lambda^{-1}|\theta|)^2dt\Bigr), \quad C = \text{const}.$$

The last equation of system (7.19) is the compatibility condition of system (7.2) and ansatz (7.18).

8 Conclusion

In this article we reduced the NSEs to systems of PDEs in three and two independent variables and systems of ODEs by means of the Lie method. Then, we investigated symmetry properties of the reduced systems of PDEs and made Lie reductions of systems which admitted non-trivial symmetry operators, i.e., operators that are not induced by operators from $A(NS)$. Some of the systems in two independent variables were reduced to linear systems of either two one-dimensional heat equations or two translational equations. We also managed to find exact solutions for most of the reduced systems of ODEs.

Now, let us give some remaining problems. Firstly, we failed, for the present, to describe the non-Lie ansatzes of form 1.6 that reduce the NSEs. (These ansatzes include, for example, the well-known ansatzes for the Karman swirling flows (see bibliography in [16]). One can also consider non-local ansatzes for the Navier-Stokes field, i.e., ansatzes containing derivatives of new unknown functions.

Second problem is to study non-Lie (i.e., non-local, conditional, and Q-conditional) symmetries of the NSEs [13].

And finally, it would be interesting to investigate compatability and to construct exact solutions of overdetermined systems that are obtained from the NSEs by means of different additional conditions. Usually one uses the condition where the nonlinearity has a simple form, for example, the potential form (see review [36]), i.e., $\mathrm{rot}((\vec{u}\cdot\vec{\nabla})\vec{u}) = \vec{0}$ (the NS fields satisfying this condition is called the generalized Beltrami flows). We managed to describe the general solution of the NSEs with the additional condition where the convective terms vanish [29, 30]. But one can give other conditions, for example,

$$\triangle\vec{u} = \vec{0}, \quad \vec{u}_t + (\vec{u}\cdot\vec{\nabla})\vec{u} = \vec{0},$$

and so on.

We will consider the problems above elsewhere.

Appendix

A Inequivalent one-, two-, and three-dimensional subalgebras of $A(NS)$

To find complete sets of inequivalent subalgebras of $A(NS)$, we use the method given, for example, in [27, 28]. Let us describe it briefly.

1. We find the commutation relations between the basis elements of $A(NS)$.

2. For arbitrary basis elements V, W^0 of $A(NS)$ and each $\varepsilon \in \mathbb{R}$ we calculate the adjoint action

$$W(\varepsilon) = \mathrm{Ad}(\varepsilon V)W^0 = \mathrm{Ad}(\exp(\varepsilon V))W^0$$

of the element $\exp(\varepsilon V)$ from the one-parameter group generated by the operator V on W^0. This calculation can be made in two ways: either by means of summing the Lie series

$$W(\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\{V^n, W^0\} = W^0 + \frac{\varepsilon}{1!}[V, W^0] + \frac{\varepsilon^2}{2!}[V,[V,W^0]] + \dots , \tag{A.1}$$

where $\{V^0, W^0\} = W^0$, $\{V^n, W^0\} = [V, \{V^{n-1}, W^0\}]$, or directly by means of solving the initial value problem

$$\frac{dW(\varepsilon)}{d\varepsilon} = [V, W(\varepsilon)], \quad W(0) = W^0. \tag{A.2}$$

3. We take a subalgebra of a general form with a fixed dimension. Taking into account that the subalgebra is closed under the Lie bracket, we try to simplify it by means of adjoint actions as much as possible.

A.1 The commutation relations and the adjoint representation of the algebra $A(NS)$

Basis elements (1.2) of $A(NS)$ satisfy the following commutation relations:

$$\begin{gathered}
[J_{12}, J_{23}] = -J_{31}, \quad [J_{23}, J_{31}] = -J_{12}, \quad [J_{31}, J_{12}] = -J_{23}, \\
[\partial_t, J_{ab}] = [D, J_{ab}] = 0, \quad [\partial_t, D] = 2\partial_t, \\
[\partial_t, R(\vec{m})] = R(\vec{m}_t), \quad [D, R(\vec{m})] = R(2t\vec{m}_t - \vec{m}), \\
[\partial_t, Z(\chi)] = Z(\chi_t), \quad [D, Z(\chi)] = Z(2t\chi_t + 2\chi), \\
[R(\vec{m}), R(\vec{n})] = Z(\vec{m}_{tt}\cdot\vec{n} - \vec{m}\cdot\vec{n}_{tt}), \quad [J_{ab}, R(\vec{m})] = R(\vec{\tilde{m}}), \\
[J_{ab}, Z(\chi)] = [Z(\chi), R(\vec{m})] = [Z(\chi), Z(\eta)] = 0,
\end{gathered} \tag{A.3}$$

where $\tilde{m}^a = m^b, \quad \tilde{m}^b = -m^a, \quad \tilde{m}^c = 0, \quad a \neq b \neq c \neq a.$

Note A.1 *Relations (A.3) imply that the set of operators (1.2) generates an algebra when, for example, the parameter-functions m^a and χ belong to $C^\infty((t_0,t_1),\mathbb{R})$ ($C_0^\infty((t_0,t_1),\mathbb{R})$, $A((t_0,t_1),\mathbb{R})$), i.e., the set of infinite-differentiable (infinite-differentiable finite, real analytic) functions from (t_0,t_1) in $\mathbb{R}$, where $-\infty \le t_0 < t_1 \le +\infty$. But the NSEs (1.1) admit operators (1.3) and (1.4) with parameter-functions of a less degree of smoothness. Moreover, the minimal degree of their smoothness depends on the smoothness that is demanded for the solutions of the NSEs (1.1). Thus, if $u^a \in C^2((t_0,t_1)\times\Omega,\mathbb{R})$ and $p \in C^1((t_0,t_1)\times\Omega,\mathbb{R})$, where Ω is a domain in $\mathbb{R}^3$, then it is sufficient that $m^a \in C^3((t_0,t_1),\mathbb{R})$ and $\chi \in C^1((t_0,t_1),\mathbb{R})$. Therefore, one can consider the "pseudoalgebra" generated by operators (1.2). The prefix "pseudo-" means that in this set of operators the commutation operation is not determined for all pairs of its elements, and the algebra axioms are satisfied only by elements, where they are defined. It is better to indicate the functional classes that are sets of values for the parameters m^a and χ in the notation of the algebra $A(NS)$. But below, for simplicity, we fix these classes, taking $m^a,\chi \in C^\infty((t_0,t_1),\mathbb{R})$, and keep the notation of the algebra generated by operators (1.2) in the form $A(NS)$. However, all calculations will be made in such a way that they can be translated for the case of a less degree of smoothness.*

Most of the adjoint actions are calculated simply as sums of their Lie series. Thus,

$$
\begin{aligned}
&\mathrm{Ad}(\varepsilon\partial_t)D = D + 2\varepsilon\partial_t, \quad \mathrm{Ad}(\varepsilon D)\partial_t = e^{-2\varepsilon}\partial_t,\\
&\mathrm{Ad}(\varepsilon Z(\chi))\partial_t = \partial_t - \varepsilon Z(\chi_t), \quad \mathrm{Ad}(\varepsilon Z(\chi))D = D - \varepsilon Z(2t\chi_t + 2\chi),\\
&\mathrm{Ad}(\varepsilon R(\vec m))\partial_t = \partial_t - \varepsilon R(\vec m_t) - \tfrac12\varepsilon^2 Z(\vec m_t\cdot\vec m_{tt} - \vec m\cdot\vec m_{ttt}),\\
&\mathrm{Ad}(\varepsilon R(\vec m))D = D - \varepsilon R(2t\vec m_t - \vec m) -\\
&\qquad\qquad \tfrac12\varepsilon^2 Z(2t\vec m_t\cdot\vec m_{tt} - 2t\vec m\cdot\vec m_{ttt} - 4\vec m\cdot\vec m_{tt}),\\
&\mathrm{Ad}(\varepsilon R(\vec m))J_{ab} = J_{ab} - \varepsilon R(\vec{\tilde m}) + \varepsilon^2 Z(m^a m^b_{tt} - m^a_{tt}m^b),\\
&\mathrm{Ad}(\varepsilon R(\vec m))R(\vec n) = R(\vec n) + \varepsilon Z(\vec m_{tt}\cdot\vec n - \vec m\cdot\vec n_{tt}), \quad \mathrm{Ad}(\varepsilon J_{ab})R(\vec m) = R(\vec{\hat m}),\\
&\mathrm{Ad}(\varepsilon J_{ab})J_{cd} = J_{cd}\cos\varepsilon + [J_{ab},J_{cd}]\sin\varepsilon \quad ((a,b)\ne(c,d)\ne(b,a)),
\end{aligned}
\tag{A.4}
$$

where

$$
\begin{aligned}
&\tilde m^a = m^b, \quad \tilde m^b = -m^a, \quad \tilde m^c = 0, \quad a\ne b\ne c\ne a,\\
&\hat m^d = m^d\cos\varepsilon + \tilde m^d\sin\varepsilon, \quad \hat m^c = m^c, \quad a\ne b\ne c\ne a, \quad d\in\{a;b\}.
\end{aligned}
$$

Four adjoint actions are better found by means of integrating a system of form (A.2). As a result we obtain that

$$
\begin{aligned}
&\mathrm{Ad}(\varepsilon\partial_t)Z(\chi(t)) = Z(\chi(t+\varepsilon)), \quad \mathrm{Ad}(\varepsilon D)Z(\chi(t)) = Z(e^{2\varepsilon}\chi(te^{2\varepsilon})),\\
&\mathrm{Ad}(\varepsilon\partial_t)R(\vec m(t)) = R(\vec m(t+\varepsilon)), \quad \mathrm{Ad}(\varepsilon D)R(\vec m(t)) = R(e^{-\varepsilon}\vec m(te^{2\varepsilon})).
\end{aligned}
\tag{A.5}
$$

Cases where adjoint actions coincide with the identical mapping are omitted.

Note A.2 *If* $Z(\chi(t)) \in A(NS)[C^\infty((t_0,t_1),\mathbb{R})]$ *with* $-\infty < t_0$ *or* $t_1 < +\infty$, *the adjoint representation* $Ad(\varepsilon\partial_t)$ $(Ad(\varepsilon D))$ *gives an equivalence relation between the operators* $Z(\chi(t))$ *and* $Z(\chi(t+\varepsilon))$ $(Z(\chi(t))$ *and* $Z(e^{2\varepsilon}\chi(te^{2\varepsilon})))$ *that belong to the different algebras*

$$A(NS)[C^\infty((t_0,t_1),\mathbb{R})] \quad \textit{and} \quad A(NS)[C^\infty((t_0-\varepsilon,t_1-\varepsilon),\mathbb{R})]$$
$$(A(NS)[C^\infty((t_0,t_1),\mathbb{R})] \quad \textit{and} \quad A(NS)[C^\infty((t_0e^{-2\varepsilon},t_1e^{-2\varepsilon}),\mathbb{R})])$$

respectively. An analogous statement is true for the operator $R(\vec{m})$. *Equivalence of subalgebras in Theorems A.1 and A.2 is also meant in this sense.*

Note A.3 *Besides the adjoint representations of operators (1.2) we make use of discrete transformation (1.6) for classifying the subalgebras of* $A(NS)$,

To prove the theorem of this section, the following obvious lemma is used.

Lemma A.1 *Let* $N \in \mathbb{N}$.

A. If $\chi \in C^N((t_0,t_1),\mathbb{R})$, *then* $\exists\eta \in C^N((t_0,t_1),\mathbb{R})$: $2t\eta_t + 2\eta = \chi$.

B. If $\chi \in C^N((t_0,t_1),\mathbb{R})$, *then* $\exists\eta \in C^N((t_0,t_1),\mathbb{R})$: $2t\eta_t - \eta = \chi$.

C. If $m^i \in C^N((t_0,t_1),\mathbb{R})$ *and* $a \in \mathbb{R}$, *then* $\exists l^i \in C^N((t_0,t_1),\mathbb{R})$:

$$2tl^1_t - l^1 + al^2 = m^1, \quad 2tl^2_t - l^2 - al^1 = m^2.$$

A.2 One-dimensional subalgebras

Theorem A.1 *A complete set of* $A(NS)$*-inequivalent one-dimensional subalgebras of* $A(NS)$ *is exhausted by the following algebras:*

1. $A^1_1(\varkappa) = < D + 2\varkappa J_{12} >$, *where* $\varkappa \geq 0$.

2. $A^1_2(\varkappa) = < \partial_t + \varkappa J_{12} >$, *where* $\varkappa \in \{0;1\}$.

3. $A^1_3(\eta,\chi) = < J_{12} + R(0,0,\eta(t)) + Z(\chi(t)) >$ *with smooth functions* η *and* χ. *Algebras* $A^1_3(\eta,\chi)$ *and* $A^1_3(\tilde\eta,\tilde\chi)$ *are equivalent if* $\exists\varepsilon,\delta \in \mathbb{R}$, $\exists\lambda \in C^\infty((t_0,t_1),\mathbb{R})$:

$$\tilde\eta(\tilde t) = e^{-\varepsilon}\eta(t), \quad \tilde\chi(\tilde t) = e^{2\varepsilon}(\chi(t) + \lambda_{tt}(t)\eta(t) - \lambda(t)\eta_{tt}(t)), \tag{A.6}$$

where $\tilde t = te^{-2\varepsilon} + \delta$.

4. $A^1_4(\vec m,\chi) = < R(\vec m(t)) + Z(\chi(t)) >$ *with smooth functions* $\vec m$ *and* χ: $(\vec m,\chi) \not\equiv (\vec 0, 0)$. *Algebras* $A^1_4(\vec m,\chi)$ *and* $A^1_4(\tilde{\vec m},\tilde\chi)$ *are equivalent if* $\exists\varepsilon,\delta \in \mathbb{R}$, $\exists C \neq 0$, $\exists B \in O(3)$, $\exists\vec l \in C^\infty((t_0,t_1),\mathbb{R}^3)$:

$$\tilde{\vec m}(\tilde t) = Ce^{-\varepsilon}B\vec m(t), \quad \tilde\chi(\tilde t) = Ce^{2\varepsilon}(\chi(t) + \vec l_{tt}(t)\cdot\vec m(t) - \vec m_{tt}(t)\cdot\vec l(t)), \tag{A.7}$$

where $\tilde t = te^{-2\varepsilon} + \delta$.

P r o o f Consider an arbitrary one-dimensional subalgebra generated by

$$V = a_1 D + a_2\partial_t + a_3 J_{12} + a_4 J_{23} + a_5 J_{31} + R(\vec{m}) + Z(\chi).$$

The coefficients a_4 and a_5 are omitted below since they always can be made to vanish by means of the adjoint representations $\mathrm{Ad}(\varepsilon_1 J_{12})$ and $\mathrm{Ad}(\varepsilon_2 J_{31})$.

If $a_1 \neq 0$ we get $\tilde{a}_1 = 1$ by means of a change of basis. Next, step-by-step we make a_2, $\vec{m}$, and χ vanish by means of the adjoint representations $\mathrm{Ad}(-\frac{1}{2}a_2 a_1^{-1}\partial_t)$, $\mathrm{Ad}(R(\vec{l}))$, and $\mathrm{Ad}(Z(\chi))$, where

$$\vec{l}\in C^\infty((t_0 + \tfrac{1}{2}a_2 a_1^{-1}, t_1 + \tfrac{1}{2}a_2 a_1^{-1}), \mathbb{R}^3), \quad \eta\in C^\infty((t_0 + \tfrac{1}{2}a_2 a_1^{-1}, t_1 + \tfrac{1}{2}a_2 a_1^{-1}), \mathbb{R}),$$

and $\vec{l}$, η are solutions of the equations

$$2t\vec{l}_t - \vec{l} + a_3 a_1^{-1}(l^2, -l^1, 0)^T = \hat{\vec{m}}, \qquad 2t\eta_t + 2\eta = \hat{\chi} + \tfrac{1}{2}(\vec{l}_{tt}\cdot\hat{\vec{m}} - \vec{l}\cdot\hat{\vec{m}}_{tt})$$

with $\hat{\vec{m}}(t) = a_1^{-1}\vec{m}(t - \frac{1}{2}a_2 a_1^{-1})$ and $\hat{\chi}(t) = a_1^{-1}\chi(t - \frac{1}{2}a_2 a_1^{-1})$. Such $\vec{l}$ and η exist in virtue of Lemma A.1. As a result we obtain the algebra $A_1^1(\varkappa)$, where $2\varkappa = a_3 a_1^{-1}$. In case $\varkappa < 0$ additionally one has to apply transformation (1.6) with $b = 1$.

If $a_1 = 0$ and $a_2 \neq 0$, we make $\tilde{a}_2 = 1$ by means of a change of basis. Next, step-by-step we make $\vec{m}$ and χ vanish by means of the adjoint representations $\mathrm{Ad}(R(\vec{l}))$ and $\mathrm{Ad}(Z(\chi))$, where $\vec{l} \in C^\infty((t_0, t_1), \mathbb{R}^3)$, $\eta \in C^\infty((t_0, t_1), \mathbb{R})$, and

$$a_2\vec{l}_t + a_3(l^2, -l^1, 0)^T = \vec{m}, \qquad a_2\eta_t = \chi + \tfrac{1}{2}(\vec{l}_{tt}\cdot\vec{m} - \vec{l}\cdot\vec{m}_{tt}).$$

If $a_3 = 0$ we obtain the algebra $A_2^1(0)$ at once. If $a_3 \neq 0$, using the adjoint representation $\mathrm{Ad}(\varepsilon D)$ and transformation (1.6) (in case of need), we obtain the algebra $A_2^1(1)$.

If $a_1 = a_2 = 0$ and $a_3 \neq 0$, after a change of basis and applying the adjoint representation $\mathrm{Ad}(R(-a_3^{-1}m^2, a_3^{-1}m^1, 0))$ we get the algebra $A_3^1(\eta, \tilde{\chi})$, where $\eta = a_3^{-1}m^3$ and $\tilde{\chi} = a_3^{-1}\chi + a_3^{-2}(m^1_{tt}m^2 - m^1 m^2_{tt})$. Equivalence relation (A.6) is generated by the adjoint representations $\mathrm{Ad}(\varepsilon D)$, $\mathrm{Ad}(\delta\partial_t)$, and $\mathrm{Ad}(R(0, 0, \lambda))$.

If $a_1 = a_2 = a_3 = 0$, at once we get the algebra $A_4^1(\vec{m}, \chi)$. Equivalence relation (A.7) is generated by the adjoint representations $\mathrm{Ad}(\varepsilon D)$, $\mathrm{Ad}(\delta\partial_t)$, $\mathrm{Ad}(R(\vec{l}))$, and $\mathrm{Ad}(\varepsilon_{ab}J_{ab})$.

A.3 Two-dimensional subalgebras

Theorem A.2 *A complete set of $A(NS)$-inequivalent two-dimensional subalgebras of $A(NS)$ is exhausted by the following algebras:*

1. $A_1^2(\varkappa) = <\partial_t,\ D + \varkappa J_{12}>$, *where* $\varkappa \geq 0$.
2. $A_2^2(\varkappa, \varepsilon) = <D,\ J_{12} + R(0, 0, \varkappa|t|^{1/2}) + Z(\varepsilon t^{-1})>$, *where* $\varkappa \geq 0$, $\varepsilon \geq 0$.
3. $A_3^2(\varkappa, \varepsilon) = <\partial_t,\ J_{12} + R(0, 0, \varkappa) + Z(\varepsilon)>$, *where* $\varkappa \in \{0; 1\}$, $\varepsilon \geq 0$ *if* $\varkappa = 1$ *and* $\varepsilon \in \{0; 1\}$ *if* $\varkappa = 0$.
4. $A_4^2(\sigma, \varkappa, \mu, \nu, \varepsilon) = <D + 2\varkappa J_{12},\ R(|t|^{\sigma+1/2}(\nu\cos\tau,\ \nu\sin\tau,\ \mu)) + Z(\varepsilon|t|^{\sigma-1})>$, *where* $\tau = \varkappa\ln|t|$, $\varkappa > 0$, $\mu \geq 0$, $\nu \geq 0$, $\mu^2 + \nu^2 = 1$, $\varepsilon\sigma = 0$, *and* $\varepsilon \geq 0$.
5. $A_5^2(\sigma, \varepsilon) = <D,\ R(0, 0, |t|^{\sigma+1/2}) + Z(\varepsilon|t|^{\sigma-1})>$, *where* $\varepsilon\sigma = 0$ *and* $\varepsilon \geq 0$.

6. $A^2_6(\sigma,\mu,\nu,\varepsilon) =< \partial_t + J_{12},\ R(\nu e^{\sigma t}\cos t, \nu e^{\sigma t}\sin t, \mu e^{\sigma t}) + Z(\varepsilon e^{\sigma t}) >$, *where* $\mu \ge 0$, $\nu \ge 0$, $\mu^2+\nu^2 = 1$, $\varepsilon\sigma = 0$, *and* $\varepsilon \ge 0$.

7. $A^2_7(\sigma,\varepsilon) =< \partial_t,\ R(0,0,e^{\sigma t}) + Z(\varepsilon e^{\sigma t}) >$, *where* $\sigma \in \{-1;0;1\}$, $\varepsilon\sigma = 0$, *and* $\varepsilon \ge 0$.

8. $A^2_8(\lambda,\psi^1,\rho,\psi^2) =< J_{12} + R(0,0,\lambda) + Z(\psi^1),\ R(0,0,\rho) + Z(\psi^2) >$ *with smooth functions (of t)* λ, ρ, *and* ψ^i: $(\rho,\psi^2) \not\equiv (0,0)$ *and* $\lambda_{tt}\rho - \lambda\rho_{tt} \equiv 0$. *Algebras* $A^2_8(\lambda,\psi^1,\rho,\psi^2)$ *and* $A^2_8(\tilde\lambda,\tilde\psi^1,\tilde\rho,\tilde\psi^2)$ *are equivalent if* $\exists C_1 \ne 0$, $\exists \varepsilon,\delta,C_2 \in \mathbb{R}$, $\exists \theta \in C^\infty((t_0,t_1),\mathbb{R})$:

$$\begin{aligned}
&\tilde\lambda(\tilde t) = e^{\varepsilon}(\lambda(t) + C_2\rho(t)), \quad \tilde\rho(\tilde t) = C_1 e^{-\varepsilon}\rho(t),\\
&\tilde\psi^1(\tilde t) = e^{2\varepsilon}(\psi^1(t) + \theta_{tt}(t)\lambda(t) - \theta(t)\lambda_{tt}(t) +\\
&\qquad\qquad + C_2(\psi^2(t) + \theta_{tt}(t)\rho(t) - \theta(t)\rho_{tt}(t))),\\
&\tilde\psi^2(\tilde t) = C_1 e^{2\varepsilon}(\psi^2(t) + \theta_{tt}(t)\rho(t) - \theta(t)\rho_{tt}(t)),
\end{aligned} \tag{A.8}$$

where $\tilde t = te^{-2\varepsilon} + \delta$.

9. $A^2_9(\vec m^1,\chi^1,\vec m^2,\chi^2) =< R(\vec m^1(t)) + Z(\chi^1(t)),\ R(\vec m^2(t)) + Z(\chi^2(t)) >$ *with smooth functions* $\vec m^i$ *and* χ^i:

$$\vec m^1_{tt}\cdot\vec m^2 - \vec m^1\cdot\vec m^2_{tt} = 0, \quad \operatorname{rank}((\vec m^1,\chi^1),(\vec m^2,\chi^2)) = 2.$$

Algebras $A^2_9(\vec m^1,\chi^1,\vec m^2,\chi^2)$ *and* $A^2_9(\vec{\tilde m}^1,\tilde\chi^1,\vec{\tilde m}^2,\tilde\chi^2)$ *are equivalent if* $\exists\varepsilon,\delta \in \mathbb{R}$, $\exists \{a_{ij}\}_{i,j=1,2} : \det\{a_{ij}\} \ne 0$, $\exists B \in O(3)$, $\exists \vec l \in C^\infty((t_0,t_1),\mathbb{R}^3)$:

$$\begin{aligned}
&\vec{\tilde m}^i(\tilde t) = e^{-\varepsilon}a_{ij}B\vec m^j(t),\\
&\tilde\chi^i(\tilde t) = e^{2\varepsilon}a_{ij}(\chi^j(t) + \vec l_{tt}(t)\cdot\vec m^j(t) - \vec l(t)\cdot\vec m^j_{tt}(t)),
\end{aligned} \tag{A.9}$$

where $\tilde t = te^{-2\varepsilon} + \delta$.

10. $A^2_{10}(\varkappa,\sigma) =< D + \varkappa J_{12},\ Z(|t|^\sigma) >$, *where* $\varkappa \ge 0$, $\sigma \in \mathbb{R}$.

11. $A^2_{11}(\sigma) =< \partial_t + J_{12},\ Z(e^{\sigma t}) >$, *where* $\sigma \in \mathbb{R}$.

12. $A^2_{12}(\sigma) =< \partial_t,\ Z(e^{\sigma t}) >$ *where* $\sigma \in \{-1;0;1\}$.

The proof of Theorem A.2 is analogous to that of Theorem A.1. Let us take an arbitrary two-dimensional subalgebra generated by two linearly independent operators of the form

$$V^i = a^i_1 D + a^i_2\partial_t + a^i_3 J_{12} + a^i_4 J_{23} + a^i_5 J_{31} + R(\vec m^i) + Z(\chi^i),$$

where $a^i_n = \mathrm{const}\,(n = \overline{1,5})$ and $[V^1,V^2] \in < V^1,V^2 >$. Considering the different possible cases we try to simplify V^i by means of adjoint representation as much as possible. Here we do not present the proof of Theorem A.2 as it is too cumbersome.

A.4 Three-dimensional subalgebras

We also constructed a complete set of $A(NS)$-inequivalent three-dimensional subalgebras. It contains 52 classes of algebras. By means of 22 classes from this set one can obtain ansatzes of codimension three for the Navier-Stokes field. Here we only give 8 superclasses that arise from unification of some of these classes:

1. $A^3_1 =< D,\ \partial_t,\ J_{12} >$.

2. $A^3_2 =< D + \varkappa J_{12},\ \partial_t,\ R(0,0,1) >$, where $\varkappa \geq 0$.

Here and below $\varkappa$, σ, ε_1, ε_2, μ, ν, and a_{ij} are real constants.

3. $A^3_3(\sigma,\nu,\varepsilon_1,\varepsilon_2) =< D,\ J_{12} + \nu(R(0,0,|t|^{1/2}\ln|t|) + Z(\varepsilon_2|t|^{-1}\ln|t|)) + Z(\varepsilon_1|t|^{-1}),$ $R(0,0,|t|^{\sigma+1/2}) + Z(\varepsilon_2|t|^{\sigma-1}) >$, where $\nu\sigma = 0$, $\varepsilon_1 \geq 0$, $\nu \geq 0$, and $\sigma\varepsilon_2 = 0$.

4. $A^3_4(\sigma,\nu,\varepsilon_1,\varepsilon_2) =< \partial_t,\ J_{12} + Z(\varepsilon_1) + \nu(R(0,0,t) + Z(\varepsilon_2 t)),\ R(0,0,e^{\sigma t}) + Z(\varepsilon_2 e^{\sigma t}) >$, where $\nu\sigma = 0$, $\sigma\varepsilon_2 = 0$, and, if $\sigma = 0$, the constants ν, ε_1, and ε_2 satisfy one of the following conditions:

$$\nu = 1,\ \varepsilon_1 \geq 0; \quad \nu = 0,\ \varepsilon_1 = 1,\ \varepsilon_2 \geq 0; \quad \nu = \varepsilon_1 = 0,\ \varepsilon_2 \in \{0;1\}.$$

5. $A^3_5(\varkappa,\vec m^1,\vec m^2,\chi^1,\chi^2) =< D + 2\varkappa J_{12},\ R(\vec m^1) + Z(\chi^1),\ R(\vec m^2) + Z(\chi^2) >$, where $\varkappa \geq 0$, $\operatorname{rank}(\vec m^1,\vec m^2) = 2$,

$$t\vec m^i_t - \tfrac{1}{2}\vec m^i + \varkappa(m^{i2}, -m^{i1}, 0)^T = a_{ij}\vec m^j,$$

$$t\chi^i_t + \chi^i = a_{ij}\chi^j, \quad a_{ij} = \text{const},$$

$$\begin{aligned}&(a_{11} + a_{22})(a_{21}\vec m^1\cdot\vec m^1 + (a_{22} - a_{11})\vec m^1\cdot\vec m^2 - a_{12}\vec m^2\cdot\vec m^2 +\\ &+ 2\varkappa(m^{12}m^{21} - m^{11}m^{22})) = 0.\end{aligned} \tag{A.10}$$

This superclass contains eight inequivalent classes of subalgebras that can be obtained from it by means of a change of basis and the adjoint actions

$$\mathrm{Ad}(\delta_1 D), \quad \mathrm{Ad}(\delta_2 J_{12}), \quad \mathrm{Ad}(R(\vec n) + Z(\eta))$$

$$(\mathrm{Ad}(\delta D), \quad \mathrm{Ad}(\varepsilon_{ab}J_{ab}), \quad \mathrm{Ad}(R(\vec n) + Z(\eta)))$$

if $\varkappa > 0$ $(\varkappa = 0)$ respectively. Here the functions $\vec n$ and η satisfy the following equations:

$$t\vec n_t - \tfrac{1}{2}\vec n + \varkappa(n^2, -n^1, 0)^T = b_i\vec m^i,$$

$$t\eta_t + \eta = b_i\chi_i + \tfrac{1}{2}t(\vec n_{ttt}\cdot\vec n - \vec n_{tt}\cdot\vec n_t) + \vec n_{tt}\cdot\vec n + \varkappa(n^1 n^2_{tt} - n^1_{tt}n^2).$$

6. $A^3_6(\varkappa,\vec m^1,\vec m^2,\chi^1,\chi^2) =< \partial_t + \varkappa J_{12},\ R(\vec m^1) + Z(\chi^1),\ R(\vec m^2) + Z(\chi^2) >$, where $\varkappa \in \{0;1\}$, $\operatorname{rank}(\vec m^1,\vec m^2) = 2$,

$$\vec m^i_t - \varkappa(m^{i2}, -m^{i1}, 0)^T = a_{ij}\vec m^j, \quad t\chi^i_t = a_{ij}\chi^j,$$

and a_{ij} are constants satisfying (A.10). This superclass contains eight inequivalent classes of subalgebras that can be obtained from it by means of a change of basis and the adjoint actions

$$\mathrm{Ad}(\delta_1\partial_t),\quad \mathrm{Ad}(\delta_2 J_{12}),\quad \mathrm{Ad}(R(\vec{n})+Z(\eta))$$

$$(\mathrm{Ad}(\delta_1\partial_t),\quad \mathrm{Ad}(\delta_2 D),\quad \mathrm{Ad}(\varepsilon_{ab}J_{ab}),\quad \mathrm{Ad}(R(\vec{n})+Z(\eta)))$$

if $\varkappa=1$ $(\varkappa=0)$ respectively. Here the functions $\vec{n}$ and η satisfy the following equations:

$$\vec{n}_t+\varkappa(n^2,-n^1,0)^T=b_i\vec{m}^i,$$

$$\eta_t=b_i\chi_i+\tfrac{1}{2}(\vec{n}_{ttt}\cdot\vec{n}-\vec{n}_{tt}\cdot\vec{n}_t)+\varkappa(n^1n^2_{tt}-n^1_{tt}n^2).$$

7. $A^3_7(\eta^1,\eta^2,\eta^3,\chi)=<J_{12}+R(0,0,\eta^3),\ R(\eta^1,\eta^2,0),\ R(-\eta^2,\eta^1,0)>,$ where

$$\eta^a\in C^\infty((t_0,t_1),\mathbb{R}),\quad \eta^1_{tt}\eta^2-\eta^1\eta^2_{tt}\equiv 0,\quad \eta^i\eta^i\not\equiv 0,\quad \eta^3\neq 0.$$

Algebras $A^3_7(\eta^1,\eta^2,\eta^3)$ and $A^3_7(\tilde\eta^1,\tilde\eta^2,\tilde\eta^3)$ are equivalent if $\exists\delta_a\in\mathbb{R},\ \exists\delta_4\neq 0$:

$$\begin{aligned}
&\tilde\eta^1(\tilde t)=\delta_4(\eta^1(t)\cos\delta_3-\eta^2(t)\sin\delta_3),\\
&\tilde\eta^2(\tilde t)=\delta_4(\eta^1(t)\sin\delta_3+\eta^2(t)\cos\delta_3),\\
&\tilde\eta^3(\tilde t)=e^{-\delta_1}\eta^3(t),
\end{aligned}\qquad\text{(A.11)}$$

where $\tilde t=te^{-2\delta_1}+\delta_2$.

8. $A^3_8(\vec{m}^1,\vec{m}^2,\vec{m}^3)=<R(\vec{m}^1),\ R(\vec{m}^2),\ R(\vec{m}^3)>,$ where

$$\vec{m}^a\in C^\infty((t_0,t_1),\mathbb{R}^3),\quad \mathrm{rank}(\vec{m}^1,\vec{m}^2,\vec{m}^3)=3,\quad \vec{m}^a_{tt}\cdot\vec{m}^b-\vec{m}^a\cdot\vec{m}^b_{tt}=0.$$

Algebras $A^3_8(\vec{m}^1,\vec{m}^2,\vec{m}^3)$ and $A^3_8(\vec{\tilde m}^1,\vec{\tilde m}^2,\vec{\tilde m}^3)$ are equivalent if $\exists\delta_i\in\mathbb{R}^3$, $\exists B\in O(3)$, $\exists\{d_{ab}\}:\det\{d_{ab}\}\neq 0$ such that

$$\vec{\tilde m}^a(\tilde t)=d_{ab}B\vec{m}^b(t),\qquad\text{(A.12)}$$

where $\tilde t=te^{-2\delta_1}+\delta_2$.

B On construction of ansatzes for the Navier-Stokes field by means of the Lie method

The general method for constructing a complete set of inequivalent Lie ansatzes of a system of PDEs are well known and described, for examle, in [27, 28]. However, in some cases when the symmetry operators of the system have a special form, this method can be modified [9]. Thus, in the case of the NSEs, coefficients of an arbitrary operator

$$Q=\xi^0\partial_t+\xi^a\partial_a+\eta^a\partial_{u^a}+\eta^0\partial_p$$

from $A(NS)$ satisfy the following conditions:

$$\begin{aligned}&\xi^0=\xi^0(t,\vec{x}),\quad \xi^a=\xi^a(t,\vec{x}),\quad \eta^a=\eta^{ab}(t,\vec{x})u^b+\eta^{a0}(t,\vec{x}),\\&\eta^0=\eta^{01}(t,\vec{x})p+\eta^{00}(t,\vec{x}).\end{aligned}\tag{B.1}$$

(The coefficients ξ^a, ξ^0, η^a, and η^0 also satisfy stronger conditions than (B.1). For example if $Q\in A(NS)$, then $\xi^0=\xi^0(t)$, $\eta^{ab}=\mathrm{const}$, and so on. But conditions (B.1) are sufficient to simplify the general method.) Therefore, ansatzes for the Navier-Stokes field can be constructing in the following way:

1. We fix a M-dimensional subalgebra of $A(NS)$ with the basis elements

$$Q^m=\xi^{m0}\partial_t+\xi^{ma}\partial_a+(\eta^{mab}u^b+\eta^{ma0})\partial_{u^a}+(\eta^{m01}p+\eta^{m00})\partial_p,\tag{B.2}$$

where $M\in\{1;2;3\}$, $m=\overline{1,M}$, and

$$\mathrm{rank}\{(\xi^{m0},\xi^{m1},\xi^{m2},\xi^{m3}),\ m=\overline{1,M}\}=M.\tag{B.3}$$

To construct a complete set of inequivalent Lie ansatzes of codimension M for the Navier-Stokes field, we have to use the set of M-dimensional subalgebras from Sec. A. Condition (B.3) is neeeded for the existance of ansatzes connected with this subalgebra.

2. We find the invariant independent variables $\omega_n=\omega_n(t,\vec{x})$, $n=\overline{1,N}$, where $N=4-M$, as a set of functionally independent solutions of the following system:

$$L^m\omega=Q^m\omega=\xi^{m0}\partial_t\omega+\xi^{ma}\partial_a\omega=0,\quad m=\overline{1,M},\tag{B.4}$$

where $L^m:=\xi^{m0}\partial_t+\xi^{ma}\partial_a$.

3. We present the Navier-Stokes field in the form:

$$u^a=f^{ab}(t,\vec{x})v^b(\bar{\omega})+g^a(t,\vec{x}),\quad p=f^0(t,\vec{x})q(\bar{\omega})+g^0(t,\vec{x}),\tag{B.5}$$

where v^a and q are new unknown functions of $\bar{\omega}=\{\omega_n,\ n=\overline{1,N}\}$. Acting on representation (B.5) with the operators Q^m, we obtain the following equations on functions f^{ab}, g^a, f^0, and g^0:

$$\begin{aligned}&L^mf^{ab}=\eta^{mac}f^{cb},\quad L^mg^a=\eta^{mab}g^b+\eta^{ma0},\quad c=\overline{1,3},\\&L^mf^0=\eta^{m01}f^0,\quad L^mg^0=\eta^{m01}g^0+\eta^{m00}.\end{aligned}\tag{B.6}$$

If the set of functions f^{ab}, f^0, g^a, and g^0 is a particular solution of (B.6) and satisfies the conditions $\mathrm{rank}\{(f^{1b},f^{2,b},f^{3b}),\ b=\overline{1,3}\}=3$ and $f^0\neq0$, formulas (B.5) give an ansatz for the Navier-Stokes field.

The ansatz connected with the fixed subalgebra is not determined in an unique manner. Thus, if

$$\begin{aligned}&\tilde{\omega}_l=\tilde{\omega}_l(\bar{\omega}),\quad \det\left\{\frac{\partial\tilde{\omega}_l}{\partial\omega_n}\right\}_{l,n=\overline{1,N}}\neq0,\\&\tilde{f}^{ab}(t,\vec{x})=f^{ac}(t,\vec{x})F^{cb}(\bar{\omega}),\quad \tilde{g}^a(t,\vec{x})=g^a(t,\vec{x})+f^{ac}(t,\vec{x})G^c(\bar{\omega}),\\&\tilde{f}^0(t,\vec{x})=f^0(t,\vec{x})F^0(\bar{\omega}),\quad \tilde{g}^0(t,\vec{x})=g^0(t,\vec{x})+f^0(t,\vec{x})G^0(\bar{\omega}),\end{aligned}\tag{B.7}$$

the formulas

$$u^a = \tilde{f}^{ab}(t,\vec{x})\tilde{v}^b(\bar{\tilde{\omega}}) + \tilde{g}^a(t,\vec{x}), \quad p = \tilde{f}^0(t,\vec{x})q(\bar{\tilde{\omega}}) + \tilde{g}^0(t,\vec{x}) \tag{B.8}$$

give an ansatz which is equivalent to ansatz (B.5). The reduced system of PDEs on the functions $\tilde{v}^a$ and $\tilde{q}$ is obtained from the system on v^a and q by means of a local transformation. Our problem is to find or "to guess", at once, such an ansatz that the corresponding reduced system has a simple and convenient form for our investigation. Otherwise, we can obtain a very complicated reduced system which will be not convenient for investigation and we can not simplify it.

Consider a simple example.

Let $M=1$ and let us give the algebra $<\partial_t + \varkappa J_{12}>$, where $\varkappa \in \{0;1\}$. For this algebra, the invariant independent variables $y_a = y_a(t,\vec{x})$ are functionally independent solutions of the equation $Ly=0$ (see (B.4)), where

$$L := \partial_t + \varkappa(x_1\partial_{x_2} - x_2\partial_{x_1}). \tag{B.9}$$

There exists an infinite set of choices for the variables y_a. For example, we can give the following expressions for y_a:

$$y_1 = \arctan\frac{x_1}{x_2} - \varkappa t, \quad y_2 = (x_1^2 + x_2^2)^{1/2}, \quad y_3 = x_3.$$

However choosing y_a in such a way, for $\varkappa \neq 0$ we obtain a reduced system which strongly differs from the "natural" reduced system for $\varkappa = 0$ (the NSEs for steady flows of a viscous fluid in Cartesian coordinates). It is better to choose the following variables y_a:

$$y_1 = x_1\cos\varkappa t + x_2\sin\varkappa t, \quad y_2 = -x_1\sin\varkappa t + x_2\cos\varkappa t, \quad y_3 = x_3.$$

The vector-functions $\vec{f}^b = (f^{1b}, f^{2b}, f^{3b})$, $b=\overline{1,3}$, should be linearly independent solutions of the system

$$Lf^1 = -\varkappa f^2, \quad Lf^2 = \varkappa f^1, \quad Lf^3 = 0$$

and the function f^0 should satisfy the equation $Lf^0 = 0$ and the condition $f^0 \neq 0$. Here the operator L is defined by (B.9). We give the following values of these functions:

$$\vec{f}^1 = (\cos\varkappa t, \sin\varkappa t, 0),\ \vec{f}^2 = (-\sin\varkappa t, \cos\varkappa t, 0),\ \vec{f}^3 = (0,0,1),\ f^0 = 1.$$

The functions g^a and g^0 are solutions of the equations

$$Lg^1 = -\varkappa g^2, \quad Lg^2 = \varkappa g^1, \quad Lg^3 = 0, \quad Lg^0 = 0.$$

We can make, for example, g^a and g^0 vanish. Then the corresponding ansatz has the form:

$$u^1 = \tilde{v}^1\cos\varkappa t - \tilde{v}^2\sin\varkappa t, \quad u^2 = \tilde{v}^1\sin\varkappa t + \tilde{v}^2\cos\varkappa t, \quad u^3 = \tilde{v}^3, \quad p = \tilde{q}, \tag{B.10}$$

where $\tilde{v}^a = \tilde{v}^a(y_1,y_2,y_3)$ and $\tilde{q} = \tilde{q}(y_1,y_2,y_3)$ are the new unknown functions. Substituting ansatz (B.10) into the NSEs, we obtain the following reduced system:

$$\begin{aligned}
&\tilde{v}^a\tilde{v}^1_a - \tilde{v}^1_{aa} + \tilde{q}_1 + \varkappa y_2\tilde{v}^1_1 - \varkappa y_1\tilde{v}^1_2 - \varkappa\tilde{v}^2 = 0,\\
&\tilde{v}^a\tilde{v}^2_a - \tilde{v}^2_{aa} + \tilde{q}_2 + \varkappa y_2\tilde{v}^2_1 - \varkappa y_1\tilde{v}^2_2 + \varkappa\tilde{v}^1 = 0,\\
&\tilde{v}^a\tilde{v}^3_a - \tilde{v}^3_{aa} + \tilde{q}_3 + \varkappa y_2\tilde{v}^3_1 - \varkappa y_1\tilde{v}^3_2 = 0,\\
&\tilde{v}^a_a = 0.
\end{aligned} \tag{B.11}$$

Here subscripts 1,2, and 3 of functions in (B.11) denote differentiation with respect to y_1, y_2, and y_3 accordingly. System (B.11), having variable coefficients, can be simplified by means of the local transformation

$$\tilde{v}^1 = v^1 - \varkappa y_2, \quad \tilde{v}^2 = v^2 + \varkappa y_1, \quad \tilde{v}^3 = v^3, \quad \tilde{q} = q + \tfrac{1}{2}(y_1^2 + y_2^2). \tag{B.12}$$

Ansatz (B.10) and system (B.11) are transformed under (B.12) into ansatz (2.2) and system (2.7), where

$$g^1 = -\varkappa x_2, \quad g^2 = \varkappa x_1, \quad g_3 = 0, \quad g^0 = \tfrac{1}{2}\varkappa^2(x_1^2 + x_2^2), \tag{B.13}$$

$\gamma_1 = -2\varkappa$, and $\gamma_2 = 0$. Therefore, we can give the values of g^a and g^0 from (B.13) and obtain ansatz (2.2) and system (2.7) at once.

The above is a good example how a reduced system can be simplified by means of modifying (complicating) an ansatz corresponding to it. Thus, system (2.7) is simpler than system (B.11) and ansatz (2.2) is more complicated than ansatz (B.10).

Finally, let us make several short notes about constructing other ansatzes for the Navier-Stokes field.

Ansatz corresponding to the algebra $A_4^1(\vec{m}, \chi)$ (see Subsec. A.2) can be constructed only for such t that $\vec{m}(t) \neq \vec{0}$. For these values of t, the parameter-function χ can be made to vanish by means of equivalence transformations (A.7).

Ansatz corresponding to the algebra $A_8^2(\lambda, \psi^1, \rho, \psi^2)$ (see Subsec. A.3) can be constructed only for such t that $\rho(t) \neq 0$. For these values of t, the parameter-function ψ^2 can be made to vanish by means of equivalence transformations (A.8). Moreover, it can be considered that $\lambda_t\rho - \lambda\rho_t \in \{0; 1\}$. The algebra obtained finally is denoted by $A_8^2(\lambda, \chi, \rho, 0)$.

Ansatz corresponding to the algebra $A_9^2(\vec{m}^1, \chi^1, \vec{m}^2, \chi^2)$ (see Subsec. A.3) can be constructed only for such t that $\operatorname{rank}(\vec{m}^1, \vec{m}^2) = 2$. For these values of t, the parameter-functions χ^i can be made to vanish by means of equivalence transformations (A.9).

The algebras $A_{10}^2(\varkappa, \sigma)$, $A_{11}^2(\sigma)$, and $A_{12}^2(\sigma)$ can not be used to construct ansatzes by means of the Lie algorithm.

In view of equivalence transformation (A.11), the functions η^i in the algebra $A_7^3(\eta^1, \eta^2, \eta^3)$ (see Subsec. A.4) can be considered to satisfy the following condition:

$$\eta_t^1\eta^2 - \eta^1\eta_t^2 \in \{0; \tfrac{1}{2}\}.$$

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