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Nonisospectral Flows on Semi-infinite Jacobi Matrices

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Abstract

It is proved that if the spectrum and spectral measure of a semi-infinite Jacobi matrix $L(t)$ change appropriately, then $L(t)$ satisfies a generalized Lax equation of the form $\dot{L}(t) = \Phi(L(t), t) + [L(t), A(L(t), t)]$, where $\Phi(\lambda, t)$ is a polynomial with t-dependent coefficients and $A(L(t), t)$ is a skew-symmetric matrix which is determined by the evolution of the spectral data. Such an equation is equivalent to a wide class of generalized Toda lattices. The theory of Jacobi matrices gives rise to the procedure of solution of the corresponding Cauchy problem by the inverse spectral problem method. The linearization of this nonlinear equation in terms of the moments is established.

0 Introduction

In brief communications [1–3] we considered non-isospectral deformations of finite and semi-infinite Jacobi matrices which are governed by a generalized Lax equation (see Section 3 below) and used the inverse spectral problem method [4, 5] to solve the corresponding Cauchy problem in the class of bounded operators.

To explain our approach recall that the famous Toda lattice

$$\dot{a}_n = \frac{1}{2} a_n (b_{n+1} - b_n), \qquad \dot{b}_n = a_n^2 - a_{n-1}^2 \tag{0.1}$$

admits the Lax representation:

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$$\dot{L}(t) = [L(t), A(t)] \ , \tag{0.2}$$

where $L(t)$ is a Jacobi matrix constructed from the solution of (0.1) and $A(t) = A(L(t), t)$ is a skew-symmetric matrix connected with $L(t)$. As it is well known [6, 7] $L(t)$ undergoes the isospectral deformations. Moreover, for a finite Toda lattice with free ends [8] or a semi-infinite Toda lattice with one free end [4, 5] it was discovered that a spectral measure $d\rho(\lambda, t)$ of $L(t)$ meets a very simple differential equation, which after a proper normalization can be written down as follows:

$$d\dot{\rho}(\lambda, t) = \lambda \ d\rho(\lambda, t) \ . \tag{0.3}$$

Thus, in order to integrate the finite or semi-infinite Toda lattice it is enough to construct the spectral measure $d\rho(\lambda, 0)$ of the initial Jacobi matrix (the direct spectral problem), to take the solution $d\rho(\lambda, t) = e^{\lambda t} d\rho(\lambda, 0)$ of (0.3), and finally, to recover a self-adjoint Jacobi matrix $L(t)$ from the obtained measure (the inverse spectral problem of the theory of Jacobi matrices).

This procedure has been used in [4, 5] to solve the Cauchy problem for (0.1) on the semi-axis $n = 0, 1, \ldots$ with a boundary condition $a_{-1} = 0$. Later, modifications of the procedure were applied to some important isospectral lattice equations, such as non-abelian Toda and Volterra lattices, discrete mKdV and NLS equations on the finite or semi-infinite intervals [1, 9–11]. In this connection we should also mention [12, 13], where semi-infinite lattice equations were treated by means of the continued fractions theory, a study of the generalized Toda flows in ℓ_2 [14], and an investigation of the semi-infinite Toda lattice with a fixed end [15]. Unlike the inverse scattering method [6, 16, 17] the approach, which is described above, allows us to construct the solution for arbitrary bounded initial values.

One of the ways to generalize the Lax equation is to describe the evolution of $L(t)$ compatible with a given evolution of the spectrum. The corresponding infinite lattice equations were first considered in [18, 19] in the framework of the inverse scattering (see also recent paper [20] and the book [21]). In the present paper we show that the more general than (0.3) evolution of the spectral measure [1–3] naturally leads to the nonisospectral flows having the following form :

$$\dot{L}(t) = \Phi(L(t), t) + [L(t), A(t)] \ , \tag{0.4}$$

where $\Phi(\lambda, t)$ is a polynomial in λ and $A(t) = A(L(t), t)$ is a skew-symmetric matrix whose form is determined by this evolution. Note that an autonomous system which is similar to (0.4) has been treated in [22]. In our derivation of (0.4) the important role is played by orthogonal polynomials on the line (see [23], where they appeared for the first time in the context of integrable systems and the inverse scattering method).

To construct solutions of the Cauchy problem for (0.4) we apply the inverse spectral method. In addition, we obtain the linearization of these nonlinear equations in terms of the properly normalized moments of the spectral measure corresponding to the Jacobi matrix $L(t)$ and prove the uniqueness of the solution of the Cauchy problem.

The paper is organized as follows. At the end of this section we introduce the necessary notations. In Section 1 well known facts of the spectral theory of Jacobi matrices are reviewed and some useful formulas are established. Section 2 is devoted to the construction of a measure transformation which plays an important role in the sequel. In Section 3 we formulate our main results. Section 4 contains two auxiliary lemmas which allow us to prove the main results in Section 5. Section 6 is devoted to the uniqueness theorem. In the last Section 7 we consider one of the possible modifications of our scheme and illustrate it by specific examples.

Some words on the notations. We denote the σ-algebra of the Borel sets on real axis by $\mathfrak{B}(\mathbb{R})$. The symbol $\mathfrak{M}$ stands for the set of all finite measures $\rho(\cdot)$ on $\mathfrak{B}(\mathbb{R})$ with compact support and infinitely many growth points. The symbol $\mathfrak{L}$ designates the set of all bounded selfadjoint semi-infinite Jacobi matrices L with real entries along the principal diagonal and positive entries along two adjacent diagonals.

If $A = (a_{jk})_{j,k=0}^{\infty}$ is a semi-infinite matrix which defines an operator acting in Hilbert space ℓ_2, then we denote

$$\{A\}_{jk} = (A\delta_k, \delta_j)_{\ell_2} = a_{jk} ,$$

where $(\delta_j)_{j=0}^{\infty}$ is a standard basis in ℓ_2. For this matrix A we denote by $\langle A\rangle$ the matrix with elements $\langle A\rangle_{jk} = a_{jk}$ if $j > k$, $\langle A\rangle_{jk} = 0$ if $j = k$, and $\langle A\rangle_{jk} = -a_{kj}$ if $j < k$. We write $\langle A\rangle_{jk}$ instead of $\{\langle A\rangle\}_{jk}$. The symbol $[\cdot,\cdot]$ designates the usual commutator, i.e., $[A, B] = AB - BA$ whenever the multiplications make sense, and $[A, B]_{jk}$ stands for $\{[A, B]\}_{jk}$. We denote the transpose of matrix A by A^{τ}, i.e., $\{A^{\tau}\}_{jk} = \{A\}_{kj}$. The dot stands for the derivative $\dot{} = \frac{d}{dt}$ or for the partial derivative $\dot{} = \frac{\partial}{\partial t}$ with respect to t, while $' = \frac{\partial}{\partial \lambda}$ denotes the partial derivative with respect to λ.

1 Preliminaries

To begin with, we review some well-known facts of the theory of semi-infinite Jacobi matrices (see e.g. [24] for more details). In the space ℓ_2 of square summable sequences $u = (u_j)_{j=0}^{\infty}$ let us consider the action of a difference expression

$$(\mathfrak{T}u)_j = a_{j-1}u_{j-1} + b_j u_j + a_j u_{j+1}, \quad a_j > 0, \quad b_j \in \mathbb{R} , \tag{1.1}$$

where $j = 0, 1, \dots$ and $a_{-1} = 0$. If the coefficients a_j and b_j are bounded, then $\mathfrak{T}$ generates in ℓ_2 a bounded selfadjoint operator L (Jacobi matrix):

$$(Lu)_j = (\mathfrak{T}u)_j , \quad u \in \ell_2 , \quad j = 0, 1, \dots ; \quad u_{-1} = 0. \tag{1.2}$$

We denote by $\mathfrak{L}$ the class of all such Jacobi matrices. Let

$$\mathfrak{B}(\mathbb{R}) \ni \Delta \mapsto \rho(\Delta) \in [0, \infty) \tag{1.3}$$

be its spectral measure. It has the following well known properties:
(1) The support is a compact set in $\mathbb{R}$,
(2) There are infinitely many points of increase,
(3) $\rho(\mathbb{R}) = 1$.
Let us denote by $\mathfrak{M}$ the set of all nonzero finite measures on $\mathfrak{B}(\mathbb{R})$ with properties (1) and (2). It turns out that every $d\rho(\lambda) \in \mathfrak{M}$ is a spectral measure (up to a scalar factor) of some bounded selfadjoint Jacobi matrix. The problem of recovering the corresponding Jacobi matrix L from the given $d\rho(\lambda) \in \mathfrak{M}$ is called the *inverse spectral problem* (ISP).

Since the ISP plays an important role in our further considerations we outline here two different approaches to its solution.

I. Let

$$P_0(\lambda) = (\rho(\mathbb{R}))^{-1/2}, \qquad P_1(\lambda), \ldots \tag{1.4}$$

be a sequence of orthonormal polynomials which is constructed by the Schmidt procedure of orthogonalization of the powers

$$1, \lambda, \lambda^2, \ldots$$

in the Hilbert space

$$L_2 := L_2(\mathbb{R}, d\rho(\lambda)) \ .$$

The orthonormality means that

$$\int_{\mathbb{R}} P_j(\lambda)\, P_k(\lambda)\, d\rho(\lambda) = \delta_{jk}, \qquad j, k = 0, 1, \ldots \tag{1.5}$$

It turns out that these polynomials satisfy a three-term recursion:

$$a_{k-1}P_{k-1}(\lambda) + b_k P_k(\lambda) + a_k P_{k+1}(\lambda) = \lambda P_k(\lambda) \ , \quad k = 0, 1, \ldots, P_{-1} = 0 \ . \tag{1.6}$$

Therefore, the coefficients a_j and b_j of the difference expression $\mathfrak{T}$ can be found by the following formulas:

$$a_j = \int_{\mathbb{R}} \lambda P_j(\lambda) P_{j+1}(\lambda)\, d\rho(\lambda), \qquad b_j = \int_{\mathbb{R}} \lambda P_j^2(\lambda)\, d\rho(\lambda) \ , \quad j = 0, 1, \ldots \tag{1.7}$$

II. Let

$$s_n = \int_{\mathbb{R}} \lambda^n\, d\rho(\lambda), \qquad n = 0, 1, \ldots \ , \tag{1.8}$$

be a moment sequence of $d\rho(\lambda) \in \mathfrak{M}$. Since $d\rho(\lambda)$ is a finite measure with bounded support, all the moments exist. Moreover, as $d\rho(\lambda)$ has infinitely many points of increase, every Hankel matrix H_k of the form

$$H_k = \begin{bmatrix} s_0 & s_1 & \ldots & s_k \\ s_1 & s_2 & \ldots & s_{k+1} \\ \cdot & \cdot & \ldots & \cdot \\ s_k & s_{k+1} & \ldots & s_{2k} \end{bmatrix}, \qquad k = 0, 1, \ldots \ , \tag{1.9}$$

is positive definite:

$$D_k = \det H_k > 0, \qquad k = 0, 1, \ldots$$

Denoting

$$\Delta_k = \det \begin{bmatrix} s_0 & s_1 & \ldots & s_{k-1} & s_{k+1} \\ s_1 & s_2 & \ldots & s_k & s_{k+2} \\ \cdot & \cdot & \ldots & \cdot & \cdot \\ s_k & s_{k+1} & \ldots & s_{2k-1} & s_{2k+1} \end{bmatrix}, \qquad k = 0, 1, \ldots \ ,$$

one can solve the ISP by the following formulas (see e.g., [25]):

$$a_n = \frac{\sqrt{D_{n-1}D_{n+1}}}{D_n}, \qquad b_n = \frac{\Delta_n}{D_n} - \frac{\Delta_{n-1}}{D_{n-1}},$$
$$n = 0, 1, \ldots, \qquad D_{-1} := 1, \qquad \Delta_{-1} := 0, \qquad \Delta_0 := s_1. \tag{1.10}$$

Note that in terms of the entries of the inverse matrices H_k^{-1}, $k = 0, 1, \ldots$, these formulas can be rewritten as follows:

$$a_n = \sqrt{\tau_n \tau_{n+1}^{-1}}, \qquad b_n = \epsilon_n \tau_n^{-1} - \epsilon_{n+1} \tau_{n+1}^{-1} , \tag{1.11}$$

where $\tau_k = \{H_k^{-1}\}_{kk}$, $\epsilon_k = \{H_k^{-1}\}_{k-1,k}$, $k = 0, 1, \ldots$, $\epsilon_0 := 0$. It is probably worth noting that a fast algorithm is known, which produces $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ from $s_0, s_1, \ldots, s_{2n+2}$ after $O(n^2)$ operations (see e.g., [26]).

Let us return to the orthogonal polynomials. Formulas (1.5)–(1.7) show that the operator of multiplication by λ in the space L_2 has the Jacobi matrix L as its matrix representation in the basis $(P_j(\lambda))_{j=0}^{\infty}$. Consider the matrix form D_L of the differential operator $\frac{\partial}{\partial \lambda}$ in the same basis:

$$D_L = (d_{jk})_{j,k=0}^{\infty}, \tag{1.12}$$

where

$$d_{jk} = (P_k'(\cdot), P_j(\cdot))_{L_2} = \int_{\mathbb{R}} P_k'(\lambda) P_j(\lambda)\, d\rho(\lambda) . \tag{1.13}$$

It follows easily from (1.5) that $d_{jk} = 0$ when $j \geq k$, i.e., D is a strictly upper triangular matrix, and therefore

$$P_k'(\lambda) = \sum_{j=0}^{k-1} d_{jk} P_j(\lambda), \qquad j, k = 0, 1, \ldots \tag{1.14}$$

By differentiation of identities (1.6) with respect to λ, one can get the recurrent formulas for the calculation of d_{jk}, and then express them in terms of a_n, b_n. For example, one has

$$d_{k-1,k} = k a_{k-1}^{-1}, \qquad d_{k-2,k} = a_{k-2}^{-1} a_{k-1}^{-1} \left(\sum_{j=1}^{k} b_{j-1} - k b_{k-1} \right) . \tag{1.15}$$

An arbitrary polynomial $C(\lambda) = c_0 + c_1 \lambda + \cdots + c_m \lambda^m$ with real coefficients $c_j \in \mathbb{R}$, $j = 0, 1, \ldots, m$, generates the operator C of multiplication by this polynomial in the space L_2 and the selfadjoint operator

$$C(L) = c_0 \mathbf{1} + c_1 L + \cdots + c_m L^m ,$$

which acts in ℓ_2. The matrix form of the operator C in the basis $(P_j(\lambda))_{j=0}^{\infty}$ coincides with the matrix form of $C(L)$ in the standard basis for ℓ_2:

$$C_{jk} = \int_{\mathbb{R}} C(\lambda) P_k(\lambda) P_j(\lambda)\, d\rho(\lambda) = \{C(L)\}_{jk}, \qquad j, k = 0, 1, \ldots \tag{1.16}$$

Note that by the spectral theorem, (1.16) holds for any function $C(\lambda)$ which is continuous on the spectrum of L. In the sequel we will need the following formula

$$\{C(L)\}_{00} = \int_{\mathbb{R}} C(\lambda) P_0^2(\lambda)\, d\rho(\lambda) = \frac{1}{\rho(\mathbb{R})} \int_{\mathbb{R}} C(\lambda)\, d\rho(\lambda) . \tag{1.17}$$

The operator $C\frac{\partial}{\partial\lambda}$ acting in L_2 can be represented in the orthonormal basis $(P_j(\lambda))_{j=0}^{\infty}$ as a matrix with jk-entry:

$$\begin{aligned}\left(C\,\frac{\partial}{\partial\lambda}\right)_{jk} &= \int_{\mathbb{R}} C(\lambda)\,P_k'(\lambda)P_j(\lambda)\,d\rho(\lambda) = \int_{\mathbb{R}} C(\lambda)\left(\sum_{i=0}^{k-1} d_{ik}P_i(\lambda)\right)P_j(\lambda)\,d\rho(\lambda) = \\ &\sum_{i=0}^{k-1} d_{ik}C_{ji} = \sum_{i=0}^{\infty} C_{ji}d_{ik} = \{C(L)D_L\}_{jk}. \end{aligned} \tag{1.18}$$

For any matrix $A=(a_{jk})_{k=0}^{\infty}$ we define $\langle A\rangle = A_{low} - A_{low}^{\tau}$, where A_{low} stands for the strictly lower triangular part of A and τ denotes the usual transpose. This definition can be rewritten as

$$\langle A\rangle_{jk} = \begin{cases} a_{jk}\ , & j>k\ , \\ 0\ , & j=k\ , \\ -a_{kj}\ , & j<k\ . \end{cases} \tag{1.19}$$

Obviously, the matrix $\langle A\rangle$ is skew-symmetric:

$$\langle A\rangle^{\tau} = -\langle A\rangle\ , \tag{1.20}$$

and

$$\langle \alpha A + \beta B\rangle = \alpha\langle A\rangle + \beta\langle B\rangle, \qquad \alpha,\beta\in\mathbb{R}\ . \tag{1.21}$$

We proceed with the following elementary

Lemma 1 *If $L=(L_{jk})_{j,k=0}^{\infty}$ is a Jacobi matrix, then for any matrix $R=(R_{jk})_{j,k=0}^{\infty}$,*

$$[L,\langle R\rangle]_{jk} = \begin{cases} [L,R]_{jk}\ , & j>k+1\ , \\ [L,R]_{kj}\ , & j<k-1\ , \end{cases} \tag{1.22}$$

where $[L,B]=LB-BL$ is a commutator. Furthermore, if $LR=RL$, then

$$[L,\langle R\rangle]_{jk} = 0 \qquad \text{for} \qquad |j-k|>1\ . \tag{1.23}$$

P r o o f Since L is a Jacobi matrix, all the elements in the j-row, except for

$$L_{j,j-1}=a_{j-1}, \qquad L_{jj}=b_j, \qquad L_{j,j+1}=a_j\ ,$$

and all the elements in the k-column, except for

$$L_{k-1,k}=a_{k-1}, \qquad L_{kk}=b_k, \qquad L_{k+1,k}=a_k\ ,$$

are zero. Hence

$$\begin{aligned}[L,\langle R\rangle]_{jk} &= \{L\langle R\rangle\}_{jk} - \{\langle R\rangle L\}_{jk} = L_{j,j-1}\langle R\rangle_{j-1,k} + L_{jj}\langle R\rangle_{jk} + \\ &L_{j,j+1}\langle R\rangle_{j+1,k} - \langle R\rangle_{j,k-1}L_{k-1,k} - \langle R\rangle_{jk}L_{kk} - \langle R\rangle_{j,k+1}L_{k+1,k}.\end{aligned}$$

If $j>k+1$, then by definition (1.19) we have

$$\begin{aligned}[L,\langle R\rangle]_{jk} &= L_{j,j-1}R_{j-1,k} + L_{jj}R_{jk} + L_{j,j+1}R_{j+1,k} - R_{j,k-1}L_{k-1,k} - \\ &R_{jk}L_{kk} - R_{j,k+1}L_{k+1,k} = \{LR\}_{jk} - \{RL\}_{jk} = [L,R]_{jk}.\end{aligned}$$

To complete the proof of (1.22), it is enough to observe that

$$\begin{aligned}[L,\langle R\rangle]^\tau &= (L\langle R\rangle)^\tau - (\langle R\rangle L)^\tau = \\ &\langle R\rangle^\tau L^\tau - L^\tau\langle R\rangle^\tau = -\langle R\rangle L + L\langle R\rangle = [L,\langle R\rangle],\end{aligned} \tag{1.24}$$

i.e., $[L,\langle R\rangle]$ is a symmetric matrix. Formula (1.23) for commuting L and R is an immediate consequence of (1.22). □

The following formula will also prove useful:

$$[A,\langle B\rangle]_{00} = 2\{AB\}_{00} - 2A_{00}B_{00}\ , \tag{1.25}$$

provided $A = A^\tau$ and all multiplications make sense. To check (1.25) it is enough to take into account definition (1.19):

$$\begin{aligned}[A,\langle B\rangle]_{00} &= \sum_{j=0}^{\infty}(A_{0j}\langle B\rangle_{j0} - \langle B\rangle_{0j}A_{j0}) = \\ &\sum_{j=1}^{\infty}(A_{0j}B_{j0} + B_{j0}A_{0j}) = 2\sum_{j=0}^{\infty}A_{0j}B_{j0} - 2A_{00}B_{00} = 2\{AB\}_{00} - 2A_{00}B_{00}.\end{aligned}$$

It immediately follows from (1.25) that if $B_{j0} = 0$, $j = 0, 1, \ldots$, then

$$[A,\langle B\rangle]_{00} = 0\ . \tag{1.26}$$

2 Measure transformation

Let $\rho(\cdot,0)$ be a finite measure of the class $\mathfrak{M}$. Our objective is to introduce a one-parametric family of measures $\rho(\cdot,t) \in \mathfrak{M}$, which depends on $\Phi(\lambda,t)$ and $\Psi(\lambda,t)$, the polynomials in λ with continuously differentiable t-dependent coefficients:

$$\Phi(\lambda,t) = \sum_{i=0}^{\ell}\varphi_i(t)\lambda^i, \tag{2.1}$$

$$\Psi(\lambda,t) = \sum_{i=0}^{m}\psi_i(t)\lambda^i, \qquad \lambda\in\mathbb{R}. \tag{2.2}$$

Let μ be any number in the compact set $M = \operatorname{supp} d\rho(\lambda,0)$ and let us consider the Cauchy problem

$$\frac{d\lambda(t)}{dt} = \Phi(\lambda(t),t), \qquad \lambda(0)=\mu. \tag{2.3}$$

It follows from the standard theory of ordinary differential equations that one can chose $T = T(M,\Phi) \in (0,+\infty)$ such that:

(1) For every fixed $\mu \in M$ there exists a solution $\lambda(t) = \lambda(t,\mu)$ of the Cauchy problem (2.3) for all $t \in [0,T]$;

(2) For every fixed $t \in [0,T]$, the function

$$M \ni \mu \mapsto \omega_t(\mu) = \lambda(t,\mu) \tag{2.4}$$

is bounded on M, and moreover, the following uniform estimate holds true:

$$|\omega_t(\mu)| \le C < +\infty, \qquad \forall\, (\mu, t) \in M \times [0, T]. \tag{2.5}$$

For every fixed $t \in [0, T]$ let us define a new finite measure $\tilde{\rho}(\cdot, t)$ by the following rule:

$$\tilde{\rho}(\Delta, t) = \rho(\omega_t^{-1}(\Delta), 0), \qquad \forall \Delta \in \mathfrak{B}(\mathbb{R}), \tag{2.6}$$

where $\omega_t^{-1}(\Delta)$ is the full preimage of the set Δ under the mapping ω_t. It follows from estimate (2.5) that for every $t \in [0, T]$ the measure $\tilde{\rho}(\cdot, t)$ has a compact support. Since the initial measure $\rho(\cdot, 0)$ has infinitely many points of increase, so does the measure $\tilde{\rho}(\cdot, t)$, and therefore, $\tilde{\rho} \in \mathfrak{M}$.

Now consider the solution

$$r(\lambda, t) = \exp\left(\int_0^t \Psi(\lambda, \tau)\, d\tau\right)$$

of the following equation

$$\frac{\partial r(\lambda, t)}{\partial t} = \Psi(\lambda, t) r(\lambda, t), \qquad t \in [0, T], \qquad \lambda \in \mathbb{R}, \tag{2.7}$$

and introduce a measure $\rho(\cdot, t)$ by putting

$$\begin{aligned} \rho(\Delta, t) \;&=\; \int_\Delta r(\lambda, t)\, d\tilde{\rho}(\lambda, t) = \int_\Delta \exp\left(\int_0^t \Psi(\lambda, \tau)\, d\tau\right) d\tilde{\rho}(\lambda, t) = \\ &\int_{\omega_t^{-1}(\Delta)} \exp\left(\int_0^t \Psi(\lambda(t, \mu), \tau)\, d\tau\right) d\rho(\mu, 0) \end{aligned} \tag{2.8}$$

for every $\Delta \in \mathfrak{B}(\mathbb{R})$ and fixed $t \in [0, T]$. One can easily check that $\rho(\cdot, t) \in \mathfrak{M}$, $t \in [0, T]$. Thus the functions Φ and Ψ define through equations (2.2) and (2.7) some measure transformation

$$\mathfrak{M} \ni \rho(\cdot, 0) \mapsto \rho(\cdot, t) \in \mathfrak{M}, \qquad t \in [0, T] \tag{2.9}$$

of the type "mapping + multiplication". This transformation will be sometimes referred to as a (Φ, Ψ)-transform of $\rho(\cdot, 0)$.

Let $F(\lambda, t) \in C^1(\mathbb{R} \times [0, T])$ (T as before) and consider the following integral:

$$f(t) = \int_{\mathbb{R}} F(\lambda, t)\, d\rho(\lambda, t), \qquad t \in [0, T].$$

Since $\rho(\cdot, t)$ is a finite measure with a compact support, the function $f(t)$ is defined for every $t \in [0, T]$. Moreover, it is continuously differentiable on $[0, T]$ and our next objective is to derive a formula for the differentiation of $f(t)$. Let us observe first that in view of (2.6) and (2.8) we have

$$\begin{aligned} f(t) \;&=\; \int_{\mathbb{R}} F(\lambda, t)\, r(\lambda, t)\, d\tilde{\rho}(\lambda, t) = \\ &\int_{\omega_t^{-1}(\mathbb{R})} F(\omega_t(\mu), t)\, r(\omega_t(\mu), t)\, d\rho(\mu, 0) = \\ &\int_{\mathbb{R}} F(\lambda(t, \mu), t)\, r(\lambda(t, \mu), t)\, d\rho(\mu, 0). \end{aligned} \tag{2.10}$$

Using (2.3), we get

$$\begin{aligned}\frac{df}{dt} &= \int_{\mathbb{R}} \Big\{ \Big(\frac{\partial F(\lambda(t,\mu),t)}{\partial \lambda} \Phi(\lambda(t,\mu),t) + \frac{\partial F(\lambda(t,\mu),t}{\partial t} \Big) r(\lambda(t,\mu),t) + \\ &\quad F(\lambda(t,\mu),t) \frac{dr(\lambda(t,\mu),t)}{dt} \Big\} d\rho(\mu,0),\end{aligned}$$

where

$$\frac{dr(\lambda(t,\mu),t)}{dt} = r(\lambda(t,\mu),t) \left(\Phi(\lambda(t,\mu),t) \int_0^t \frac{\partial \Psi(\lambda(t,\mu),\tau)}{\partial \lambda} \, d\tau + \Psi(\lambda(t,\mu),t) \right) \tag{2.11}$$

by (2.3), (2.7) and the formula

$$\begin{aligned}\frac{\partial r(\lambda,t)}{\partial \lambda} &= \frac{\partial}{\partial \lambda} (\exp \left(\int_0^t \Psi(\lambda,\tau) d\tau \right) = \\ &\quad r(\lambda,t) \int_0^t \frac{\partial \Psi(\lambda,\tau)}{\partial} \lambda\tau, \qquad \lambda \in \mathbb{R}.\end{aligned}$$

As a result we obtain

$$\begin{aligned}\frac{df}{dt} &= \int_{\mathbb{R}} \Big\{ \frac{\partial F(\lambda,t)}{\partial \lambda} \Phi(\lambda,t) + \frac{\partial F(\lambda,t)}{\partial t} + \\ &\quad F(\lambda,t) \left(\Psi(\lambda,t) + \Phi(\lambda,t) \int_0^t \frac{\partial \Psi(\lambda,\tau)}{\partial \lambda} d\tau \right) \Big\} d\rho(\lambda,t).\end{aligned}$$

This important formula can be written down as follows:

$$\frac{d}{dt} \int_{\mathbb{R}} F(\lambda,t) \, d\rho(\lambda,t) = \int_{\mathbb{R}} \left(F'(\lambda,t)\Phi(\lambda,t) + \dot{F}(\lambda,t) + F(\lambda,t)\Theta(\lambda,t) \right) d\rho(\lambda,t), \tag{2.12}$$

where

$$\Theta(\lambda,t) = \Psi(\lambda,t) + \Phi(\lambda,t) \int_0^t \frac{\partial \Psi(\lambda,\tau)}{\partial \lambda} \, d\tau = \sum_{i=0}^{\ell+m-1} \theta_i(t)\lambda^i, \qquad \lambda \in \mathbb{R}, \tag{2.13}$$

is a polynomial in λ. Note that its coefficients $\theta_i(t)$, which can be expressed in term of $\varphi_j(t)$, smoothly depend on t.

Note that if the initial measure $\rho(\cdot,0)$ is concentrated on a finite (or countable) set with no limit points, then one should consider instead of (2.3) and (2.7) the following equations [2, 3]:

$$\frac{d\lambda_j(t)}{dt} = \Phi(\lambda_j(t),t), \qquad \frac{d\rho_j(t)}{dt} = \Psi(\lambda_j(t),t)\rho_j(t), \qquad j = 1,2,\ldots,N \le \infty. \tag{2.14}$$

After solving (2.14) one can construct the measure $\rho(\cdot,t)$, which is concentrated at the points $\lambda_j(t)$ by the rule

$$\rho(\Delta,t) = \sum_j \rho_j(t), \qquad \forall\, \Delta \in \mathfrak{B}(\mathbb{R}), \tag{2.15}$$

where the summation is taken over those j for which $\lambda_j(t) \in \Delta$. It is easy to understand that the rule of differentiation (2.12) remains valid for such a measure $\rho(\cdot, t)$ with Ψ in place of Θ.

3 Nonisospectral flows. The Cauchy problem

Assume that the entries of the Jacobi matrix $L \in \mathfrak{L}$ are continuously differentiable functions of t on the interval $[0, T]$:

$$a_j = a_j(t)\ , \quad b_j = b_j(t)\ , \quad L = L(t)\ , \quad j = 0, 1, \dots\ , \quad t \in [0, T]\ . \tag{3.1}$$

The functions $\Phi(\lambda, t)$ and $\Psi(\lambda, t)$, as in (2.1), have the polynomial dependence on λ and therefore,

$$\Phi(L(t), t) = \sum_{i=0}^{\ell} \varphi_i(t)(L(t))^i, \qquad \Theta(L(t), t) = \sum_{i=0}^{\ell+m-1} \theta_i(t)(L(t))^i \tag{3.2}$$

are polynomials of the Jacobi matrix $L(t)$ ($\Theta(\lambda, t)$ is given by (2.13)).

Let us consider the following differential equation:

$$\dot{L}(t) = \Phi(L(t), t) + [L(t), \langle \Phi(L(t), t) D_{L(t)} + \frac{1}{2} \Theta(L(t), t) \rangle], \tag{3.3}$$

where $\cdot = \frac{d}{dt}$, $[\cdot, \cdot]$ denotes the commutator, the matrix operation $\langle \cdot \rangle$ is defined by (1.19) and $D_{L(t)}$ is given by formulas (1.12) and (1.15).

We shall see in Section 5 that equation (3.3) is equivalent to the following differential-difference equations in variables $a_n(t)$, $b_n(t)$, $n = 0, 1, \dots$:

$$\begin{aligned}
\dot{a}_n(t) &= \{\Phi(L(t), t)\}_{n+1,n} + \frac{1}{2} a_n (\{\Theta(L(t), t)\}_{n+1,n+1} - \{\Theta(L(t), t)\}_{nn}) + \\
&\quad a_{n+1} \{\Phi(L(t), t) D_{L(t)}\}_{n+2,n} - a_{n-1} \{\Phi(L(t), t) D_{L(t)}\}_{n+1,n-1} + \\
&\quad (b_{n+1} - b_n) \{\Phi(L(t), t) D_{L(t)}\}_{n+1,n}, \qquad (3.4) \\
\dot{b}_n(t) &= \{\Phi(L(t), t)\}_{nn} + a_n \{2\Phi(L(t), t) D_{L(t)} + \Theta(L(t), t)\}_{n+1,n} - \\
&\quad a_{n-1} \{2\Phi(L(t), t) D_{L(t)} + \Theta(L(t), t)\}_{n,n-1}, \qquad (3.5)
\end{aligned}$$

where $n = 0, 1, \dots;\ a_{-1} = 0$.

Remark 1 Since $D_{L(t)}$ is a strictly upper triangular matrix and the matrices $\Phi(L(t), t)$ and $\Theta(L(t), t)$ have only a finite number of nonzero diagonals, it follows that in (3.3) all the matrix multiplications make sense.

Remark 2 When taking $\Phi(\lambda, t) = 0$ (isospectral deformation) and $\Psi(\lambda, t) = \lambda$, equation (3.3) becomes

$$\dot{L}(t) = [L(t), \frac{1}{2} \langle L(t) \rangle], \tag{3.6}$$

which is the Lax form of the semi-infinite Toda lattice

$$\dot{a}_n(t) = \frac{1}{2} a_n (b_{n+1} - b_n), \quad \dot{b}_n(t) = a_n^2 - a_{n-1}^2, \quad n = 0, 1, \dots\ ; \quad a_{-1} = 0. \tag{3.7}$$

The Cauchy problem for the differential equation (3.3) can be stated as follows. Given $L_0 \in \mathfrak{L}$, i.e., a bounded selfadjoint Jacobi matrix with $a_j > 0$, $b_j \in \mathbb{R}$, $j = 0, 1, \ldots$, find a $\mathfrak{L}$-valued function $L(t)$ with continuously differentiable entries $a_j(t), b_j(t)$ such that: $L(t)$ is a (weak) solution of the equation (3.3) for $t \in [0, T]$ ($T > 0$ depends only on L_0 and $\Phi(\lambda, t)$) and

$$L(0) = L_0 . \tag{3.8}$$

Theorem 1 *A solution of the Cauchy problem (3.3), (3.8) exists, and can be found in the following way. Let $\rho(\cdot) \in \mathfrak{M}$ be the spectral measure of the Jacobi matrix $L_0 \in \mathfrak{L}$. Chose T (as in Section 2) and construct $\rho(\cdot, t) \in \mathfrak{M}$, $t \in [0, T]$, the (Φ, Ψ)-transform of $\rho(\cdot, 0) = \rho(\cdot)$. Obtain the solution $L(t)$, $t \in [0, T]$, by solving the ISP via formulas (1.7) or (1.10).*

It is not always possible to implement Theorem 1, since the construction of the (Φ, Ψ)-transform requires the solution of the nonlinear equation (2.3). Another approach to the Cauchy problem takes advantage of the description of a Jacobi matrix L through its moments $s_k = \{L^k\}_{00}$. It turns out that the nonlinear equation (3.3) can be linearized in terms of the appropriately normalized moments s_k.

Theorem 2 *If $L(t) \in \mathfrak{L}$, $t \in [0, T]$, is a solution of the nonlinear equation (3.3), then its moments*

$$s_k(t) = \{L^k(t)\}_{00}, \qquad k = 0, 1, \ldots, \tag{3.9}$$

after normalization

$$h_k(t) = s_k(t) \exp\left(\int_0^t \{\Theta(L(\tau), \tau)\}_{00}\, d\tau\right) , \tag{3.10}$$

satisfy the following linear system:

$$\begin{aligned}\frac{dh_k(t)}{dt} &= k\varphi_0(t)h_{k-1}(t) + \sum_{i=0}^{\ell-1}(k\varphi_{i+1}(t) + \theta_i(t))h_{k+i}(t) + \\ &\sum_{i=l}^{\ell+m-1} \theta_i(t)h_{k+i}(t), \qquad k = 0, 1, \ldots; \quad h_{-1} = 0,\end{aligned} \tag{3.11}$$

where $\varphi_i(t)$ and $\theta_i(t)$ are the coefficients of the polynomials $\Phi(\lambda, t)$ and $\Theta(\lambda, t)$ respectively.

The converse is also true, as the following theorem shows.

Theorem 3 *For all $t \in [0, T]$ let the sequence $(h_k(t))_{k=0}^\infty$ be a moment sequence of some measure $\rho(\cdot, t) \in \mathfrak{M}$, i.e.,*

$$h_k(t) = \int_{\mathbb{R}} \lambda^k d\rho(\lambda, t) .$$

Assume further that the moments $h_k(t)$ satisfy the linear system (3.11) with continuously differentiable coefficients $\varphi_i(t)$, $(i = 0, 1, \ldots, \ell)$, and $\theta_i(t)$, $(i = 0, 1, \ldots, \ell + m - 1)$. Then the corresponding Jacobi matrix $L(t)$ (which is built via formulas (1.10) with $h_k(t)$ instead of s_k) meets the generalized Lax equation (3.3) on the same interval.

Remark 3 Theorems 2 and 3 suggest another procedure for the construction of solutions to the Cauchy problem (3.3), (3.8) in the class $\mathfrak{L}$. Namely, starting from an initial value $L(0) = L_0 \in \mathfrak{L}$:

(1) Construct initial moments

$$h_k(0) = s_k = \{L_0^k\}_{00} \ , \qquad . \tag{3.12}$$

(2) Find the solution $(h_k(t))_{k=0}^{\infty}$ of the Cauchy problem (3.11), (3.12),

(3) For those t for which $(h_k(t))_{k=0}^{\infty}$ is a moment sequence of some measure $\rho(\cdot, t) \in \mathfrak{M}$, solve the ISP via formulas (1.10), to obtain the sought-for solution $L(t)$.
Note that at least for t from the interval $[0, T]$, where T is chosen as in Section 2, the solution $L(t)$ does exist.

4 Relations for orthogonal polynomials $P_j(\lambda, t)$

Let $\rho(\cdot, t) \in \mathfrak{M}$ be a (Φ, Ψ)-transform of some spectral measure $\rho(\cdot, 0)$, where $\Phi(\lambda, t)$ and $\Psi(\lambda, t)$ are polynomials in λ, as in (2.1) and (2.2). Consider a sequence $(P_j(\lambda, t))_{j=0}^{\infty}$ of the corresponding orthogonal polynomials which is an orthonormal basis in $L_2(\mathbb{R}, d\rho(\lambda, t))$:

$$\int_{\mathbb{R}} P_j(\lambda, t) P_k(\lambda, t) d\rho(\lambda, t) = \delta_{jk}, \qquad j, k = 0, 1, \ldots \tag{4.1}$$

The evolution of these polynomials in time t is complicated. Nevertheless, for all $j, k = 0, 1, \ldots$ the following two important quantities,

$$I_{jk}(t) = \int_{\mathbb{R}} P_j(\lambda, t) \frac{\partial P_k(\lambda, t)}{\partial t} d\rho(\lambda, t) \tag{4.2}$$

and

$$E_{jk}(t) = \int_{\mathbb{R}} \lambda \frac{\partial P_j(\lambda, t)}{\partial t} P_k(\lambda, t) d\rho(\lambda, t) \tag{4.3}$$

can be expressed in terms of the Jacobi matrix $L(t)$ corresponding to the measure $\rho(\cdot, t)$.

Lemma 2 *If $I_{jk}(t)$ is given by (4.2), then the following equalities hold:*

(a) $\quad I_{jk}(t) = 0 \ , \quad j > k \ ,$ $\hfill (4.4)$

(b) $\quad I_{kk}(t) = -\{\Phi D_{L(t)}\}_{kk} - \dfrac{1}{2}\Theta_{kk} \ ,$ $\hfill (4.5)$

(c) $\quad I_{jk}(t) = -(\{\Phi D_{L(t)}\}_{jk} + \{\Phi D_{L(t)}\}_{kj} + \Theta_{jk}) \ , \quad j < k \ ,$ $\hfill (4.6)$

where $\quad \Phi = \Phi(L(t), t) \ , \quad \Theta_{jk} = \{\Theta(L(t), t)\}_{jk}.$

P r o o f By differentiation of the polynomial $P_k(\lambda, t)$ with respect to t, we obtain the polynomial in λ of the same degree k. Hence the following expansion according to the basis $(P_j(\lambda, t))_{j=0}^{\infty}$ holds for some coefficients $\beta_i(t)$, $i = 0, \ldots, k$,

$$\frac{\partial P_k(\lambda, t)}{\partial t} = \sum_{i=0}^{k} \beta_i(t) P_i(\lambda, t) \ .$$

Now it follows immediately from (4.1) that

$$I_{jk}(t) = 0,$$

when $j > k$. To prove relations (4.5) and (4.6), let us differentiate identities (4.1) with respect to t, using formula (2.12):

$$\begin{aligned} 0 &= \frac{d}{dt}\int_{\mathbb{R}} P_j(\lambda,t)P_k(\lambda,t)d\rho(\lambda,t) = \int_{\mathbb{R}} \frac{\partial}{\partial\lambda}(P_j(\lambda,t)P_k(\lambda,t))\Phi(\lambda,t)d\rho(\lambda,t) + \\ & \int_{\mathbb{R}} \frac{\partial}{\partial t}(P_j(\lambda,t)P_k(\lambda,t))\, d\rho(\lambda,t) + \int_{\mathbb{R}} P_j(\lambda,t)P_k(\lambda,t)\Theta(\lambda,t)\, d\rho(\lambda,t)\ . \qquad (4.7)\end{aligned}$$

By (1.16), (1.18), and by definition (4.2) of $I_{jk}(t)$, it follows from (4.7) that

$$\{\Phi D_{L(t)}\}_{jk} + \{\Phi D_{L(t)}\}_{kj} + I_{jk}(t) + I_{kj}(t) + \Theta_{jk} = 0\ . \qquad (4.8)$$

If now $j = k$, then

$$2I_{kk}(t) + 2\{\Phi D_{L(t)}\}_{kk} + \Theta_{kk} = 0\ ,$$

which implies (4.5). If $j < k$, then $I_{kj}(t) = 0$, and (4.6) follows easily from (4.4). □

In order to express $E_{jk}(t)$ in terms of $L(t)$, it is convenient to denote

$$\Omega = \Phi(L(t),t)D_{L(t)}, \qquad \Theta = \Theta(L(t),t)\ . \qquad (4.9)$$

Lemma 3 *If $E_{jk}(t)$ are given by (4.3), then one has:*

(a) $E_{jk}(t) = 0, \qquad j < k-1\ ,$

(b) $E_{k-1,k}(t) = -a_{k-1}\{\frac{1}{2}\Theta+\Omega\}_{k-1,k-1}\ ,$

(c) $E_{kk}(t) = -a_{k-1}\{\Omega+\Omega^{\tau}+\Theta\}_{k-1,k} - b_k\{\Omega+\frac{1}{2}\Theta\}_{kk}\ ,$

(d) $E_{k+1,k}(t) = -\{\Omega L+(L\Omega)^{\tau}+\Theta L\}_{k+1,k} + a_k\{\Omega+\frac{1}{2}\Theta\}_{k+1,k+1}\ ,$

(e) $E_{jk}(t) = -\{\Omega L+(L\Omega)^{\tau}+\Theta L\}_{jk}, \qquad j > k+1\ ,$

where a_k, b_k are elements of $L = L(t)$, and Ω and Θ are given by (4.9).

P r o o f Recall that the orthogonal polynomials $(P_k(\lambda,t))_{k=0}^{\infty}$ satisfy the three-term recursion

$$\lambda P_k(\lambda,t) = a_{k-1}(t)P_{k-1}(\lambda,t) + b_k(t)P_k(\lambda,t) + a_k(t)P_{k+1}(\lambda,t), \qquad \lambda\in\mathbb{R}, \quad t\in[0,T],$$

where $a_k(t) = \{L(t)\}_{k,k+1} = \{L(t)\}_{k+1,k}$, $b_k(t) = \{L(t)\}_{kk}$, $k = 0,1,\ldots$ This allows us to rewrite $E_{jk}(t)$ in the following form:

$$\begin{aligned} E_{jk}(t) &= \int_{\mathbb{R}} \frac{\partial P_j(\lambda,t)}{\partial t}(\lambda P_k(\lambda,t))d\rho(\lambda,t) = \\ & a_{k-1}(t)I_{k-1,j}(t) + b_k(t)I_{kj}(t) + a_k(t)I_{k+1,j}(t). \end{aligned}$$

In view of (4.9) we have: $I_{jk}(t) = 0$, if $j > k$, while $I_{kk}(t) = -\{\Omega + \frac{1}{2}\Theta\}_{kk}$, and $I_{jk}(t) = -\{\Omega + \Omega^\tau + \Theta\}_{jk}$, if $j < k$. To complete the proof, it is enough to take into account formulas (1.16) and the three-diagonal structure of the Jacobi matrix $L(t)$. □

5 Proofs

Now we are ready to prove the results formulated in Section 3.

Proof of Theorem 1 Let $\rho(\cdot) \in \mathfrak{M}$ be a spectral measure of the initial value $L_0 \in \mathfrak{L}$, and let $\rho(\cdot, t) \in \mathfrak{M}$, $t \in [0, T]$, be the (Φ, Ψ) transform of $\rho(\cdot)$. Thus, we have

$$\rho(\cdot, 0) = \rho(\cdot) \ . \tag{5.1}$$

Let $(P_j(\lambda, t))_{j=0}^{\infty}$ be the orthogonal polynomials in $L_2(\mathbb{R}, d\rho(\lambda, t))$. Then the entries $L_{jk}(t)$ of the Jacobi matrix $L(t)$ corresponding to $\rho(\cdot, t)$ can be expressed as follows:

$$L_{jk}(t) = \int_{\mathbb{R}} \lambda P_j(\lambda, t) P_k(\lambda, t) d\rho(\lambda, t), \qquad j, k = 0, 1, \ldots \tag{5.2}$$

It follows from (5.1), (5.2) that $L(0) = L_0$. Hence it remains to show that $L(t)$ satisfies the generalized Lax equation (3.3)

$$\dot{L}(t) = \Phi(L(t), t) + [L(t), \langle \Phi(L(t), t) D_{L(t)} + \frac{1}{2}\Theta(L(t), t)\rangle] \ , \quad t \in [0, T] \ . \tag{5.3}$$

Using formula (2.12), let us write

$$\begin{aligned} \dot{L}_{jk}(t) &= \frac{d}{dt}\int_{\mathbb{R}} \lambda P_j(\lambda, t) P_k(\lambda, t) d\rho(\lambda, t) = \int_{\mathbb{R}} \frac{\partial}{\partial \lambda}(\lambda P_j(\lambda, t) P_k(\lambda, t)) \Phi(\lambda, t) d\rho(\lambda, t) + \\ & \int_{\mathbb{R}} \lambda \frac{\partial P_j(\lambda, t)}{\partial t} P_k(\lambda, t) d\rho(\lambda, t) + \int_{\mathbb{R}} \lambda P_j(\lambda, t) \frac{\partial P_k(\lambda, t)}{\partial t} d\rho(\lambda, t) + \\ & \int_{\mathbb{R}} \lambda P_j(\lambda, t) P_k(\lambda, t) \Theta(\lambda, t) d\rho(\lambda, t) = A_{jk}(t) + E_{jk}(t) + E_{kj}(t) + B_{jk}(t), \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} A_{jk}(t) &= \int_{\mathbb{R}} \frac{\partial}{\partial \lambda}(\lambda P_j(\lambda, t) P_k(\lambda, t)) \Phi(\lambda, t) d\rho(\lambda, t), \\ B_{jk}(t) &= \int_{\mathbb{R}} \lambda P_j(\lambda, t) P_k(\lambda, t) \Theta(\lambda, t) d\rho(\lambda, t) \end{aligned}$$

and $E_{jk}(t)$, $E_{kj}(t)$ are given by Lemma 3.

Taking into account formulas (1.16), (1.18) one can find that

$$B_{jk}(t) = \{L(t)\Theta(L(t), t)\}_{jk}, \qquad j, k = 0, 1, \ldots, \tag{5.5}$$

$$\begin{aligned} A_{jk}(t) &= \int_{\mathbb{R}} P_j(\lambda, t) P_k(\lambda, t) \Phi(\lambda, t) d\rho(\lambda, t) + \int_{\mathbb{R}} \lambda \frac{\partial P_j(\lambda, t)}{\partial \lambda} P_k(\lambda, t) \Phi(\lambda, t) d\rho(\lambda, t) + \\ & \int_{\mathbb{R}} \lambda P_j(\lambda, t) \frac{\partial P_k(\lambda, t)}{\partial \lambda} \Phi(\lambda, t) d\rho(\lambda, t) = \\ & \{\Phi(L(t), t)\}_{jk} + \{L(t)\Omega\}_{kj} + \{L(t)\Omega\}_{jk}, \qquad j, k = 0, 1, \ldots, \end{aligned} \tag{5.6}$$

where $\Omega = \Phi(L(t), t)D_{L(t)}$ as in (4.9).

We proceed in steps.

Step 1 The nontrivial entries

$$a_n(t) = \{L(t)\}_{n+1,n} = L_{n+1,n}(t), \qquad b_n(t) = \{L(t)\}_{nn} = L_{nn}(t)$$

of the Jacobi matrix $L(t)$ satisfy equations (3.4), (3.5):

$$\begin{aligned}
\dot{a}_n &= \Phi_{n+1,n} + \frac{1}{2}a_n(\Theta_{n+1,n+1} - \Theta_{nn}) + a_{n+1}\{\Phi D_{L(t)}\}_{n+2,n} - \\
&\quad a_{n-1}\{\Phi D_{L(t)}\}_{n+1,n-1} + (b_{n+1} - b_n)\{\Phi D_{L(t)}\}_{n+1,n}, &(5.7)\\
\dot{b}_n &= \Phi_{nn} + 2a_n\{\Phi D_{L(t)}\}_{n+1,n} - 2a_{n-1}\{\Phi D_{L(t)}\}_{n,n-1} + \\
&\quad a_n\Theta_{n+1,n} - a_{n-1}\Theta_{n,n-1}\ , &(5.8)
\end{aligned}$$

where $\Phi = \Phi(L(t), t)$, $\quad \Theta = \Theta(L(t), t), \quad n = 0, 1, \ldots;\ a_{-1} = 0.$

P r o o f of Step 1 By (5.4), (5.5), (5.6), and Lemma 3 (c) it is easy to verify that

$$\begin{aligned}
\dot{b}_n &= \dot{L}_{nn}(t) = A_{nn}(t) + 2E_{nn}(t) + B_{nn}(t) = (\Phi_{nn} + 2\{L\Omega\}_{nn}) + \\
&\quad 2(-a_{n-1}\{\Omega + \Omega^\tau + \Theta\}_{n-1,n} - b_n\{\Omega + \frac{1}{2}\Theta\}_{nn}) + \{L\Theta\}_{nn} = \\
&\quad \Phi_{nn} + 2(a_{n-1}\Omega_{n-1,n} + b_n\Omega_{nn} + a_n\Omega_{n+1,n}) - 2(a_{n-1}\Omega_{n-1,n} + b_n\Omega_{nn}) - \\
&\quad 2a_{n-1}\Omega_{n,n-1} - 2a_{n-1}\Theta_{n-1,n} - b_n\Theta_{nn} + a_{n-1}\Theta_{n-1,n} + b_n\Theta_{nn} + a_n\Theta_{n+1,n} = \\
&\quad \Phi_{nn} + 2a_n\Omega_{n+1,n} - 2a_{n-1}\Omega_{n,n-1} + a_n\Theta_{n+1,n} - a_{n-1}\Theta_{n,n-1}, &(5.9)
\end{aligned}$$

which proves (5.8); Ω is given by (4.9).

Similar arguments work for (5.7) where now Lemma 3 (b, d) is invoked:

$$\begin{aligned}
\dot{a}_n &= \dot{L}_{n+1,n}(t) = A_{n+1,n}(t) + E_{n,n+1}(t) + E_{n+1,n}(t) + B_{n+1,n}(t) = \\
&\quad (\Phi_{n+1,n} + \{L\Omega\}_{n,n+1} + \{L\Omega\}_{n+1,n}) + (-a_n\{\Omega + \frac{1}{2}\Theta\}_{nn}) + &(5.10)\\
&\quad (-\{\Omega L + (L\Omega)^\tau + \Theta L\}_{n+1,n} + a_n\{\Omega + \frac{1}{2}\Theta\}_{n+1,n+1}) + \{L\Theta\}_{n+1,n}.
\end{aligned}$$

Taking into account that $\Theta L = L\Theta$ and $\{L\Omega\}_{n,n+1} = \{(L\Omega)^\tau\}_{n+1,n}$, we continue (5.10):

$$\begin{aligned}
\dot{a}_n &= \Phi_{n+1,n} + \frac{1}{2}a_n(\Theta_{n+1,n+1} - \Theta_{nn}) + \{L\Omega\}_{n+1,n} - a_n\Omega_{nn} - \\
&\quad \{\Omega L\}_{n+1,n} + a_n\Omega_{n+1,n+1} = \\
&\quad \Phi_{n+1,n} + \frac{1}{2}a_n(\Theta_{n+1,n+1} - \Theta_{nn}) + (a_n\Omega_{nn} + b_{n+1}\Omega_{n+1,n} + a_{n+1}\Omega_{n+2,n}) - \\
&\quad a_n\Omega_{nn} - (\Omega_{n+1,n-1}a_{n-1} + \Omega_{n+1,n}b_n + \Omega_{n+1,n+1}a_n) + a_n\Omega_{n+1,n+1} = \\
&\quad \Phi_{n+1,n} + \frac{1}{2}a_n(\Theta_{n+1,n+1} - \Theta_{nn}) + a_{n+1}\Omega_{n+2,n} - \\
&\quad a_{n-1}\Omega_{n+1,n-1} + (b_{n+1} - b_n)\Omega_{n+1,n}, &(5.11)
\end{aligned}$$

which completes the proof of Step 1.

Step 2 Equations (5.7), (5.8) can be rewritten as

$$\dot{L}_{jk} = \{\Phi(L(t),t) + [L(t), \langle \Phi(L(t),t)D_{L(t)} + \frac{1}{2}\Theta(L(t),t)\rangle]\}_{jk},$$
$$k = 0,1,\dots\ , \quad j = k, k+1. \tag{5.12}$$

P r o o f of Step 2 Let us denote for a moment

$$R = \Omega + \frac{1}{2}\Theta = \Phi(L(t),t)D_{L(t)} + \frac{1}{2}\Theta(L(t),t)\ , \tag{5.13}$$

and rewrite (5.8) in the following form:

$$\begin{aligned}\dot{L}_{nn}(t) \quad &= \quad \dot{b}_n = \Phi_{nn} + (a_{n-1}(-R_{n,n-1}) + b_n \cdot 0 + a_n R_{n+1,n}) - (R_{n,n-1}a_{n-1} + 0 \cdot b_n + \\ &(-R_{n+1,n})a_n) = \Phi_{nn} + (a_{n-1}\langle R\rangle_{n-1,n} + b_n\langle R\rangle_{nn} + a_n\langle R\rangle_{n+1,n}) - \\ &(\langle R\rangle_{n,n-1}a_{n-1} + \langle R\rangle_{nn}b_n + \langle R\rangle_{n,n+1}a_n) = \{\Phi + [L, \langle R\rangle]\}_{nn},\end{aligned}$$

where $\langle\cdot\rangle$ is defined by (1.19), $\Phi = \Phi(L(t),t)$. To rewrite (5.7) as required, it is convenient to use its equivalent form (5.11):

$$\begin{aligned}\dot{L}_{n+1,n}(t) \quad &= \quad \dot{a}_n(t) = \Phi_{n+1,n} + \frac{1}{2}a_n(\Theta_{n+1,n+1} - \Theta_{nn}) + \\ &\{L\Omega\}_{n+1,n} - a_n\Omega_{nn} - \{\Omega L\}_{n+1,n} + a_n\Omega_{n+1,n+1} = \\ &\Phi_{n+1,n} + a_n(R_{n+1,n+1} - R_{nn}) + \{LR\}_{n+1,n} - \{RL\}_{n+1,n},\end{aligned} \tag{5.14}$$

where it is worth to remember that R is defined by (5.13) and $\Theta L = L\Theta$.

Since

$$\{LR\}_{n+1,n} = a_n R_{nn} + b_{n+1}R_{n+1,n} + a_{n+1}R_{n+2,n}$$

and

$$\{RL\}_{n+1,n} = R_{n+1,n-1}a_{n-1} + R_{n+1,n}b_n + R_{n+1,n+1}a_n,$$

it follows from (5.14) that

$$\begin{aligned}\dot{L}_{n+1,n}(t) \quad &= \quad \Phi_{n+1,n} + b_{n+1}R_{n+1,n} + a_{n+1}R_{n+2,n} - R_{n+1,n-1}a_{n-1} - R_{n+1,n}b_n = \\ &\Phi_{n+1,n} + (a_n\langle R\rangle_{nn} + b_{n+1}\langle R\rangle_{n+1,n} + a_{n+1}\langle R\rangle_{n+2,n}) - \\ &(\langle R\rangle_{n+1,n-1}a_{n-1} + \langle R\rangle_{n+1,n}b_n + \langle R\rangle_{n+1,n+1}a_n) = \\ &\Phi_{n+1,n} + \{L\langle R\rangle\}_{n+1,n} - \{\langle R\rangle L\}_{n+1,n} = \{\Phi + [L, \langle R\rangle]\}_{n+1,n},\end{aligned} \tag{5.15}$$

as needed.

Step 3 Equation (5.12) holds for $j > k+1$, i.e.,

$$\{\Phi + [L, \langle R\rangle]\}_{jk} = 0, \qquad j > k+1. \tag{5.16}$$

P r o o f of Step 3 It follows from (5.4), (5.5), (5.6) and Lemma 3(a,e) that for $j > k+1$,

$$\begin{aligned}0 \quad &= \quad \dot{L}_{jk}(t) = A_{jk}(t) + E_{jk}(t) + E_{kj}(t) + B_{jk}(t) = \\ &(\Phi_{jk} + \{L\Omega\}_{kj} + \{L\Omega\}_{jk}) + (-\{\Omega L + (L\Omega)^\tau + \\ &\Theta L\}_{jk}) + 0 + \{L\Theta\}_{jk}.\end{aligned} \tag{5.17}$$

Using again the fact that $\{L\Omega\}_{kj} = \{(L\Omega)^\tau\}_{jk}$ and $L\Theta = \Theta L$, we have

$$\begin{aligned} 0 &= \Phi_{jk} + \{L\Omega\}_{jk} - \{\Omega L\}_{jk} = \Phi_{jk} + [L,\Omega]_{jk} = \\ &\Phi_{jk} + [L,\Omega]_{jk} + [L,\frac{1}{2}\Theta]_{jk} = \\ &\Phi_{jk} + [L,R]_{jk} = \{\Phi + [L,\langle R\rangle]\}_{jk}, \end{aligned}$$

where the last equality is justified by Lemma 1.

Step 4 is to complete the proof of the theorem. It remains to show that (5.12) holds for $j < k$. Both sides of equation (5.3) are symmetric, since $\dot{L}(t)$ and $\Phi(L(t),t)$ are symmetric in view of the symmetricity of $L(t)$, and $[L,\langle R\rangle]$ is symmetric by (1.24). Therefore, the equality (5.12) holds for $j < k$ since in view of Steps 2 and 3 it is valid for $j > k$. □

To prove Theorem 2, we need the following

Lemma 4 *Let a Jacobi matrix $L(t) \in \mathfrak{L}$ is such that*

$$\dot{L}(t) = \sum_{i=0}^{\ell} \varphi_i(t) L^i(t) + [L(t), S(t)], \qquad t \in [0,T] \tag{5.18}$$

for some matrix-valued function $S(t)$ and scalar-valued $\varphi_i(t)$, $i = 0,\dots,\ell$. Then its powers $L^k(t)$ meet the following equation:

$$\frac{d}{dt}(L^k(t)) = k\sum_{i=0}^{\ell} \varphi_i(t) L^{i+k-1}(t) + [L^k(t), S(t)], \qquad t \in [0,T]\ , \quad k = 1,2,\dots \tag{5.19}$$

P r o o f Let us check (5.19) for $k=2$. Denoting $\Phi(t) = \sum_{i=0}^{\ell} \varphi_i(t) L^i(t)$, we can write

$$\begin{aligned} \frac{d}{dt}(L^2(t)) &= \left(\frac{d}{dt}L(t)\right) L(t) + L(t)\frac{d}{dt}L(t) = (\Phi(t) + L(t)S(t) - S(t)L(t))L(t) + \\ &L(t)(\Phi(t) + L(t)S(t) - S(t)L(t)) = \Phi(t)L(t) + L(t)\Phi(t) + \\ &L^2(t)S(t) - S(t)L^2(t) = 2\sum_{i=0}^{\ell} \varphi_i(t) L^{i+1}(t) + [L^2(t), S(t)]. \end{aligned}$$

Now the standard induction arguments serve to complete the proof. □

P r o o f of Theorem 2 If $L(t)$ satisfies the generalized Lax equation (3.3), then its moments $s_k(t) = \{L^k(t)\}_{00}$ satisfy, in view of Lemma 4, the following system, $k = 0,1,\dots$:

$$\dot{s}_k(t) = \{\frac{d}{dt}L^k(t)\}_{00} = k\sum_{i=0}^{\ell} \varphi_i(t) s_{i+k-1}(t) + [L^k(t), S(t)]_{00}\ , \tag{5.20}$$

where now

$$S(t) = \langle \Phi(L(t),t) D_{L(t)} + \frac{1}{2}\Theta(L(t),t)\rangle,$$

$$\begin{aligned}\Phi(L(t),t) &= \sum_{i=0}^{\ell}\varphi_i(t)L^i(t),\\ \Theta(L(t),t) &= \sum_{j=0}^{\ell+m-1}\theta_j(t)L^j(t);\ s_{-1}=0.\end{aligned}$$

Using (1.21) let us rewrite

$$[L^k(t),S(t)]_{00} = [L^k(t),\langle\Phi(L(t),t)D_{L(t)}\rangle]_{00} + [L^k(t),\frac{1}{2}\langle\Theta(L(t),t)\rangle]_{00}\ . \tag{5.21}$$

The first term on the right-hand side of (5.21) is equal to zero by formula (1.26), while the second term can be computed via formula (1.25):

$$\begin{aligned}[L^k(t),\frac{1}{2}\langle\Theta(L(t),t)\rangle]_{00} &= \{L^k(t)\Theta(L(t),t)\}_{00} - \{L^k(t)\}_{00}\cdot\{\Theta(L(t),t)\}_{00} =\\ &\left\{\sum_{j=0}^{\ell+m-1}\theta_j(t)L^{j+k}(t)\right\}_{00} - \{L^k(t)\}_{00}\Theta_{00}(t) =\\ &\sum_{j=0}^{\ell+m-1}\theta_j(t)s_{j+k}(t) - s_k(t)\Theta_{00}(t),\end{aligned}$$

where $\Theta_{00}(t)$ is a short notation for $\{\Theta(L(t),t)\}_{00}$. Thus (5.20) can be rewritten as follows:

$$\dot{s}_k(t) = k\sum_{i=0}^{\ell}\varphi_i(t)s_{i+k-1}(t) + \sum_{j=0}^{\ell+m-1}\theta_j(t)s_{j+k}(t) - s_k(t)\Theta_{00}(t)\ . \tag{5.22}$$

Multiplying both sides of (5.22) by

$$N(t) = \exp\left(\int_0^t\Theta_{00}(\tau)d\tau\right)\ , \tag{5.23}$$

and denoting $h_k(t) = N(t)s_k(t)$, one can easily obtain

$$\dot{h}_k(t) = k\sum_{i=0}^{\ell}\varphi_i(t)h_{i+k-1}(t) + \sum_{j=0}^{\ell+m-1}\theta_j(t)h_{j+k}(t), \qquad k=0,1,\ldots;\ h_{-1}=0\ , \tag{5.24}$$

which is the same as (3.11). □

P r o o f of Theorem 3 Let us consider $(h_k(t))_{k=0}^{\infty}$ which is a moment sequence for an arbitrary measure $\rho(\cdot,t)\in\mathfrak{M}$, and which satisfies (3.11) or the equivalent system (5.24). Let us take any polynomial $F(\lambda,t)=\sum\limits_{k=0}^{s}f_k(t)\lambda^k$, with continuously differentiable coefficients $f_k(t)$, and prove that the basic differentiation formula (2.12) still holds. Indeed,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}F(\lambda,t)d\rho(\lambda,t) &= \frac{d}{dt}\int_{\mathbb{R}}\left(\sum_{k=0}^{s}f_k(t)\lambda^k\right)d\rho(\lambda,t) =\\ &\frac{d}{dt}\left(\sum_{k=0}^{s}f_k(t)h_k(t)\right) = \sum_{k=0}^{s}\frac{df_k(t)}{dt}h_k(t) + \sum_{k=0}^{s}f_k(t)\dot{h}_k(t)\end{aligned}$$

and, after substituting $\dot{h}_k(t)$ from (5.24), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} F(\lambda,t)d\rho(\lambda,t) = \int_{\mathbb{R}} \frac{\partial F(\lambda,t)}{\partial t} d\rho(\lambda,t) +$$
$$\sum_{k=0}^{s} f_k(t)\left(k\sum_{i=0}^{\ell}\varphi_i(t)h_{i+k-1}(t) + \sum_{j=0}^{\ell+m-1}\theta_j(t)h_{j+k}(t)\right) =$$
$$\int_{\mathbb{R}} \frac{\partial F(\lambda,t)}{\partial t} d\rho(\lambda,t) + \sum_{k=0}^{s}\sum_{i=0}^{\ell}\int_{\mathbb{R}}(f_k(t)\cdot k\lambda^{k-1})(\varphi_i(t)\lambda^i)d\rho(\lambda,t) +$$
$$\sum_{k=0}^{s}\sum_{j=0}^{\ell+m-1}\int_{\mathbb{R}}(f_k(t)\lambda^k)(\theta_j(t)\lambda^j)d\rho(\lambda,t) =$$
$$\int_{\mathbb{R}} \frac{\partial F(\lambda,t)}{\partial t} d\rho(\lambda,t) + \int_{\mathbb{R}} \frac{\partial F(\lambda,t)}{\partial \lambda}\Phi(\lambda,t) d\rho(\lambda,t) + \int_{\mathbb{R}} F(\lambda,t)\Theta(\lambda,t)d\rho(\lambda,t),$$

as needed. Since the ISP has a unique solution, formulas (1.7) and (1.10) lead to the same Jacobi matrix $L(t) \in \mathfrak{L}$. Thus, by repeating the arguments of the proof of Theorem 1, we can conclude that $L(t)$ meets (3.3). □

6 Uniqueness of solutions of the Cauchy problem

Let us return to the Cauchy problem (3.3), (3.8). We consider its solutions in the following class: the entries of the Jacobi matrix $L(t) \in \mathfrak{L}$ are continuously differentiable functions with respect to t, and the sequences $a_n(t) > 0$, $b_n(t) \in \mathbb{R}$ $(n = 0, 1, \ldots)$ are uniformly bounded on $[0, T]$, i.e., there exists a positive number Q such that

$$\|L(t)\| \le Q < +\infty, \quad t \in [0,T] . \tag{6.1}$$

Theorem 4 *Let the coefficients $\varphi_i(t), \psi_i(t)$ of polynomials $\Phi(\lambda,t), \Psi(\lambda,t)$ be (real) analytic functions, $t \in [0,T]$. Then any two solutions $L_1(t), L_2(t)$ of the Cauchy problem (3.3), (3.8) from the class defined just above, for which $L_1(0) = L_2(0)$, are identically equal.*

Remark 4 If the coefficients $\varphi_i(t), \psi_i(t)$ are analytic or smooth (i.e., belong to C^∞) then every solution of (3.3) from our class is automatically smooth. This is easily seen by inspecting the right hand side of the equivalent system (3.4), (3.5).

Before proving Theorem 4 we need several auxiliary results.

Let Jacobi matrix $L(t)$ be any solution of the generalized Lax equation (3.3):

$$\dot{L}(t) = \Phi(L(t),t) + [L(t), \langle\Phi(L(t),t)D_{L(t)} + \frac{1}{2}\Theta(L(t),t)\rangle], \qquad t \in [0,T] , \tag{6.2}$$

and let $d\tilde{\rho}(\lambda,t)$ be its unique spectral measure such that $\tilde{\rho}(\mathbb{R},\ t) = 1$, $t \in [0,T]$. It is convenient to normalize the spectral measure of $L(t)$ (which is defined for every $t \in [0,T]$ up to a scalar factor) as follows:

$$\begin{aligned} d\rho(\lambda,t) &= q(t)d\tilde{\rho}(\lambda,t), \qquad\qquad \rho(\mathbb{R},\ t) = q(t), \\ q(t) &= \exp\left(\int_0^t\int_{\mathbb{R}}\Theta(\lambda,\tau)d\tilde{\rho}(\lambda,\tau)\ d\tau\right), \quad t \in [0,T]. \end{aligned} \tag{6.3}$$

Thus, without lost of generality we can assume that the spectral measure $d\rho(\lambda, t)$ of $L(t)$ is such that

$$\dot{\rho}(\mathbb{R},\ t) = \int_{\mathbb{R}} \Theta(\lambda, t) d\rho(\lambda, t), \qquad t \in [0, T]. \tag{6.4}$$

In view of (6.1) there exists a finite interval $[a, b] \in \mathbb{R}$ which contains for every $t \in [0, T]$ the support of $d\rho(\lambda, t)$:

$$\text{supp } d\rho(\lambda, t) \subset [a, b], \tag{6.5}$$

(in fact, we can take $a = -Q, \quad b = Q$).

Lemma 5 *For every complex number $z \in \mathbb{C}\backslash[a, b]$, the following formula holds*

$$\frac{d}{dt} \int_{\mathbb{R}} \frac{1}{\lambda - z} d\rho(\lambda, t) = \int_{\mathbb{R}} \Big\{ -\frac{\Phi(\lambda, t)}{(\lambda - z)^2} + \frac{\Theta(\lambda, t)}{\lambda - z} \Big\} d\rho(\lambda, t), \qquad t \in [0, T]. \tag{6.6}$$

P r o o f Fix $z \in \mathbb{C}\backslash[a, b]$ and let $R_z(t) = (L(t) - z\mathbf{1})^{-1}$ be a resolvent of the bounded operator $L(t)$ in the space ℓ_2. The operator-valued function $R_z(t)$ is continuously differentiable with respect to t on $[0, T]$ in a strong sense. Therefore, formulas (6.2) and (1.21) imply that

$$\begin{aligned}
\dot{R}_z(t) \quad &= \quad -R_z(t)(L(t) - z\mathbf{1})^{\cdot} R_z(t) = -R_z(t)\dot{L}(t)R_z(t) = \\
&\quad -R_z(t)\{\Phi(L(t), t) + [L(t), \langle \Phi(L(t), t) D_{L(t)} + \frac{1}{2}\Theta(L(t), t)\rangle]\}R_z(t) = \\
&\quad -R_z^2(t)\Phi(L(t), t) - R_z(t)[L(t) - z\mathbf{1}, \langle \Phi(L(t), t) D_{L(t)} + \frac{1}{2}\Theta(L(t), t)\rangle]R_z(t) = \\
&\quad -R_z^2(t)\Phi(L(t), t) + [R_z(t), \langle \Phi(L(t), t) D_{L(t)} + \frac{1}{2}\Theta(L(t), t)\rangle] = \\
&\quad -R_z^2(t)\Phi(L(t), t) + [R_z(t), \langle \Phi(L(t), t) D_{L(t)}\rangle] + [R_z(t), \langle \frac{1}{2}\Theta(L(t), t)\rangle].
\end{aligned} \tag{6.7}$$

In particular, for the upper left hand side entry of corresponding matrix we get:

$$\begin{aligned}
\{\dot{R}_z(t)\}_{00} \quad &= \quad -\{R_z^2(t)\Phi(L(t), t)\}_{00} + [R_z(t), \langle \Phi(L(t), t) D_{L(t)}\rangle]_{00} + \\
&\quad [R_z(t), \langle \frac{1}{2}\Theta(L(t), t)\rangle]_{00} = -\{R_z^2(t)\Phi(L(t), t)\}_{00} + \\
&\quad 2\{R_z(t)\Phi(L(t), t) D_{L(t)}\}_{00} - 2\{R_z(t)\}_{00}\{\Phi(L(t), t) D_{L(t)}\}_{00} + \\
&\quad \{R_z(t)\Theta(L(t), t)\}_{00} - \{R_z(t)\}_{00}\{\Theta(L(t), t\}_{00},
\end{aligned}$$

where formula (1.25) is invoked. Since $D_{L(t)}$ is a strictly upper triangular matrix, the entry $\{C D_{L(t)}\}_{00}$ vanishes for an arbitrary bounded operator C . Hence we can rewrite the last equation as follows:

$$\begin{aligned}
\{\dot{R}_z(t)\}_{00} \quad &= \quad -\{R_z^2(t)\Phi(L(t), t)\}_{00} + \{R_z(t)\Theta(L(t), t)\}_{00} - \\
&\quad \{R_z(t)\}_{00}\{\Theta(L(t), t)\}_{00}, \qquad t \in [0, T], \quad z \in \mathbb{C}\backslash[a, b].
\end{aligned} \tag{6.8}$$

Using the general formula (1.17) and taking into account (6.4), we obtain

$$\frac{d}{dt}\Big(\frac{1}{\rho(\mathbb{R}, t)} \int_{\mathbb{R}} \frac{1}{\lambda - z} d\rho(\lambda, t)\Big) = \frac{d}{dt}\{R_z(t)\}_{00} = \{\dot{R}_z(t)\}_{00} =$$

$$-\frac{1}{\rho(\mathbb{R},t)}\int_{\mathbb{R}}\frac{\Phi(\lambda,t)}{(\lambda-z)^2}d\rho(\lambda,t)+\frac{1}{\rho(\mathbb{R},t)}\int_{\mathbb{R}}\frac{\Theta(\lambda,t)}{\lambda-z}d\rho(\lambda,t)-$$
$$\frac{1}{\rho^2(\mathbb{R},t)}\int_{\mathbb{R}}\frac{1}{\lambda-z}d\rho(\lambda,t)\cdot\int_{\mathbb{R}}\Theta(\lambda,t)d\rho(\lambda,t)=$$
$$\frac{1}{\rho(\mathbb{R},t)}\int_{\mathbb{R}}\Big\{-\frac{\Phi(\lambda,t)}{(\lambda-z)^2}+\frac{\Theta(\lambda,t)}{\lambda-z}\Big\}d\rho(\lambda,t)-$$
$$\frac{\dot\rho(\mathbb{R},t)}{\rho^2(\mathbb{R},t)}\int_{\mathbb{R}}\frac{1}{\lambda-z}d\rho(\lambda,t),$$

which yields the required formula (6.6). □

Using Theorems 2 and 3 one can show that the basic formula (2.12) is valid under the assumption that $\rho(\cdot,t)$ is the spectral measure of the solution $L(t)$ (before we proved (2.12) when assuming that $\rho(\cdot,t)$ is the (Φ,Ψ)-transform of $\rho(\cdot,0)$). Below we formulate this result precisely and give an alternative proof on the base of Lemma 5.

Lemma 6 *Let $F(\lambda,t)$ be a complex valued function which is analytic with respect to λ in some neighborhood $G\subset\mathbb{C}$ of the interval $[a,b]$ (for every fixed $t\in[0,T]$) and continuously differentiable with respect to t in $[0,T]$ (for every fixed $\lambda\in G$). Then the function*

$$f(t)=\int_{\mathbb{R}}F(\lambda,t)d\rho(\lambda,t),\qquad t\in[0,T],\tag{6.9}$$

is continuously differentiable and its derivative has the form

$$\frac{d}{dt}\int_{\mathbb{R}}F(\lambda,t)d\rho(\lambda,t)=\int_{\mathbb{R}}\Big(F'(\lambda,t)\Phi(\lambda,t)+\dot F(\lambda,t)+F(\lambda,t)\Theta(\lambda,t)\Big)\,d\rho(\lambda,t).\tag{6.10}$$

P r o o f Let $\Gamma\subset G$ be some contour which encloses $[a,b]$. Then it follows from the Cauchy formula that

$$\int_{\mathbb{R}}F(\lambda,t)\;d\rho(\lambda,t)=\int_{\mathbb{R}}\Big(\frac{1}{2\pi i}\oint_{\Gamma}\frac{F(\zeta,t)}{\zeta-\lambda}\;d\zeta\Big)\,d\rho(\lambda,t)=$$
$$-\frac{1}{2\pi i}\oint_{\Gamma}\Big(\int_{\mathbb{R}}\frac{1}{\lambda-\zeta}d\rho(\lambda,t)\Big)\,F(\zeta,t)d\zeta,\quad t\in[0,T].$$

Using this representation and formula (6.6) we get:

$$\frac{d}{dt}\int_{\mathbb{R}}F(\lambda,t)\,d\rho(\lambda,t)=-\frac{1}{2\pi i}\oint_{\Gamma}\Big\{\Big(\int_{\mathbb{R}}\frac{1}{\lambda-\zeta}\,d\rho(\lambda,t)\Big)^{\cdot}F(\zeta,t)+$$
$$\Big(\int_{\mathbb{R}}\frac{1}{\lambda-\zeta}d\rho(\lambda,t)\Big)\dot F(\zeta,t)\Big\}d\zeta=-\frac{1}{2\pi i}\oint_{\Gamma}\Big\{\Big(\int_{\mathbb{R}}\Big(-\frac{\Phi(\lambda,t)}{(\lambda-\zeta)^2}+$$
$$\frac{\Theta(\lambda,t)}{\lambda-\zeta}\Big)\,d\rho(\lambda,t)\Big)F(\zeta,t)+\Big(\int_{\mathbb{R}}\frac{1}{\lambda-\zeta}d\rho(\lambda,t)\Big)\dot F(\zeta,t)\Big\}d\zeta=$$
$$\int_{\mathbb{R}}\Big\{\frac{1}{2\pi i}\oint_{\Gamma}\frac{F(\zeta,t)}{(\lambda-\zeta)^2}d\zeta\cdot\Phi(\lambda,t)+\frac{1}{2\pi i}\oint_{\Gamma}\frac{F(\zeta,t)}{\zeta-\lambda}d\zeta\cdot\Theta(\lambda,t)+$$
$$\frac{1}{2\pi i}\oint_{\Gamma}\frac{\dot F(\zeta,t)}{\zeta-\lambda}d\zeta\Big\}\,d\rho(\lambda,t)=\int_{\mathbb{R}}\{F'(\lambda,t)\Phi(\lambda,t)+$$
$$F(\lambda,t)\Theta(\lambda,t)+\dot F(\lambda,t)\}d\rho(\lambda,t),$$

as needed. □

REMARK 5 The assertions of Lemmas 5 and 6 hold true for polynomials $\Phi(\lambda,t)$, $\Psi(\lambda,t)$ whose t dependent coefficients are continuously differentiable (as in Section 2).

Lemma 7 *Assume that the coefficients $\varphi_i(t), \psi_i(t)$ of polynomials $\Phi(\lambda,t), \Psi(\lambda,t)$, and the function $F(\cdot,t)$ from Lemma 6 are smooth, i.e., belong to the class $C^\infty([0,T])$. Then the function (6.9) is also smooth and its derivatives are given by the following formula:*

$$f^{(n)}(t) := \frac{d^n}{dt^n}\left(\int_{\mathbb{R}} F(\lambda,t)d\rho(\lambda,t)\right) = \int_{\mathbb{R}} (M^n F)(\lambda,t)d\rho(\lambda,t), \qquad n = 0,1,\dots, \tag{6.11}$$

where the differential expression M is of the form

$$\begin{aligned} (MF)(\lambda,t) &= \Phi(\lambda,t)\left(\frac{\partial F}{\partial\lambda}\right)(\lambda,t) + \left(\frac{\partial F}{\partial t}\right)(\lambda,t) + \Theta(\lambda,t)F(\lambda,t), \\ & \lambda \in [a,b], \quad t \in [0,T]; \quad M^0 F := F. \end{aligned} \tag{6.12}$$

Proof For $n = 0$ and $n = 1$ formula (6.11) is the same as (6.9) and (6.10), respectively. Under our assumptions the function $(MF)(\lambda,t)$, $(\lambda,t) \in G \times [0,T]$ has the same differential properties as $F(\lambda,t)$. It follows from Lemma 6 that the function

$$\left(\frac{df}{dt}\right)(t) = \int_{\mathbb{R}} (MF)(\lambda,t)d\rho(\lambda,t)$$

is differentiable, and for its derivative the representation (6.10) is valid. This justifies (6.11) for $n = 2$. One can easily complete the proof by induction. □

Proof of Theorem 4 Let $L_1(t)$ and $L_2(t)$ be solutions of the Cauchy problem (3.3), (3.8) such that $L_1(0) = L_2(0)$. Let us take the corresponding spectral measures $\rho_1(\lambda,t)$ and $d\rho_2(\lambda,t)$, which are normalized as in (6.3). This means that they both meet the condition (6.4). The inverse spectral problem has a unique solution which can be found via formulas (1.8)–(1.10). Hence, to prove the theorem it is sufficient to show that the moment sequences of measures $\rho_1(\cdot,t)$ and $\rho_2(\cdot,t)$ are the same on $[0,T]$. Introduce for every $t \in [0,T]$ a real-valued measure $\mathfrak{B}(\mathbb{R}) \ni \Delta \mapsto \omega(\Delta,t) = \rho_1(\Delta,t) - \rho_2(\Delta,t) \in \mathbb{R}$. We need to prove that all the moments of the measure $\omega(\cdot,t)$ vanish:

$$\int_{\mathbb{R}} \lambda^q d\omega(\lambda,t) = 0\ , \qquad q = 0,1,\dots\ , \quad t \in [0,T]\ . \tag{6.13}$$

To do this, let us take any nonnegative integer q and prove that the function

$$f(t) = \int_{\mathbb{R}} \lambda^q\ d\omega(\lambda,t)\ , \tag{6.14}$$

is identically equal to 0 on $[0,T]$.

By Lemma 7 we have $f \in C^\infty([0,T])$, and

$$f^{(n)}(t) = \int_{\mathbb{R}} (M^n F)(\lambda,t)d\omega(\lambda,t)\ , \quad F(\lambda,t) = \lambda^q\ , \quad n = 0,1,\dots\ , \quad t \in [0,T]\ . \tag{6.15}$$

Under our assumptions $d\rho_1(\lambda,0) = d\rho_2(\lambda,0)$, i.e., $\omega(\cdot,0)=0$, and hence $f^{(n)}(0)=0$, $n=0,1,\cdots$. We shall prove below that for some constant $K>0$

$$|f^{(n)}(t)| \le K^n n!, \qquad n=1,2,\ldots, \qquad t\in[0,T]. \tag{6.16}$$

These estimates ensure the analyticity of $f(t)$ in some neighborhood of $[0,T]$, and therefore the conditions $f^{(n)}(0)=0$, $n=0,1,\ldots$, will give the equality $f(t)=0$ for all $t\in[0,T]$.

Let us prove the estimates (6.16). Bearing in mind (6.15), (2.1), and (2.13), we take r enough large and write down the polynomials F, Φ, and Θ in the following form:

$$F(\lambda,t)=\sum_{i=0}^{r-1} f_i(t)\lambda^i\ , \qquad \Phi(\lambda,t)=\sum_{i=0}^{r-1}\varphi_i(t)\lambda^i\ , \qquad \Theta(\lambda,t)=\sum_{i=0}^{r-1}\theta_i(t)\lambda^i\ , \tag{6.17}$$

where the coefficients $f_i(t),\varphi_i(t),\theta_i(t)$, $i=0,1,\cdots,r-1$, are analytic in some domain $D\supset[0,T]$ of the complex plane (clearly, some of the coefficients are zero). One can see that the expressions $(M^nF)(\lambda,t)$ are polynomials in λ with analytic in the same domain D t-dependent coefficients:

$$\begin{aligned}
F(\lambda,t) &= \lambda^q\ ,\\
(MF)(\lambda,t) &= \Phi(\lambda,t)(\frac{\partial}{\partial\lambda}F)(\lambda,t)+(\frac{\partial}{\partial t}F)(\lambda,t)+\Theta(\lambda,t)F(\lambda,t)=\\
&\quad \sum_{i=0}^{\ell} q\varphi_i(t)\lambda^{i+q-1}+\sum_{i=0}^{\ell+m-1}\theta_i(t)\lambda^{i+q}\ , \qquad (6.18)\\
(M^2F)(\lambda,t) &= \Phi(\lambda,t)(\frac{\partial}{\partial\lambda}MF)(\lambda,t)+(\frac{\partial}{\partial t}MF)(\lambda,t)+\Theta(\lambda,t)(MF)(\lambda,t)=\ldots\ ,\\
(M^3F)(\lambda,t) &= \Phi(\lambda,t)(\frac{\partial}{\partial\lambda}M^2F)(\lambda,t)+(\frac{\partial}{\partial t}M^2F)(\lambda,t)+\Theta(\lambda,t)(M^2F)(\lambda,t)=\ldots\ ,
\end{aligned}$$

and so forth. The estimate for derivatives of any analytic function $[0,T]\ni t\mapsto g(t)\in\mathbb{C}$

$$\begin{aligned}
\exists C_g>0: \qquad & |g^{(n)}(t)| = \Big|\frac{n!}{2\pi i}\oint_\Gamma \frac{g(\zeta)}{(\zeta-t)^{n+1}}d\zeta\Big| \le C_g^{n+1}n!\ ,\\
& n=0,1,\ldots\ , \quad t\in[0,T]\ , \qquad (6.19)
\end{aligned}$$

(fixed Γ encloses $[0,T]$, $\Gamma\subset D$) ensures the existence of constants $A,B\ge 1$ such that the absolute value of every derivative $\partial^n/\partial t^n$ of the terms

$$f_i(t)\lambda^j k\ , \qquad \varphi_i(t)\lambda^j k\ , \qquad \theta_i(t)\lambda^j k\ , \tag{6.20}$$

is not greater than $AB^n n!$, $n=0,1,\ldots$, $(\lambda,t)\in[a,b]\times[0,T]$, $i,j,k=0\ldots,r-1$. For example, we can take

$$A=(r-1)\max\{1,\ Q^{r-1}\},\quad B=\max_i\{C_{f_i},\ C_{\varphi_i},\ C_{\theta_i}\},\qquad i=0,1,\ldots,r-1,$$

where Q is given by (6.1), while the constants C come from (6.19).

The polynomial F from (6.17) contains r terms of the first type from (6.20), therefore $|F(\lambda,t)|\le r\cdot A$, $(\lambda,t)\in[a,b]\times[0,T]$. The polynomial MF, according to (6.12), contains less than $r\cdot r\cdot 3$ terms, where every such term is a product of two terms of the type (6.20) or some of their first derivatives $\partial/\partial t$. Therefore, one can write $|(MF)(\lambda,t)| \le 3r^2 \cdot A \cdot AB1!$, $(\lambda,t) \in [a,b] \times [0,T]$. Similarly, the polynomial M^2F contains less than $r \cdot 3r^2 \cdot 3$ terms, each of which is a product of three terms of type (6.20) or some of their derivatives $\partial/\partial t, \partial^2/\partial^2 t$. Hence, one has $|(M^2F)(\lambda,t)| \le 3^2r^3 \cdot A \cdot AB \cdot AB^2 2!$, $(\lambda,t) \in [a,b] \times [0,T]$. Continuing this procedure, it is easy to understand that for every $n = 3,4,\ldots$ the polynomial M^nF contains less than $r \cdot 3^{n-1}r^n \cdot 3 = 3^n r^{n+1}$ terms, each of which is a product of $n+1$ terms of type (6.20) or some of their derivatives $\partial^{\nu_1}/\partial t^{\nu_1}, \ldots, \partial^{\nu_k}/\partial t^{\nu_k}$, where $\nu_1 + \cdots + \nu_k \le n$. Therefore, the absolute value of every such product does not exceed $A^{n+1}B^{\nu_1}\cdots B^{\nu_k}\nu_1!\ldots\nu_k! \le A^{n+1}B^n n!$, in $[a,b] \times [0,T]$, and we have

$$|(M^nF)(\lambda,t)| \le 3^n r^{n+1} \cdot A^{n+1}B^n n! \ , \qquad (\lambda,t) \in [a,b] \times [0,T] \ , \quad n = 0,1,\ldots \ .$$

This estimate and (6.15) prove (6.16), which completes the proof of the theorem. □

Remark 6 The application of Theorem 4 to the linear system of Theorems 2 and 3 gives the uniqueness of the solution for the Cauchy problem (3.11), (3.12) in the class of moment sequences $(h_k(t))_{k=0}^{\infty}$ of measures $\rho(\cdot,t) \in \mathfrak{M}$. Because of the factor k on the right hand side of (3.11), in general the operator of this linear system is unbounded.

7 Modification and examples

Let us consider instead of (2.7) the following partial differential equation:

$$\frac{\partial s(\lambda,t)}{\partial \lambda}\Phi(\lambda,t) + \frac{\partial s(\lambda,t)}{\partial t} = \Psi(\lambda,t)s(\lambda,t) \ . \tag{7.1}$$

Let $s(\lambda,t)$ be its nonnegative solution, such that

$$s(\lambda,0) = 1, \qquad \lambda \in \mathbb{R} \ . \tag{7.2}$$

Replacing $r(\lambda,t)$ by $s(\lambda,t)$ in the procedure of Section 2, we can construct a new measure $\sigma(\cdot,t) \in \mathfrak{M}$: $d\sigma(\lambda,t) = s(\lambda,t)\tilde\rho(\lambda,t)$. It follows from (7.1) that if $\dot\lambda(t) = \Phi(\lambda(t),t)$ then one has:

$$\frac{d}{dt}s(\lambda(t),t) = \Psi(\lambda(t),t)s(\lambda(t),t) \ . \tag{7.3}$$

The basic formula (2.12) (with Ψ in place of Θ) remains true for this measure (compare (2.11), (2.13) and (7.3)). Hence, *the corresponding Jacobi matrix $L(t)$ meets the appropriately modified equation (3.3):*

$$\dot L(t) = \Phi(L(t),t) + [L(t), \langle \Phi(L(t),t)D_{L(t)} + \frac{1}{2}\Psi(L(t),t)\rangle] \ . \tag{7.4}$$

Remark 7 Since $\Theta(\lambda,t)$ in (2.13) has a special structure, equation (7.4) is more general than our original equation (3.3). However, in order to solve (7.4) we have to integrate not only equation (2.3) but also (7.1) with initial condition (7.2).

Remark 8 The moment equation (3.11) (or, which is the same, (5.24)) should be replaced by the following equation:

$$\dot h_k(t) = k\sum_{i=0}^{\ell}\varphi_i(t)h_{i+k-1}(t) + \sum_{j=0}^{m}\psi_j(t)h_{j+k}(t) \ , \tag{7.5}$$

where $\varphi_i(t)$ and $\psi_j(t)$ are the coefficients of $\Phi(\lambda,t)$ and $\Psi(\lambda,t)$, as in (2.1).

We are going to write down the differential-difference equations corresponding to (7.4) for the case when

$$\Phi(\lambda,t)=\varphi_0+\varphi_1\lambda+\varphi_2\lambda^2, \qquad \Psi(\lambda,t)=\psi_0+\psi_1\lambda\ , \tag{7.6}$$

where φ_i and ψ_j are real constants. The following terms appearing in (3.4), (3.5) are to be expressed through a_n and b_n:

$$\begin{aligned}
\Phi_{jk} &= \{\Phi(L(t),t)\}_{jk}=\{\varphi_0\mathbf{1}+\varphi_1L(t)+\varphi_2L^2(t)\}_{jk}\ , \quad j=k,k+1,k+2\ ,\\
\Psi_{jk} &= \{\Psi(L(t),t)\}_{jk}=\{\psi_0\mathbf{1}+\psi_1L(t)\}_{jk}\ , \quad j=k,k+1\ ,\\
\Omega_{jk} &= \{\Phi(L(t),t)D_{L(t)}\}_{jk}=\{\varphi_0D_{L(t)}+\varphi_1L(t)D_{L(t)}+\varphi_2L^2(t)D_{L(t)}\}_{jk}\ ,\\
&\qquad j=k+1,k+2\ .
\end{aligned}$$

We have

$$\begin{aligned}
\Phi_{k+1,k} &= a_k(\varphi_1+\varphi_2b_k+\varphi_2b_{k+1})\ ,\\
\Phi_{k+1,k-1} &= \varphi_2a_{k-1}a_k\ ,\\
\Phi_{kk} &= \varphi_0+\varphi_1b_k+\varphi_2(a_{k-1}^2+b_k^2+a_k^2)\ ,\\
\Psi_{kk} &= \psi_1b_k+\psi_0\ ,\\
\Psi_{k+1,k} &= \psi_1a_k\ .
\end{aligned}$$

Since $\Phi(L(t),t)$ is a 5-diagonal matrix and $D_{L(t)}$ is a strictly upper triangular matrix, one has: $\Omega_{k+2,k}=0$. Moreover, it follows from (1.15) that

$$\Omega_{k+1,k}=\Phi_{k+1,k-1}\{D_{L(t)}\}_{k-1,k}=\varphi_2a_{k-1}a_k\frac{k}{a_{k-1}}=k\varphi_2a_k\ .$$

Substituting all these expressions into (3.4) and (3.5) (where Θ is replaced by Ψ), we obtain the following nonlinear equations:

$$\begin{aligned}
\dot a_n &= \frac{1}{2}a_n(2\varphi_1+2\varphi_2(b_n+b_{n+1})+\psi_1(b_{n+1}-b_n)+2n\varphi_2(b_{n+1}-b_n))\ , && (7.7)\\
\dot b_n &= \varphi_0+\varphi_1b_n+a_{n-1}^2(\varphi_2-\psi_1-2\varphi_2(n-1))+a_n^2(\varphi_2+\psi_1+2\varphi_2n)+\varphi_2b_n^2\ , && (7.8)\\
&\qquad n=0,1,\ldots;\quad a_{-1}=0\ .
\end{aligned}$$

Let us consider several concrete examples. The procedure of integration of the corresponding equations will be explained at the end of this section.

Example 1 For the following choice of coefficients,

$$\varphi_0=-4\delta, \qquad \varphi_1=\psi_0=0\ , \qquad \varphi_2=\delta\ , \qquad \psi_1=\delta-1\ , \tag{7.9}$$

after the change of variables

$$A_n=a_n^2, \qquad\qquad B_n=b_n\ , \tag{7.10}$$

we get from (7.7) and (7.8) the so-called inhomogeneous Toda lattice [20] on semi-axis:

$$\begin{aligned} \dot{A}_n &= A_n\{B_n(1-\delta(2n-1)) - B_{n+1}(1-\delta(2n+3))\} \, , \\ \dot{B}_n &= A_{n-1}(1-2\delta(n-1)) - A_n(1-2\delta(n+1)) + \delta(B_n^2-4) \, , \\ & n=0,1,\dots; \quad A_{-1}=0 \, . \end{aligned} \tag{7.11}$$

Example 2 If we take $\varphi_0 = \varphi_1 = \psi_0 = 0$ and $\varphi_2 = \psi_1 = 1$, we get from (7.7) and (7.8) the following equations:

$$\begin{aligned} \dot{a}_n &= \frac{1}{2} a_n(b_{n+1}(3+2n) - b_n(2n-1)) \, , \\ \dot{b}_n &= 2a_{n-1}^2(1-n) + 2a_n^2(1+n) + b_n^2 \, , \\ & n=0,1,\dots; \quad a_{-1}=0 \, . \end{aligned} \tag{7.12}$$

It follows from (7.5) that the corresponding moments are subject to the equation $\dot{h}_k(t) = (k+1)h_{k+1}(t) \quad k=0,1,\dots$, which can be easily integrated:

$$\begin{aligned} h_0(t) &= \sum_{j=0}^{\infty} h_j(0)t^j = \{(1-tL_0)^{-1}\}_{00} \, , \\ h_k(t) &= \frac{1}{k!}\,\frac{d^k h_0(t)}{dt^k} = \sum_{j=0}^{\infty} \frac{(k+j)!}{k!} h_{k+j}(0)\cdot t^j \, , \end{aligned}$$

where $h_j(0) = \{L_0^j\}_{00}$, $j = 0,1,\cdots$, are initial moments of $L_0 \in \mathfrak{L}$.

In view of the estimate $h_j(0) \le \|L_0\|^j$, $j=0,1,\dots,$ all the series converge absolutely and uniformly on the interval $t \in [0,T]$ for every positive T such that $T < \|L_0\|^{-1}$. Therefore, the solution of the Cauchy problem on this interval can be found via formulas (1.10) or (1.11) as will be explained below.

Let polynomials

$$\Phi(\lambda,t) = \sum_{j=0}^{\ell} \varphi_{2i+1}(t)\lambda^{2i+1} \quad , \qquad \Psi(\lambda,t) = \sum_{i=0}^{m} \psi_{2i}(t)\lambda^{2i}$$

be odd and even functions of λ, respectively. Assume that all the diagonal entries $b_n(0)$ of the initial Jacobi matrix L_0 are zero. This is equivalent to the fact that the corresponding initial measure $\rho(\cdot,0)$ is even, i.e., its support is a symmetric subset of $\mathbb{R}$ and for any odd integrable function $g(\lambda)$ the following integral vanishes:

$$\int_{\mathbb{R}} g(\lambda)d\rho(\cdot,0) = 0.$$

In the present situation the function $s(\lambda,t)$ is even, as one can see from (2.3), (7.1) under our assumptions on $\Phi(\lambda,t)$ and $\Psi(\lambda,t)$. Therefore, the mapping (2.4) (more precisely, its counterpart for our modification) produces the measure $\rho(\cdot,t)$, which is also even. Thus, the solution $L(t)$ of the Cauchy problem corresponding to equation (7.4) preserves this extra structure for all t for which it exists (cf. [2]):

$$b_n(t) = 0, \qquad\qquad n=0,1,\dots \, , \tag{7.13}$$

and equations (3.4), (3.5) are reduced to the following system:

$$\begin{aligned}\dot{a}_n(t) \quad = \quad & \{\Phi(L(t),t)\}_{n+1,n} + \frac{1}{2}a_n(\{\Psi(L(t),t)\}_{n+1,n+1} - \{\Psi(L(t),t)\}_{nn}) + \\ & a_{n+1}\{\Phi(L(t),t)D_{L(t)}\}_{n+2,n} - a_{n-1}\{\Phi(L(t),t)D_{L(t)}\}_{n+1,n-1}, \\ & \quad n = 0,1,\ldots; \quad a_{-1} = 0\ .\end{aligned} \tag{7.14}$$

One can show that the right hand side of (7.14) is a product of $a_n(t)$ and some polynomial function in variables $a_k(t)$, and therefore, these equations can be regarded as generalizations of the classical Lotka-Volterra system.

Example 3 a) If we take $\Phi(\lambda,t) = 0$ and $\Psi(\lambda,t) = \lambda^2$, and substitute $x_n = a_n^2$, then equation (7.14) transforms into the Kac-van Moerbeke system:

$$\dot{x}_n(t) = x_n(x_{n+1} - x_{n-1}), \quad n = 0,1,\cdots;\ x_{-1} = 0$$

(see e.g. [7, 16, 23, 4]).
b) Consider the following equation :

$$\dot{x}_n(t) = (x_n + n)(x_{n+1} - x_{n-1}), \qquad n = 0,1,\ldots; \quad x_{-1} = 1\ . \tag{7.15}$$

Let us rewrite (7.15) in variables $a_n = \sqrt{x_n + n}, \quad n = 0,1,\ldots$:

$$\dot{a}_n(t) = -a_n + \frac{1}{2}a_n(a_{n+1}^2 - a_{n-1}^2), \qquad n = 0,1,\ldots; a_{-1} = 0\ . \tag{7.16}$$

It can be easily checked that this equation is of the form (7.14) with $\Phi(\lambda,t) = -\lambda$ and $\Psi(\lambda,t) = \lambda^2$. In view of Theorem 1 this allows us to integrate (7.16) and to find the solution of (7.15) (the details will be provided below).
c) By taking $\Phi(\lambda,t) = \lambda^3$ and $\Psi(\lambda,t) = \lambda^2$ we get from (7.14):

$$\dot{a}_n(t) = \frac{1}{2}a_n\{a_{n-1}^2(3-2n) + 2a_n^2 + a_{n+1}^2(3+2n)\}, \qquad n = 0,1,\ldots; a_{-1} = 0\ .$$

After the change of variables $x_n = 2na_n^2$ we obtain another kind of "nonisospectral Kac-van Moerbeke equation":

$$\dot{x}_n(t) = x_n(x_{n+1} - x_{n-1}) + \frac{1}{2n}x_n\left(\frac{n}{n+1}x_{n+1} + x_n + \frac{n}{n-1}x_{n-1}\right), \ n = 0,1,\cdots;\ x_{-1} = 0\ . \tag{7.17}$$

Note that the Cauchy problem for (7.17) can be locally resolved by the ISP method provided the initial values $x_n(0)$ grow with n not too fast (see below).

Let us return to Example 1 and present the ISP method of integration of the inhomogeneous Toda lattice (7.11). In this case, according to (7.9), the polynomials Φ and Ψ have the form $\Phi(\lambda,t) = \delta(\lambda^2 - 4)$, $\Psi(\lambda,t) = (\delta - 1)\lambda$. Therefore the Cauchy problems (2.3) and (7.1), (7.2) are as follows:

$$\frac{d\lambda(t)}{dt} = \delta(\lambda^2(t) - 4)\ , \qquad \lambda(0) = \mu; \tag{7.18}$$

$$\begin{aligned}\frac{\partial s(\lambda,t)}{\partial\lambda}\delta(\lambda^2 - 4) \quad + \quad & \frac{\partial s(\lambda,t)}{\partial t} = (\delta - 1)\lambda s(\lambda,t)\ , \\ s(\lambda,0) \quad = \quad & 1, \qquad \lambda \in \mathbb{R}\ .\end{aligned} \tag{7.19}$$

Let supp $d\rho(\cdot,\lambda)$ be a spectrum set of the initial Jacobi matrix. It is easy to calculate the solution $\lambda(t,\mu)$ of (7.18):

$$\lambda(t,\mu)=2\frac{\mu+2+(\mu-2)e^{4\delta t}}{\mu+2-(\mu-2)e^{4\delta t}}\;,\qquad \mu\in\text{supp }d\rho(\cdot,\lambda)\;,\qquad t\in[0,T]\;,\tag{7.20}$$

where $T>0$ should be chosen in such a manner that for all $\mu\in$ supp $d\rho(\cdot,\lambda)$ formula (7.20) makes sense. Note that a small enough number $T>0$ with this property always exists.

The measure $\tilde\rho(\cdot,t)$ is constructed as the image of $\rho(\cdot,0)$ under the mapping $\mathbb{R}\ni\mu\mapsto\omega_t(\mu)=\lambda(t,\mu)\in\mathbb{R}$. To solve the Cauchy problem (7.19), we explore the method of characteristics (see e.g., [27]). We rewrite equation (7.19) using the classical variables (x,y,z) in place of (λ,t,s) :

$$\delta(x^2-4)\frac{\partial z(x,y)}{\partial x}+\frac{\partial z(x,y)}{\partial y}=(\delta-1)\;x\;z(x,y)\;.\tag{7.21}$$

Our next objective is to find a surface $z=z(x,y)$ passing through the curve $\ell: z(x,0)=1$, $x\in\mathbb{R},\; y=0$, in the space (x,y,z). Let us represent the curve ℓ in the parametric form $\ell=\{(x,y,z): x=v,\; y=0,\; z=1;\; v\in\mathbb{R}\}$, and consider the characteristic system of equation (7.21):

$$\frac{dx}{du}=\delta(x^2-4),\qquad \frac{dy}{du}=1,\qquad \frac{dw}{du}=(\delta-1)\;x\;w\;.\tag{7.22}$$

By integrating (7.22) with the initial data $x(0)=v,\;\; y(0)=0,\;\; w(0)=1$ we get the (u,v)-parametric equations of our integral surface:

$$\begin{aligned}&x=x(u,v)=2\frac{v+2+(v-2)e^{4\delta u}}{v+2-(v-2)e^{4\delta u}}\;,\qquad y=y(u,v)=u\;,\\ &z=w(u,v)=e^{2(\delta-1)u}\left(\frac{1}{4}(v+2-(v-2)e^{4\delta u})\right)^{\frac{1-\delta}{\delta}}\;,\qquad u,v\in\mathbb{R}\;.\end{aligned}\tag{7.23}$$

It follows by elimination of u,v from (7.23) that the solution of the Cauchy problem (7.19) is

$$z(x,y)=2^{\frac{2(1-\delta)}{\delta}}e^{2(\delta-1)y}\left(x+2-(x-2)e^{-4\delta y}\right)^{\frac{\delta-1}{\delta}}\;,\qquad x,y\in\mathbb{R}\;,\tag{7.24}$$

or, in the old variables,

$$s(\lambda,t)=2^{\frac{2(1-\delta)}{\delta}}e^{2(\delta-1)t}\left(\lambda+2-(\lambda-2)e^{-4\delta t}\right)^{\frac{\delta-1}{\delta}}\;,\qquad \lambda\in\mathbb{R}\;,\qquad t\in[0,T]\;.\tag{7.25}$$

The corresponding spectral measure $d\sigma(\lambda,t)$ has the form $d\sigma(\lambda,t)=s(\lambda,t)d\tilde\rho(\lambda,t)$, where $\tilde\rho(\cdot,t)$ is constructed via (7.20). The function $s(\lambda,t)$ should be nonnegative on supp $\rho(\cdot,t)$, therefore we take a small enough $T>0$.

The solution of the Cauchy problem (7.11) is unique (by Theorem 4) and can be found by the procedure of Theorem 1. One should replace $\rho(\cdot,t)$ by $\sigma(\cdot,t)$, take into account relations (7.10), and compute $a_n(t)$ and $b_n(t)$) via formulas (1.7) or (1.10).

To calculate integrals with respect to the measure $d\sigma(\lambda,t)$ (this is a necessary step when using Theorem 1), it is convenient to apply the following formula for enough arbitrary function $F(\lambda)$:

$$\int_{\mathbb{R}} F(\lambda)\, d\sigma(\lambda,t) = 2^{\frac{2(1-\delta)}{\delta}} e^{2(\delta-1)t} \int_{\mathbb{R}} F\left(2\frac{\mu+2+(\mu-2)e^{4\delta t}}{\mu+2-(\mu-2)e^{4\delta t}}\right) \times$$
$$(\mu+2-(\mu-2)e^{4\delta t})^{\frac{1-\delta}{\delta}} d\rho(\mu,0)\ , \qquad t\in[0,T]\ , \tag{7.26}$$

where $\rho(\cdot,0)$ is a spectral measure of the initial Jacobi matrix. This formula is a consequence of (2.10), (7.20) and (7.25). The value of $T>0$ is small enough and depends on supp $\rho(\cdot,0) : T$ is such that $\mu+2-(\mu-2)e^{4\delta t}>0$ for $\mu\in \mathrm{supp}\rho(\cdot,0)$ and $t\in[0,T]$.

Remark 9 If $\delta=0$ then the inhomogeneous Toda lattice (7.11) becomes the classical one (3.7) up to a simple change of variables. It follows from (7.20) that in this case for all $t\in[0,T]:\ \ \lambda(t,\mu)=\mu$, and therefore $\tilde\rho(\cdot,t)=\rho(\cdot,0)$. It is also easy to verify that the solution (7.25) tends to $e^{-\lambda t}$ when $\delta\to 0$, which is consistent with the classical situation.

For Example 2, when $\Phi=\lambda^2$, $\Psi=\lambda$, the calculations are even simplier, and one can easily get instead of (7.20), (7.25) and (7.26), the following formulas:

$$\begin{aligned} \lambda(t,\mu) &= \frac{\mu}{1-\mu t}, \qquad s(\lambda,t)=1+\lambda t\ ,\\ \int_{\mathbb{R}} F(\lambda)\, d\sigma(\lambda,t) &= \int_{\mathbb{R}} F(\frac{\mu}{1-\mu t})\frac{1}{1-\mu t}\, d\rho(\mu,0)\ , \qquad t\in[0,T]\ . \end{aligned} \tag{7.27}$$

If the initial Jacobi matrix is stable (the spectrum is in the left semi-axis), then $T>0$ can be chosen arbitrarily, while in general it has to be small enough to ensure that $1-\mu t>0$ for $\mu\in$ supp $(\rho\cdot,0)$ and $t\in[0,T]$. By Theorem 4 the solution of the Cauchy problem (7.12) is unique and can be found by means of Theorem 1 for $t\in[0,T]$.

Concerning Example 3, the situation is as follows. In case (a) we have the classical isospectral equation. As for cases (b) and (c), we can apply the scheme of Example 1 and get the solutions by means of Theorem 1. The formulas (7.20), (7.25), and (7.26) in case (b) should be replaced by

$$\begin{aligned} \lambda(t,\mu) &= \mu e^{-t}\ , \qquad s(\lambda,t)=e^{\frac12\lambda^2(e^{2t}-1)}\ ,\\ \int_{\mathbb{R}} F(\lambda)\, d\sigma(\lambda,t) &= \int_{\mathbb{R}} F(\mu e^{-t})e^{\frac12\mu^2(1-e^{-2t})}d\rho(\mu,0)\ , \qquad t\in[0,T]\ , \end{aligned}$$

respectively, where $T>0$ can be chosen arbitrarily.

In case (c) we have

$$\begin{aligned} \lambda(t,\mu) &= \frac{\mu}{\sqrt{1-2\mu^2 t}}\ , \qquad s(\lambda,t)=\sqrt{1+2\lambda^2 t}\ ,\\ \int_{\mathbb{R}} F(\lambda)d\sigma(\lambda,t) &= \int_{\mathbb{R}} F\left(\frac{\mu}{\sqrt{1-2\mu^2 t}}\right)\frac{1}{\sqrt{1-2\mu^2 t}}\, d\rho(\mu,0)\ , \qquad t\in[0,T]\ , \end{aligned}$$

$T>0$ is chosen small enough, such that $1-2\mu^2 t>0$ for $\mu\in$ supp $\rho(\cdot,0)$ and $t\in[0,T]$.

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